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# On the Okounkov–Olshanski formula for standard tableaux of skew shapes

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**Abstract.** The classical hook-length formula counts the number of standard tableaux of straight shapes, but there is no known product formula for skew shapes. Okounkov–Olshanski (1996) and Naruse (2014) found new positive formulas for the number of standard Young tableaux of a skew shape. We prove various properties of the Okounkov–Olshanski formula: a reformulation similar to the Naruse formula, determinantal formulas for the number of terms, and a $q$-analogue extending the formula to reverse plane partitions, which complements work by Chen and Stanley for semistandard tableaux.

**Résumé.** La formule classique des équerres compte le nombre de tableaux standard de formes droites, mais il n'existe aucune formule de produit pour les formes gauches. Okounkov–Olshanski (1996) et Naruse (2014) ont découvert de nouvelles formules positives pour le nombre de tableaux de Young standard de forme gauche. Nous prouvons diverses propriétés de la formule d'Okounkov–Olshanski: une reformulation similaire à la formule de Naruse, des formules déterminantes pour le nombre de termes et un $q$-analogue pour les partitions planes inversées, ce qui complète les travaux de Chen et Stanley pour semi-standard tableaux.

**Keywords:** standard tableaux, skew shapes, Okounkov–Olshanski formula, skew reverse plane partitions

## 1 Introduction

Standard Young tableaux are fundamental objects in algebraic and enumerative combinatorics with origins in representation theory and numerous applications elsewhere: semistandard tableaux, an extension of standard tableaux, are inherent in the representation theory of the general linear group, and they define Schur functions, which are one of the key bases of the ring of symmetric functions. A further extension of these ideas to skew shapes yields a rich theory related to descents of permutations, jeu de taquin,

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and Littlewood-Richardson coefficients, which have further applications to areas such as Schubert calculus.

In 1954, Frame, Robinson, and Thrall [3] discovered the hook-length formula, a product formula that counts the number of standard Young tableaux $f^\lambda$ of a certain shape $\lambda$:

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in[\lambda]} h(u)},$$

where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the cell $u = (i, j)$. The structural simplicity of the formula leads to a wide variety of applications; for instance, the fact that it involves only products allows analytic methods to be applied, yielding shapes $\lambda$ for which $f^\lambda$ is maximized when $|\lambda|$ is kept fixed [15]. In 1971, Stanley [13] found a $q$-analogue of the hook-length formula for the generating function of semistandard tableaux:

$$s_\lambda(1, q, q^2, \dots) = q^{b(\lambda)} \prod_{u\in[\lambda]} \frac{1}{1 - q^{h(u)}},$$

where $b(\lambda) = \sum_i \binom{\lambda'_i}{2}$. A $q$-analogue for the generating function of reverse plane partitions that we denote by $\mathrm{rpp}_\lambda(q)$ can be shown to satisfy the following:

$$\mathrm{rpp}_\lambda(q) = \prod_{u\in[\lambda]} \frac{1}{1 - q^{h(u)}}.$$

Considering skew shapes $\lambda/\mu$, there is no known product formula that gives the number $f^{\lambda/\mu}$ of standard Young tableaux of skew shape. However, there are recent formulas for $f^{\lambda/\mu}$ as nonnegative sums of products indexed by combinatorial objects that come from rules for *equivariant Littlewood–Richardson coefficients*. In particular, Okounkov and Olshanski [12] discovered the following formula, which will be our focus:

**Theorem 1.1** (Okounkov–Olshanski [12])**.**

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in[\lambda]} h(u)} \sum_{T\in \mathrm{SSYT}(\mu,d)} \prod_{u\in[\mu]} (\lambda_{d+1-T(u)} - c(u)), \tag{OOF}$$

*where $c(u) = j - i$ is the* content *of the cell $u = (i, j)$, $d = \ell(\lambda)$, and $\mathrm{SSYT}(\mu, d)$ is the set of SSYT of shape $\mu$ with entries $\leq d$ (see Example 3.16).*

Here, in a similar manner to Morales–Pak–Panova's study [7, 8, 9] of the Naruse hook length formula, we prove various properties of the Okounkov–Olshanski formula.

