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Bijective link between Chapoton's new intervals and bipartite planar maps

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Abstract. In 2006, Chapoton defined a class of Tamari intervals called "new intervals" in his enumeration of Tamari intervals, and he found that these new intervals are equinumerous with bipartite planar maps. We present here a direct bijection between these two classes of objects using a new object called "degree tree". Our bijection also gives an intuitive proof of an unpublished equi-distribution result of some statistics on new intervals given by Chapoton and Fusy.

Résumé. En 2006, Chapoton a défini une classe d'intervalles de Tamari nommés "intervalles nouveaux" dans son travail de comptage des intervalles de Tamari. Il a découvert que ces intervalles nouveaux sont équi-énumérés avec les cartes planaires biparties. Nous proposons une bijection directe entre ces deux classes d'objets en utilisant un nouvel objet appelé "arbre des degrés". Notre bijection donne aussi une preuve intuitive d'un résultat non publié de Chapoton et Fusy sur l'équi-distribution de certains statistiques sur les intervalles nouveaux.

Keywords: Bijection, Tamari lattice, new interval, bipartite planar map, degree tree

On classical Catalan objects, such as Dyck paths and binary trees, we can define the famous *Tamari lattice*, first proposed by Tamari [13]. This partial order was later found woven into the fabric of other more sophisticated objects. A notable example is diagonal coinvariant spaces, which have led to several generalizations of the Tamari lattice [1, 12], and also incited the interest in intervals in such Tamari-like lattices. Recently, there is a surge of interest in the enumeration [3, 4, 8] and the structure [2, 6] of different families of Tamari-like intervals. In particular, several bijective relations were found between various families of Tamari-like intervals and planar maps [2, 7, 8]. The current work is a natural extension of this line of research.

In [3], Chapoton introduced a subclass of Tamari intervals called *new intervals*, which are irreducible elements in a grafting construction of intervals. Definitions of these objects and related statistics are postponed to the next section. The number of new intervals in the Tamari lattice of order $n \geq 2$ was given in [3] as

$$\frac{3 \cdot 2^{n-2}(2n-2)!}{(n-1)!(n+1)!}.$$

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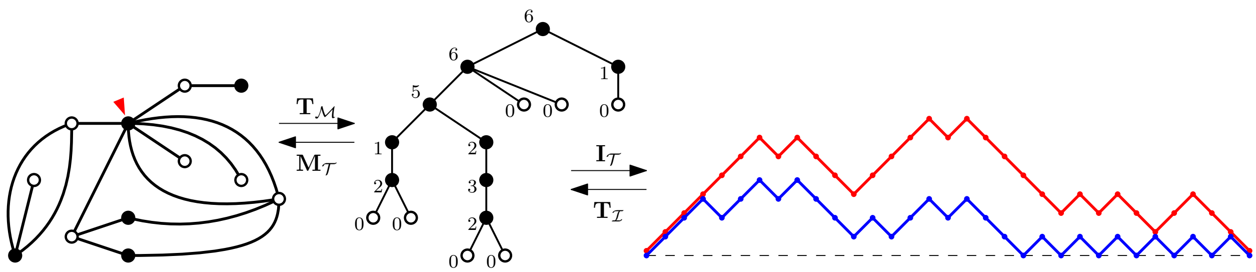

Figure 1: Our bijections between bipartite planar maps, degree trees and new intervals

This is also the number of bipartite planar maps with $n-1$ edges. Furthermore, in a more recent unpublished result of Chapoton and Fusy (see [9]), a symmetry in three statistics on new intervals was observed then proven by identifying the corresponding generating function of new intervals with that of bipartite planar maps recording the numbers of faces, of black and of white vertices, and those are well-known to be equi-distributed. These results strongly hint a bijective link between the two classes of objects.

In this article, we give a direct bijection between new intervals and bipartite planar maps (see Figure 1) explaining the results above. Our bijection also generalizes a bijection on trees given in [11] in the study of random maps. We have the following theorem.