## 1.1 Number of nonzero terms

We examine properties of nonzero terms in the formula, allowing their number, denoted $\mathrm{OOT}(\lambda/\mu)$, to be counted by a determinant.

**Theorem 1.2.** *The number of nonzero terms of the Okounkov–Olshanski formula is*

$$\mathrm{OOT}(\lambda/\mu) = \det\left[\binom{\lambda_i - \mu_j + j - 1}{i-1}\right]_{i,j=1}^{d} = \det\left[\binom{\lambda'_i}{\mu'_j + i - j}\right]_{i,j=1}^{\mu_1}.$$

These results allow $\mathrm{OOT}(\lambda/\mu)$ to be evaluated in certain special cases. Most notably, in the case of a zigzag skew shape, the number of nonzero terms is given by the *Genocchi numbers* denoted by $G_n$:

**Corollary 1.3** (Conjecture by Morales–Pak–Panova, unpublished)**.** *For the zigzag* $\sigma_n = (n, n-1, \dots, 1)/(n-2, n-3, \dots, 1)$, *we have* $\mathrm{OOT}(\sigma_n) = G_n$.

Moreover, the bijection used in the proof of Theorem 1.2 can be used to give a formulation of (OOF) in terms of reverse excited diagrams: we start with cells of $[\lambda/\mu]$ and apply reverse excited moves of $\lambda/\mu$ viewed as shifted skew shape. See Example 3.16.

**Corollary 1.4** (Okounkov–Olshanski — excited diagram formulation)**.**

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in[\lambda]} h(u)} \sum_{D\in\mathcal{RE}(\lambda/\mu)} \prod_{u\in B(D)} \mathrm{arm}(u), \tag{1.1}$$

*where* $B(D)$ *are certain cells of* $[\lambda/\mu]$ *(viewed as a shifted skew shape) associated to* $D$ *and* $\mathrm{arm}(u)$ *is the length of the arm of cell* $u$.

## 1.2 A $q$-analogue for skew reverse plane partitions

There has also been work to find $q$-analogues of the Okounkov–Olshanski formula. Chen and Stanley [1] proved the following result for the generating function of skew semistandard tableaux:

**Theorem 1.5** (Chen–Stanley [1])**.**

$$\frac{s_{\lambda/\mu}(1, q, q^2, \dots)}{s_\lambda(1, q, q^2, \dots)} = \sum_{T\in\mathrm{SSYT}(\mu,d)} \prod_{u\in[\mu]} q^{T(u)-d}(1 - q^{w(u,T(u))}),$$

*where* $w(u,k) = \lambda_{d+1-k} + c(u)$.

We announce a $q$-analogue of the Okounkov–Olshanski formula for skew reverse plane partitions that is different from the skew SSYT $q$-analogue for skew shapes. We sketch a proof using identities. There is another proof using equivariant $K$-theory of Grassmannians. This result has a few reformulations and so in order to show similarity to the Okounkov–Olshanski formula and the Chen–Stanley $q$-analogue, we give a version in terms of tableaux. See Example 4.5.

**Theorem 1.6.**
$$\frac{\mathrm{rpp}_{\lambda/\mu}(q)}{\mathrm{rpp}_{\lambda}(q)} = \sum_{T\in \mathrm{SSYT}(\mu,d)} q^{p(T)} \prod_{u\in[\mu]} (1-q^{w(u,T(u))}),$$
*where* $w(u,k) = \lambda_{d+1-k} - c(u)$, $p(T) = \sum_{u\in[\mu], m_T(u)\le k< T(u)} w(u,k)$, *and* $m_T(u)$ *is the minimum* $k \le T(u)$ *such that replacing* $T(u)$ *with* $k$ *still results in a semistandard tableau.*

The full version of this abstract will appear in [10].

# 2 Preliminaries

## 2.1 Skew partitions and tableaux

A partition is denoted by $\lambda$, its size is denoted by $|\lambda|$ and its Young diagram is denoted by $[\lambda]$. Given a cell $u = (i,j) \in [\lambda]$, define the *content* $c(u) = j - i$, the arm $\mathrm{arm}(u) = \lambda_i - i + 1$, and the *hook-length* $h(u) = \lambda_i + \lambda'_j - i - j + 1$.