Theorem 0.1. *There is a bijection $\mathbf{I}_\mathcal{M}$, with $\mathbf{M}_\mathcal{I}$ its inverse, such that, for a bipartite planar map M with n edges and $I = \mathbf{I}_\mathcal{M}(M)$, which is a new interval of size $n+1$, we have*

$$\mathrm{white}(M) = \mathrm{c}_{00}(I), \quad \mathrm{black}(M) = \mathrm{c}_{01}(I), \quad \mathrm{face}(M) = 1 + \mathrm{c}_{11}(I).$$

Here, $\mathrm{black}(M)$, $\mathrm{white}(M)$ *and* $\mathrm{face}(M)$ *are the number of black vertices, white vertices and faces of M respectively, while* $\mathrm{c}_{00}(I)$, $\mathrm{c}_{01}(I)$ *and* $\mathrm{c}_{11}(I)$ *are some natural statistics on new intervals that we will define in Section 1.*

These bijections use a new family of objects called *degree trees*, and are in the same line as some previous work of the author [7, 8]. While the symmetry in statistics of new intervals is known to Chapoton and Fusy [9], our bijection intuitively captures this symmetry. Due to space limit, some proofs are omitted.

1 Preliminaries

A *Dyck path* P is a lattice path of up steps $u = (1,1)$ and down steps $d = (1,-1)$, starting from $(0,0)$, ending on the x-axis without falling below. The *size* (also called the *semilength*) of P is half of its length. We denote by $\mathcal{D}_n$ the set of Dyck paths of size n. A *rising contact* of P is an up step on the x-axis. We can also see P as a word in $\{u,d\}$ whose prefixes all have at least as many up steps than down steps.

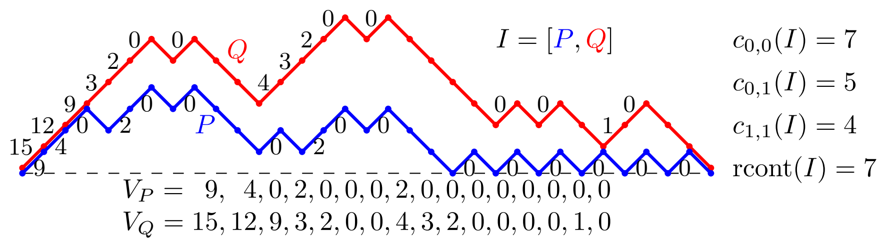

Figure 2: An example of Chapoton's new interval with bracket vectors for both paths and related statistics.

We now define the Tamari lattice as a partial order on $\mathcal{D}_n$ in the spirit of [10]. Given a Dyck path P seen as a word, its i-th up step u_i *matches* with a down step d_j if the factor P_i of P strictly between u_i and d_j is a Dyck path. Clearly, there is a unique match for every u_i. We define the *bracket vector* V_P of P by taking $V_P(i)$ to be the size of P_i. The *Tamari lattice* of order n is the partial order $\preceq$ on $\mathcal{D}_n$ where $P \preceq Q$ if and only if $V_P(i) \leq V_Q(i)$ for all i. See Figure 2 for an example. A *Tamari interval* of size n can be viewed as a pair of Dyck paths $[P, Q]$ of size n with $P \preceq Q$.

In [3], Chapoton defined a subclass of Tamari intervals called "new intervals". Originally defined on pairs of binary trees, this notion can also be defined on pairs of Dyck paths (see [9]). The example in Figure 2 is also a new interval. Given a Tamari interval $[P, Q]$, it is a *new interval* if and only if the following conditions hold:

(i) $V_Q(1) = n - 1$;

(ii) For all $1 \leq i \leq n$, if $V_Q(i) > 0$, then $V_P(i) \leq V_Q(i+1)$.

We denote by $\mathcal{I}_n$ the set of new intervals of size $n \geq 1$.

We now define some statistics on new intervals. Given a Dyck path P of size n, its *type* $\mathrm{Type}(P)$ is a word w such that, if the i^{th} up step u_i is followed by an up step in P, then $w_i = 1$, otherwise $w_i = 0$. Since the last up step is always followed by a down step, we have $w_n = 0$. Note that our definition here is slightly different from that in, *e.g.*, [8], where the last letter is not taken into account. Given a new interval $I = [P, Q] \in \mathcal{I}_n$, if $\mathrm{Type}(P)_i = 1$ and $\mathrm{Type}(Q)_i = 0$, then we have $V_P(i) > 0 = V_Q(i)$, violating the condition for Tamari interval. Hence, we have only three possibilities for $(\mathrm{Type}(P)_i, \mathrm{Type}(Q)_i)$. We define $\mathrm{c}_{00}(I)$ (resp. $\mathrm{c}_{01}(I)$ and $\mathrm{c}_{11}(I)$) to be the number of indices i such that $(\mathrm{Type}(P)_i, \mathrm{Type}(Q)_i) = (0,0)$ (resp. $(0,1)$ and $(1,1)$). Figure 2 also shows such statistics in the example. We define the generating function $F_\mathcal{I} \equiv F_\mathcal{I}(t; u, v, w)$ of new intervals as

$$F_\mathcal{I}(t; u, v, w) = \sum_{n \geq 1} t^n \sum_{I \in \mathcal{I}_n} u^{\mathrm{c}_{00}(I)} v^{\mathrm{c}_{01}(I)} w^{\mathrm{c}_{11}(I)}. \quad (1.1)$$

For the other side of the bijection, a *bipartite planar map* M is a drawing of a bipartite graph on a plane (in which all edges link a black vertex to a white one), defined up to

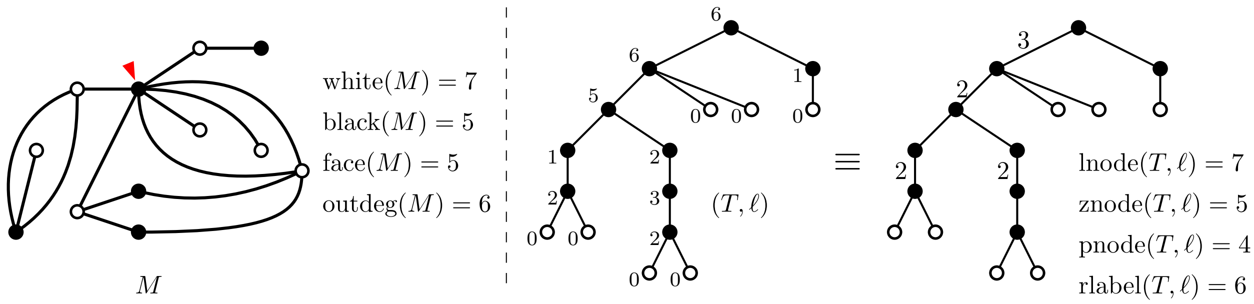

Figure 3: Left: an example of bipartite map. Right: an example of degree trees and the corresponding edge labels (zeros are omitted). Both with related statistics.

continuous deformation, such that edges intersect only at their ends. Edges in M cut the plane into *faces*, and the *outer face* is the infinite one. The *size* of M is its number of edges; a map of size zero consists of only one black vertex. In the following, we only consider *rooted* bipartite planar maps, which have a distinguished corner c called the *root corner* of the outer face on a black vertex, called the *root vertex*. See the left part of Figure 3 for an example. We denote by $\mathcal{M}_n$ the set of (rooted) bipartite planar maps of size n.

We now define the generating function $F_{\mathcal{M}} \equiv F_{\mathcal{M}}(t; u, v, w)$ of bipartite planar maps recording these statistics by

$$F_{\mathcal{M}} \equiv F_{\mathcal{M}}(t; u, v, w) = \sum_{n \geq 0} t^n \sum_{M \in \mathcal{M}_n} u^{\mathrm{black}(M)} v^{\mathrm{white}(M)} w^{\mathrm{face}(M)}. \tag{1.2}$$

It is well-known that $\mathrm{black}(M), \mathrm{white}(M), \mathrm{face}(M)$ are jointly equi-distributed in $\mathcal{M}_n$, meaning that $F_{\mathcal{M}}$ is symmetric in u, v, w. This can be seen with the bijection between bipartite maps and bicubic maps by Tutte [14].