A *skew partition* is denoted by $\lambda/\mu$ for $[\mu] \subseteq [\lambda]$. For a strict partition $\lambda^* = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ its Young diagram is denoted by $[\lambda^*]$. We can similarly define shifted skew shapes. Given an ordinary skew shape $\lambda/\mu$ of length $d$, we denote by $\lambda^*/\mu^*$ the shifted skew shape $(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d)/(\mu_1 + d - 1, \mu_2 + d - 2, \ldots, \mu_d)$.

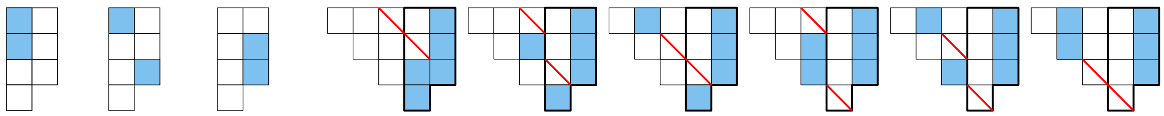
Given a (possibly skew) partition $\theta$, we denote the set of all reverse plane partitions of shape $\theta$ by $\mathrm{RPP}(\theta)$. We denote the generating function of RPP of shape $\lambda/\mu$ by $\mathrm{rpp}_\theta(q) := \sum_{T\in\mathrm{RPP}(\theta)} q^{|T|}$, where $|T|$ denotes the sum of the entries in $T$. The set of all *semistandard Young tableaux* of shape $\theta$ is denoted by $\mathrm{SSYT}(\theta)$. Let $\mathrm{SSYT}(\theta, L)$ be the set of semistandard Young tableaux with all entries at most $L$. A *set-valued semistandard Young tableau* is a filling of $\theta$ with nonempty sets of positive integers, such that for every way to choose an element from the entry of each cell, the chosen elements form a valid semistandard tableau.

## 2.2 Schur functions and generalizations

If $\mathbf{x} = (x_1, x_2, \ldots)$, $s_\theta(\mathbf{x})$ denotes the *Schur function*. If $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a finite sequence of variables and $\mathbf{a} = (a_1, a_2, \ldots)$ is an infinite sequence of variables, define the *factorial Schur function*

$$s_\theta(\mathbf{x} \mid \mathbf{a}) := \sum_{T\in\mathrm{SSYT}(\theta)} \prod_{u\in[\theta]} (x_{T(u)} - a_{T(u)+c(u)}).$$

Let $a \oplus b = a + b - ab$ and $\ominus a = \frac{a}{a-1}$ be the unique value so that $a \oplus (\ominus a) = 0$.



(a) (b)

**Figure 1:** Excited diagrams and reverse excited diagrams with broken diagonals of the shape $\lambda/\mu = 2221/11$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a finite sequence of variables and $\mathbf{a} = (\ldots, a_{-1}, a_0, a_1, \ldots)$ is an infinite sequence of variables, define the *factorial Grothendieck polynomial* [6] to be

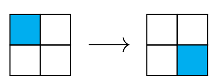
$$G_\mu(\mathbf{x} \mid \mathbf{a}) := \sum_T (-1)^{|T|-|\mu|} \prod_{\substack{u \in [\mu] \\ r \in T(u)}} (x_r \oplus a_{r+c(u)})$$

where we sum over all set-valued semistandard tableaux $T$ with all entries at most $d$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and an infinite sequence of variables $\mathbf{y} = (y_1, y_2, \ldots)$, define $\mathbf{y}_\lambda = (y_{\lambda_1+d}, y_{\lambda_2+d-1}, \ldots, y_{\lambda_d+1})$.

## 2.3 (Reverse) excited diagrams

Fix a skew shape $\lambda/\mu$. Given a subset $D$ of $[\lambda]$, consider a subset of $D'$ obtained from $D$ by applying the following move to an element of $D$ (represented in blue):



($\searrow$)

(This is only allowed if the white cells on the left side of ($\searrow$) are not in $D$ and exist in $[\lambda]$.) We call this process an *excited move*. Then, we define an *excited diagram* of $\lambda/\mu$ to be any set of $|\mu|$ cells obtained by starting with the cells of $[\mu] \subset [\lambda]$ and applying any number of excited moves. We let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of $\lambda/\mu$.