To describe our bijection, we propose an intermediate class of objects called "degree tree". An example is given in the right part of Figure 3. We can also see degree trees as a variant of description trees (see [5]). A *degree tree* is a pair (T, ℓ), where T is a plane tree, and ℓ is a labeling function defined on nodes of T such that

- If v is a leaf, then $\ell(v) = 0$;
- If v is an internal node with k children $v_1, v_2, \ldots, v_k$, then $\ell(v) = k - a + \ell(v_1) + \ell(v_2) + \cdots + \ell(v_k)$, where $0 \leq a \leq \ell(v_1)$.

We observe that the leftmost child of a node v is special when computing $\ell(v)$. This is different from the case of description trees. The size of a degree tree (T, ℓ) is the number of edges. We denote by $\mathcal{T}_n$ the set of degree trees (T, ℓ) of size n.

Given a degree tree (T, ℓ), we can replace ℓ by a labeling function on *edges*. More precisely, for an internal node v, we label its leftmost descending edge (*i.e.*, the edge to its leftmost child) by the value a used in the computation of $\ell(v)$, and all other edges by 0. We denote this edge labeling function by $\Lambda(\ell)$. It is clear that Λ is an injection. Given $\lambda = \Lambda(\ell)$, the vertex labeling ℓ can be easily recovered.

We also define several natural statistics on degree trees, illustrated in Figure 3, using its edge labeling. Let (T, ℓ) be a degree tree with $\lambda = \Lambda(\ell)$, and v a node in T. If v is a leaf, then it is called a *leaf node*. Otherwise, let e be the leftmost descending edge of v. If $\lambda(e) = 0$, then v is a *zero node*, otherwise it is a *positive node*. We denote by $\mathrm{lnode}(T, \ell)$, $\mathrm{znode}(T, \ell)$ and $\mathrm{pnode}(T, \ell)$ the number of leaf nodes, zero nodes and positive nodes in (T, ℓ) respectively. If $T \in \mathcal{T}_n$, we have $\mathrm{lnode}(T, \ell) + \mathrm{znode}(T, \ell) + \mathrm{pnode}(T, \ell) = n + 1$.

Lemma 1.1. *Let (T, ℓ) be a degree tree, and $\lambda = \Lambda(\ell)$ the related edge labeling. We have*

1. *If v has m descendants, then $\ell(v) = m - \sum_{e \in T_v} \lambda(e)$, with T_v the subtree induced by v;*
2. *$\ell(v) \geq 0$, and $\ell(v) = 0$ if and only if v has no descendant.*

Proof sketch. We proceed by induction on the size of induced subtrees. □

2 Degree trees and bipartite maps

Our bijection from bipartite maps to new intervals is relayed by degree trees, in which the related statistics are transferred. We start by the bijection from maps to trees.

2.1 From bipartite maps to degree trees

In [11], Janson and Stefánsson described a bijection between plane trees with n nodes and k leaves and plane trees with n nodes in which k of them are of even depth, giving another interpretation of Narayana numbers. We now introduce a bijection between $\mathcal{M}_n$ and $\mathcal{T}_n$, which can be seen as a generalization of the bijection in [11].

We first define a transformation $\mathbf{T}_{\mathcal{M}}$ from $\mathcal{M}_n$ to $\mathcal{T}_n$ for all n. Let $M \in \mathcal{M}_n$. If $n = 0$, we define $\mathbf{T}_{\mathcal{M}}(M)$ to be the tree with one node. Otherwise, we perform the following exploration procedure to obtain a tree T with a labeling λ on its edges. In this procedure, we distinguish edges in M, which will be deleted one by one, and edges in T that we add. We start from the root vertex, with the edge next to the root corner in clockwise order as the pending edge. Suppose that the current vertex is u and the pending edge is e_M, which is always in M. We repeat two steps, *advance* and *prepare*, until termination. The advance step comes in the following cases illustrated in Figure 4:

(A1) If e_M is a bridge to a vertex v of degree 1, then we delete e_M in M and add $e_T = e_M$ in T. The new current vertex is $u' = u$, and we define $\lambda(e_T) = 0$.