Next, we define a variant of excited diagrams. Given a skew shape $\lambda/\mu$, its *reverse excited diagrams* are the diagrams obtained by starting with the cells of the shifted skew shape $\lambda^*/\mu^*$ and applying *reverse excited moves*:

($\nwarrow$)

(we ignore momentarily the red diagonals on certain cells). We let $\mathcal{RE}(\lambda/\mu)$ denote the set of reverse excited diagrams of $\lambda/\mu$.

**Example 2.1.** The skew shape $\lambda/\mu = 2221/11$ has two excited diagrams and six reverse excited diagrams as illustrated in Figure 1.

Lastly, in each reverse excited $D$ diagram of $\mathcal{RE}(\lambda/\mu)$ we distinguish $|\mu|$ cells of the complement of $D$ as follows.

**Definition 2.2** (Broken diagonals)**.** For the reverse excited diagram $[\lambda^*/\mu^*]$, we define the diagonals $\mathsf{d}_1, \ldots, \mathsf{d}_{\ell(\mu')}$ so that $\mathsf{d}_i$ contains the cells in $[\mu^*]$ with contents $c(\mu'_i, i) = \mu'_i - i$ for $i = 1, \ldots, \ell(\mu')$. Then, iteratively, if $D$ is a reverse excited diagram with broken diagonals $\mathsf{d}_1(D), \ldots, \mathsf{d}_{\ell(\mu')}(D)$ and $D'$ is obtained from $D$ by doing the reverse excited move $(i,j) \to (i-1, j-1)$, then $(i, j-1)$ is in some $\mathsf{d}_t(D)$ (see the red diagonals in (↖)). Let

$$\mathsf{d}_r(D') = \begin{cases} \mathsf{d}_r(D) & \text{if } r \neq t, \\ \mathsf{d}_t(D) \setminus (i, j-1) \cup (i,j) & \text{if } r = t. \end{cases}$$

We denote by $B(D)$ the cells of the broken diagonals of $D$[1]. See Figure 1(b).

## 2.4 Genocchi numbers

A *pistol* is a sequence of positive integers $a_1, a_2, \ldots, a_n$ so that $a_k \leq \frac{k+1}{2}$ for all $1 \leq k \leq n$. A pistol is *strictly alternating* if $a_k \geq a_{k+1}$ for $k$ odd and $a_k < a_{k+1}$ for $k$ even, for all $1 \leq k < n$. Let the $n$th *Genocchi number* $G_n$ be the number of strictly alternating pistols of length $2n-1$ [2]. Let the $n$th *median Genocchi number* $H_n$ be the number of strictly alternating pistols of length $2n$ [16] [11, A110501], [11, A005439].

# 3 Positive Terms in the Okounkov–Olshanski Formula

## 3.1 Nonnegativity of the formula

A quick look at the formula (OOF) suggests that there could be negative terms. However, we next show that every term in the formula is nonnegative.

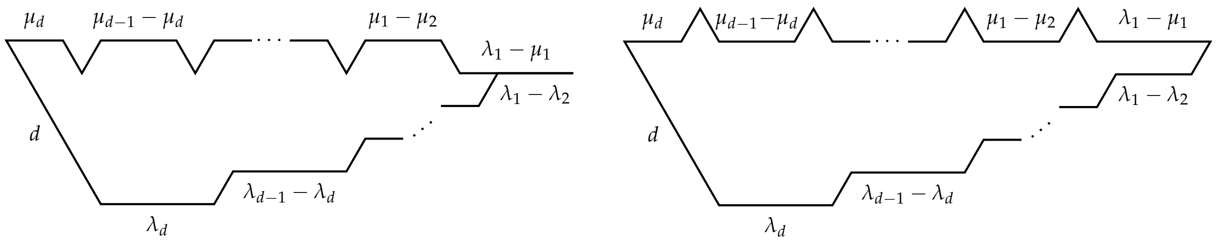
**Proposition 3.1.** *Every term in the Okounkov–Olshanski formula is nonnegative. Moreover, every positive term has all weights positive.*

*Proof.* Suppose that there exist $i, j$ so that $\lambda_{d+1-T(i,j)} - c(i,j) < 0$. It suffices to show that there exists some $i', j'$ so that $\lambda_{d+1-T(i',j')} - c(i', j') = 0$.

Since $c(i,j) > 0$, $j > i$. For $i \leq k \leq j$, consider the quantity $a_k = \lambda_{d+1-T(i,k)} - c(i,k)$. Note that $a_{k+1} - a_k = -1 + \lambda_{d+1-T(i,k+1)} - \lambda_{d+1-T(i,k)} \geq -1$. Thus, since $a_i = \lambda_{d+1-T(i,i)} \geq 0$, there must be some $k$ with $a_k = 0$, as desired. □

Though all terms are nonnegative, not all of them have a nonzero contribution. The aim of this section is to examine the properties of the positive terms. We start with a definition and a reduction which follows from Proposition 3.1.

[1] The notion of broken diagonals of excited diagrams also appears in [7, Section 7]




(a) (b)

**Figure 2:** The regions whose lozenge tilings $\nabla_{\lambda/\mu}$ and $\nabla^*_{\lambda/\mu}$ both correspond to nonzero Okounkov–Olshanski terms.

**Definition 3.2.** For a skew shape $\lambda/\mu$ of length $d$ we let $\mathcal{OOT}(\lambda/\mu)$ be the set of SSYT $T$ in $\mathrm{SSYT}(\mu, d)$ such that $c(u) < \lambda_{d+1-T(u)}$ for all $u \in [\mu]$.

**Corollary 3.3.** *The nonzero terms in the Okounkov–Olshanski formula for the shape $\lambda/\mu$ correspond to SSYT $T$ in $\mathcal{OOT}(\lambda/\mu)$.*

## 3.2 Enumerating nonzero terms of the formula

In this subsection we prove Theorem 1.2. To do so, it is convenient to define the following sets of *lozenge tilings*, tilings of a region in the triangular lattice with tiles of two adjacent equilateral triangles joined together.

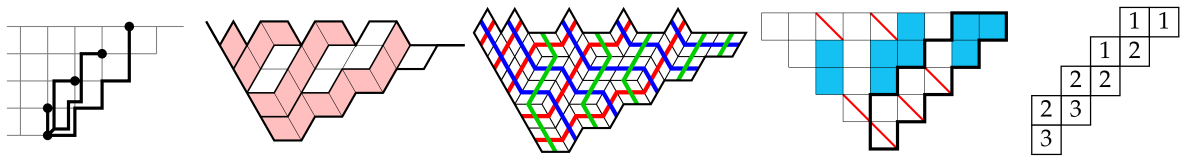
**Definition 3.4.** Let $\nabla_{\lambda/\mu}$ be the set of lozenge tilings of the region shown in Figure 2(a) corresponding to the skew shape $\lambda/\mu$. Let $\nabla^*_{\lambda/\mu}$ be the set of lozenge tilings of the region shown in Figure 2(b) corresponding to the shape $\lambda/\mu$.

The proof of the theorem now proceeds in four steps. We first prove three propositions establishing a bijection between nonzero Okounkov–Olshanski terms and $\nabla^*_{\lambda/\mu}$, passing through non-crossing paths and lozenge tilings in $\nabla_{\lambda/\mu}$. Then the theorem follows from two different applications of the Lindström–Gessel–Viennot lemma.

Throughout this section, we will refer to the following reccurring example:

**Example 3.5.** We consider the following skew shape $\lambda/\mu$ where $\lambda = (54321)$, $\mu = (32100)$, and the following SSYT of shape $\mu$: $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$. The associated objects that are in correspondence with this SSYT can be seen in Figure 3.