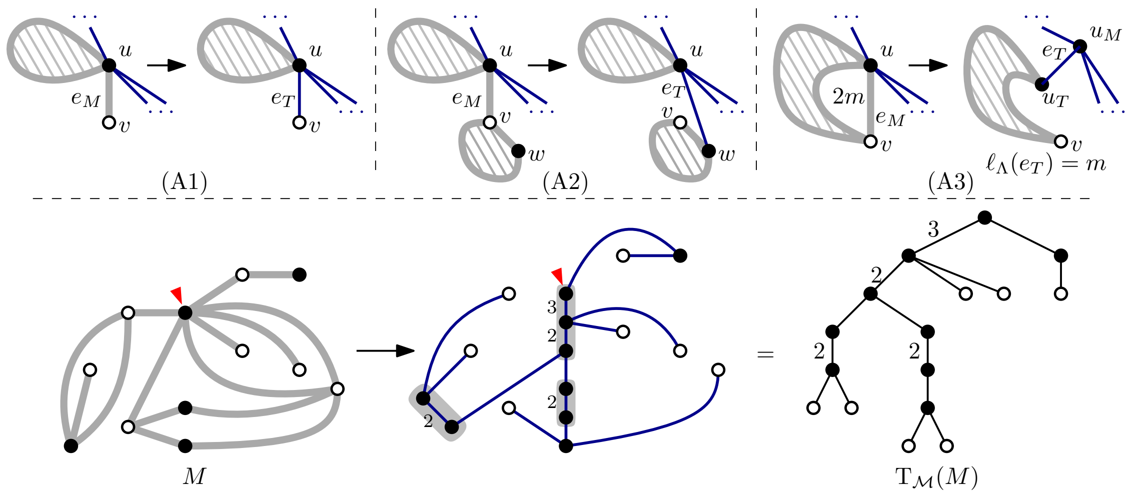

Figure 4: Cases in the advance step of $\mathbf{T}_\mathcal{M}$ and an example of the bijection $\mathbf{T}_\mathcal{M}$. Nodes in the same shaded pack come from the same vertex in the map.

(A2) If e_M is a bridge to a vertex v of degree at least 2, let e_1 be the edge adjacent to v next to e_M in clockwise order, and w the other end of e_1. We draw a new edge e_T in T from u to w such that e_M, e_1, e_T form a face with u, v, w in counter-clockwise order. The next current vertex is $u' = w$. We delete e_M, and define $\lambda(e_T) = 0$.

(A3) If e_M is not a bridge, we split u into u_M and u_T, with u_T taking all edges in T and u_M taking the rest. We add a new edge e_T in T from u_M to u_T. Since e_M is not a bridge, by planarity, it is between the outer face and a face of degree $2m$ with $m > 0$. We define $\lambda(e_T) = m$ and delete e_M. The next current vertex is $u' = u_T$.

In the prepare step, let u' be the new current vertex, which is adjacent to the new edge e_T. The next pending edge is the next remaining edge in M starting from e_T in the clockwise order around u'. If no such edge exists, we backtrack in the tree T until finding a vertex u'' with such an edge e''_M, and we set u'' as the current vertex, and e''_M the pending edge. If no such vertex exists, the procedure terminates, and we shall obtain a tree T with an edge label function λ. We define $\mathbf{T}_\mathcal{M}(M)$ as the degree tree $(T, \Lambda^{-1}(\lambda))$. See Figure 4 for an example of $\mathbf{T}_\mathcal{M}$. The bijection in [11] is simply $\mathbf{T}_\mathcal{M}$ applied to a plane tree, where Case (A3) never applies, and the degree tree (T, ℓ) obtained has $\lambda = 0$ for all edges.