**Proposition 3.6.** *The SSYT in $\mathcal{OOT}(\lambda/\mu)$ are in bijection with systems of non-crossing lattice paths from $(0,0)$ to $(\mu_i, d-i)$ for $1 \le i \le d$ which stay within the region $x \le \lambda_{d-y}$.*




(a) (b) (c) (d) (e)

**Figure 3:** Constructions related to the bijection described in Section 3.

*Sketch of Proof of Proposition 3.6.* The result follows by an argument similar to the nonintersecting lattice path poof of the Jacobi-Trudi identity for Schur functions. □

**Example 3.7.** For the SSYT in Example 3.5, the corresponding non-crossing path system is found in Figure 3(a).

**Proposition 3.8.** *Systems of non-crossing lattice paths from $(0,0)$ to $(\mu_i, d-i)$ for $1 \leq i \leq d$ that stay within the region $x \leq \lambda_{d-y}$ are in bijection with the lozenge tilings in $\nabla_{\lambda/\mu}$.*

*Proof of Proposition 3.8.* Given a path system, one can apply an affine transformation to the triangular lattice, and expand each path so that the segments turn into lozenges with border on their SE-NW sides. Because of the $x \leq \lambda_{d-y}$ condition, the lozenge tilings must stay inside to the left of the right-side boundary in Figure 2(a). Since thickening the paths into lozenges pushes the paths about it up, every path must end at the same height, and at a place that corresponds to the top boundary in Figure 2(a). See Example 3.9 for an illustration of this bijection. □

**Example 3.9.** In our example, this bijection sends Figure 3(a) to Figure 3(b).

**Proposition 3.10.** *There is a bijection between the lozenge tilings in $\nabla_{\lambda/\mu}$ and in $\nabla^*_{\lambda/\mu}$.*

*Proof.* The correspondence is as follows. To go from an element of $\nabla_{\lambda/\mu}$ to $\nabla^*_{\lambda/\mu}$, add vertical lozenges in the triangular gaps on the top border and add a row of lozenges in the top-right. It is easy to see that this process is reversible. □

**Example 3.11.** In our example, this bijection sends Figure 3(b) to Figure 3(c).

We combine the three propositions above in the following lemma.

**Lemma 3.12.** *For a skew shape $\lambda/\mu$ there is a bijection between the SSYT in in $\mathcal{OOT}(\lambda/\mu)$ and lozenge tilings in $\nabla^*_{\lambda/\mu}$.*

*Proof sketch of Theorem 1.2.* By Lemma 3.12 we have that $\mathrm{OOT}(\lambda/\mu) = |\nabla^*_{\lambda/\mu}|$. Next, we use the Lindström–Gessel–Viennot lemma (e.g. see [14, Thm. 2.7.1]) applied to two different path systems associated with each element of $\nabla^*_{\lambda/\mu}$. For the first determinant, one considers paths of lozenges that border along SW-NE sides. Excluding the vertical lozenges that fit into the triangular gaps, which are forced, from the $i$th highest SW-NE edge on the right boundary to the $j$th rightmost SW-NE edge on the top boundary. A path must go up $i-1$ times out of $\lambda_i - (\mu_j - (j-1)) = \lambda_i - \mu_j + j - 1$ steps in total, which can be done in $\binom{\lambda_i - \mu_j + j - 1}{i-1}$ ways.

For the second determinant, one considers paths of lozenges that border along horizontal sides, ignoring the last $\lambda_1 - \mu_1$ steps from the right, which are clearly forced. To go from the $i$th horizontal edge from the left on the bottom to the $j$th horizontal edge from the left on the top requires $\lambda'_i$ steps up.

In both cases, the endpoints can be connected in only one way, so the Lindström–Gessel–Viennot lemma can be applied. □

**Example 3.13.** The blue paths in Figure 3(c) are an instance of the paths counted by the first determinant. The green paths are an instance of the paths counted by the second determinant.

## 3.3 Other objects counted by $\mathrm{OOT}(\lambda/\mu)$

We have given bijections between the SSYT of shape $\mu$ indexing nonzero Okounkov Olshanski terms of the shape $\lambda/\mu$ and lozenge tilings. In this section we give other objects that are in bijection with the tableaux in $\mathcal{OOT}(\lambda/\mu)$.