We now prove that $\mathbf{T}_\mathcal{M}(M)$ is well-defined. We start by describing the structure of the map in intermediate steps. The *leftmost branch* of a tree is the path starting from the root node and taking the leftmost descending edge at each node till a leaf.

Lemma 2.1. *Let $M \in \mathcal{M}_n$ and $T = \mathbf{T}_\mathcal{M}(M)$. Let M_i^+ be the map after the i-th prepare step, with u_i the current vertex and e_i the pending edge. We denote by T_i the partially constructed T in M_i^+ , and by M_i the remaining of M. Clearly T_i and M_i form a partition of edges in M_i^+.*

For every i, T_i is a tree, and M_i^+ is T_i with connected components of M_i attached to the left of nodes on the leftmost branch of T_i, one component to only one vertex, with u_i the deepest such vertex and e_i its first edge in M_i in clockwise order from the leftmost branch of T_i.

Sketch of proof. We proceed by induction on i. It clearly holds when $i = 0$, and we check that the properties are kept for each case of each step in the procedure of $\mathbf{T}_{\mathcal{M}}$. □

We now prove that trees obtained in $\mathbf{T}_{\mathcal{M}}$ are degree trees.

Proposition 2.2. *Given $M \in \mathcal{M}_n$, then $(T, \ell) = \mathbf{T}_{\mathcal{M}}(M)$ is a degree tree of size n.*

Proof sketch. By Lemma 2.1, T is a tree of size n, and the first time a node u is explored, there is a component M_u of remaining edges in M attached to u. We prove by induction and case checking that the outer face of M_u is of degree $2\ell(u)$, hence $(T, \ell) \in \mathcal{T}_n$. □

The transformation $\mathbf{T}_{\mathcal{M}}$ transfers some statistics from $\mathcal{M}_n$ to $\mathcal{T}_n$ as follows.

Proposition 2.3. *Given $M \in \mathcal{M}_n$, let $(T, \ell) = \mathbf{T}_{\mathcal{M}}(M)$. We have*

$$\mathrm{white}(M) = \mathrm{lnode}(T, \ell), \quad \mathrm{black}(M) = \mathrm{znode}(T, \ell), \quad \mathrm{face}(M) = 1 + \mathrm{pnode}(T, \ell).$$

Proof. By the definition of $\mathbf{T}_{\mathcal{M}}$, all leaves in T are from white vertices, which are never split. Hence $\mathrm{white}(M) = \mathrm{lnode}(T, \ell)$. Then at each occurrence of Case (A3), we lost a face but gain a positive node in T, thus $\mathrm{face}(M) = 1 + \mathrm{pnode}(T, \ell)$, with 1 for the outer face. Now for $\mathrm{black}(M) = \mathrm{znode}(T, \ell)$, we note that a new black vertex in M is reached only in Case (A2), which leads to a zero edge. □

2.2 From degree trees to bipartite maps

We now define a transformation $\mathbf{M}_{\mathcal{T}}$ from $\mathcal{T}_n$ to $\mathcal{M}_n$. Let $(T, \ell) \in \mathcal{T}_n$ and $\lambda = \Lambda(\ell)$. We now perform a procedure that deals with nodes in T in postorder (*i.e.*, first visit the subtrees induced by children from left to right, then the parent). For each node u, let u^* be its parent and e_u the edge between u and u^*. By construction, when we deal with u, its induced subtree has already been dealt with and transformed into a bipartite planar map M_u attached to u. We have three cases, illustrated in Figure 5.

- **Case (A1′)**: If u is a leaf, then we delete e_u from T and add it to M.
- **Case (A2′)**: If u is not a leaf but $\lambda(e_u) = 0$, let e' be the edge next to e_u around u in counterclockwise order, and v the other end of e'. As M_u is bipartite, $v \neq u$. We add a new edge e_M from u^* to v such that the triangle formed by e_u, e', e_M has vertices u^*, u, v in clockwise order, without any edge inside. We then delete e_u.