**Theorem 3.14.** *The SSYT in $\mathcal{OOT}(\lambda/\mu)$ are in bijection with each of the following objects:*

*(a) SSYT of shape $\lambda/\mu$ such that all the entries in row $i$ are at most $i$,*

*(b) reverse excited diagrams in $\mathcal{RE}(\lambda/\mu)$.*

*Proof (sketch).* By Lemma 3.12 the SSYT of shape $\mu$ indexing the nonzero Okounkov–Olshanski terms for the shape $\lambda/\mu$ are in bijection with lozenge tilings in $\nabla^*_{\lambda/\mu}$. Next we show a bijection between these lozenge tilings and the SSYT from (a).

Given an element $\mathsf{t}$ of $\nabla^*_{\lambda/\mu}$, consider paths of lozenges created by following the SW-NE side lengths; call the $i$th path from the northeast corner $\beta_i$. Similarly, call the $j$th path marked by the horizontal edges from the left $\gamma_j$.

There exists a correspondence $\psi$ between cells $[\lambda/\mu]$ and the SW-NE lozenges given by $\psi(i,j) = \beta_i \cap \gamma_j$. Now we construct a tableau $T$ from $\mathsf{t}$ as follows. Let $T(i,j) = \mathrm{ht}(\psi(i,j))$, where ht is the distance from a lozenge to the top boundary. (The highest possible lozenge $x$ has $\mathrm{ht}(x) = 1$, and descending by one level increases $\mathrm{ht}(x)$ by one.) By considering paths $\beta_i$ ($\gamma_j$) the entries in row $i$ (column $j$) are nondecreasing (increasing). Since $\beta_i$ does not contain a lozenge $x$ where $\mathrm{ht}(x) > i$, $T$ is a semistandard tableau of shape $\lambda/\mu$ where the entries in row $i$ are at most $i$. This map can be reversed.

Lastly, the reverse excited diagrams in $\mathcal{RE}(\lambda/\mu)$ are in bijection with the SSYT of shape $\lambda/\mu$ in (a) by the expected modification of the map $\varphi$ between excited diagrams and semistandard tableaux given in [7, Section 3]. □

**Example 3.15.** For the lozenge tiling in Figure 3(c), the paths $\beta_i$ are colored in blue and the paths $\gamma_j$ are colored in green. See Figure 3(d) for the corresponding reverse excited diagram. The corresponding SSYT of shape $\lambda/\mu$ is in Figure 3(e).

We give two corollaries of these bijections, including a reformulation of the Okounkov–Olshanskii formula.

*Proof of Corollary 1.4.* The bijection in the proof of Theorem 3.14 goes from the SSYT in $\mathcal{OOT}(\lambda/\mu)$ to lozenge tilings in $\nabla^*_{\lambda/\mu}$ and then to reverse excited diagrams in $\mathcal{RE}(\lambda/\mu)$. In the lozenge tiling, the weighted objects are the northwest-southeast rhombi, with a weight equal to the product of the distance of each rhombi to the right edge of the shape. Under the correspondence, those rhombi become cells of broken diagonals of the excited diagram. The weight of each such cell $u$ is the length of the arm $\mathrm{arm}(u)$. □

**Example 3.16.** For the shape $\lambda/\mu = 2221/11$, we have $\mathrm{OOT}(\lambda/\mu) = \left\{ \begin{smallmatrix}3\\4\end{smallmatrix}, \begin{smallmatrix}2\\4\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\4\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \right\}$. The reverse excited diagrams of $\lambda/\mu = 2221/11$ are in Figure 1(b) and include their respective broken diagonals (in red). The reformulation (1.1) of (OOF) gives

$$f^{\lambda/\mu} = \frac{5!}{2\cdot 3\cdot 3\cdot 4\cdot 5}\,(2\cdot 3 + 2\cdot 3 + 2\cdot 3 + 1\cdot 3 + 1\cdot 3 + 1\cdot 3) = 9. \tag{3.1}$$

*Proof of Corollary 1.3.* By Theorem 3.14(a), $\mathrm{OOT}(\sigma_n)$ is the number of SSYT of shape $\sigma_n$ so that the entries in row $i$ are at most $i$. By considering the reverse row word of such tableaux (reading entries from the top right to the bottom left), $\mathrm{OOT}(\sigma_n)$ is the number of sequences $a_1, a_2, \ldots, a_{2n-1}$ so that $a_{2i-1} \geq a_{2i} < a_{2i+1}$ for $1 \leq i < n$, $a_{2i-1} \leq i$ for $1 \leq i \leq n$, and $a_{2i} \leq i$ for $1 \leq i < n$. These are the strictly alternating pistols of length $2n-1$. □

# 4 $q$-Analogues of the Okounkov–Olshanski Formula

The aim of this section is to generalize this reverse plane partition result to skew shapes in a manner similar to the Okounkov–Olshanski formula. We first state a version of such a result using the language of Grothendieck polynomials. To do so we need the following notation, let $G_{\lambda/\mu}(\mathbf{y}) := G_\mu(\ominus \mathbf{y}_\lambda \mid \mathbf{y})$.

**Theorem 4.1.** $\frac{\mathrm{rpp}_{\lambda/\mu}(q)}{\mathrm{rpp}_\lambda(q)} = G_{\lambda/\mu}(1-q^{-1}, 1-q^{-2}, \ldots)$.

*Proof sketch of Theorem 4.1.* For this section only let $[a] := 1-q^a$, $[n]! := \prod_{i=1}^n [i]$, $\ell_i := \lambda_i + d - i$, and $m_i := \mu_i + d - i$.

We write the Grothendieck polynomial in a similar determinantal form as the Jacobi–Trudi identity for $s_{\lambda/\mu}(1, q, \ldots)$.

**Lemma 4.2.** $G_{\lambda/\mu}(1-q^{-1}, 1-q^{-2}, \ldots) \cdot \mathrm{rpp}_\lambda(q) = \det\left[\frac{q^{\ell_i(j-i)}}{[\ell_i - m_j]!}\right]_{i,j=1}^{d}$.

To prove this lemma we use a result of Ikeda–Naruse [4] and a determinantal formula of Krattenthaler [5, Corollary 8] for $\mathrm{rpp}_{\lambda/\mu}(q)$. Finally, we combine Lemma 4.2 and a known determinantal formula for $\mathrm{rpp}_{\lambda/\mu}(q)$ to prove Theorem 4.1. □

To rewrite Theorem 4.1 in a form similar to (OOF), we need a technical result that categorizes set-valued semistandard tableaux by the maximum entries in each cell.

**Definition 4.3.** Given a semistandard tableau $T$ of shape $\mu$ and a cell $u \in [\mu]$, let $m_T(u)$ be the minimum $k \le T(u)$ such that replacing $T(u)$ with $k$ still results in a semistandard tableau.

**Lemma 4.4.** *The set of all set-valued tableaux of shape $\mu$ can be decomposed as follows:*

$$\bigsqcup_{T_0 \in \mathrm{SSYT}(\mu)} \{T \mid T(u) = S_u \cup \{T_0(u)\},\ S_u \subseteq [m_{T_0}(u), T(u))\}$$

*Here we let $[a,b) = \{n \in \mathbb{Z} \mid a \le n < b\}$.*

Finally, by evaluating the Grothendieck polynomial we can obtain Theorem 1.6.

*Proof sketch of Theorem 1.6.* Since factorial Grothendieck polynomials are symmetric [6]

$$\begin{aligned}\frac{\mathrm{rpp}_{\lambda/\mu}(q)}{\mathrm{rpp}_\lambda(q)} &= G_\mu(1-q^{\lambda_d+1}, 1-q^{\lambda_{d-1}+2}, \ldots \mid 1-q^{-1}, 1-q^{-2}, \ldots)\\ &= \sum_T (-1)^{|\mu|} \prod_{u \in [\mu]} \prod_{r \in T(u)} (q^{w(u,r)} - 1).\end{aligned}$$

We use Lemma 4.4 to rewrite this in terms of SSYT to obtain the desired formula. □

**Example 4.5.** Continuing with Example 3.16 for the shape $\lambda/\mu = 2221/11$, the reverse plane partition $q$-analogue of the Okounkov–Olshanski formula (OOF) gives

$$\frac{\mathrm{rpp}_{\lambda/\mu}(q)}{\mathrm{rpp}_\lambda(q)} = \Big((q^3+q^4+q^1)(1-q^2)(1-q^3) + (q^6+q^3+1)(1-q^1)(1-q^3)\Big). \tag{4.1}$$

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