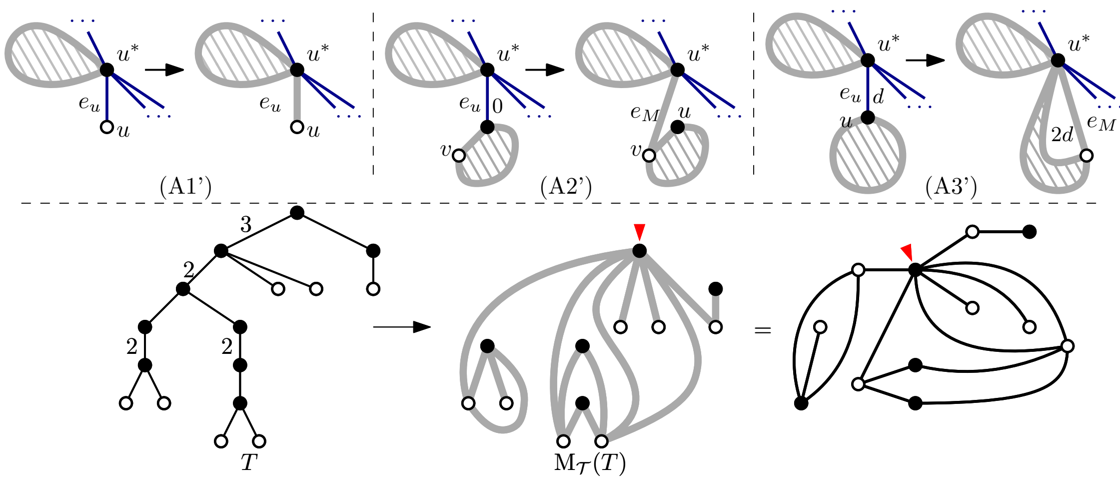

Figure 5: Cases in the procedure of $\mathbf{M}_\mathcal{T}$, and an example of $\mathbf{M}_\mathcal{T}$

- **Case (A3')**: If $\lambda(e_u) > 0$, let d be the degree of the outer face of M_u. If $2\lambda(e_u) \geq d$, then the procedure fails. Otherwise, we start from the corner of M_u to the right of e_u and walk clockwise along edges for $2\lambda(e_u) - 1$ times to another corner, and we connect the two corners by a new edge e_M in M, making a new face of degree $2\lambda(e_u)$. The component remains planar and bipartite. We finish by contracting e_u.

In the end, we obtain a planar bipartite map M with the same root corner as T. We define $\mathbf{M}_\mathcal{T}(T, \ell) = M$. We see that (A1'), (A2') and (A3') are exactly the opposite of (A1), (A2), (A3) in the definition of $\mathbf{T}_\mathcal{M}$.

Proposition 2.4. *Given (T, ℓ) a degree tree, for a node $u \in T$, let M_u be the map obtained in the procedure of $\mathbf{M}_\mathcal{T}(T, \ell)$ from the subtree T_u induced by u. Then the degree of the outer face of M_u is $2\ell(u)$, and the procedure never fails.*

Proof sketch. We proceed by induction on u in reverse postorder with case checking. □

Proposition 2.5. *For (T, ℓ) a degree tree, $M = \mathbf{M}_\mathcal{T}(T, \ell)$ is a bipartite planar map.*

Proof. Planarity is easily checked through the definition of $\mathbf{M}_\mathcal{T}$. Faces in M are only created in Case (A3'), thus all even. Along with planarity, M is bipartite. □

It is also clear that $\mathbf{M}_\mathcal{T}$ is the inverse of $\mathbf{T}_\mathcal{M}$.

Proposition 2.6. *The transformation $\mathbf{T}_\mathcal{M}$ is a bijection from $\mathcal{M}_n$ to $\mathcal{T}_n$, with $\mathbf{M}_\mathcal{T}$ its inverse.*

Proof sketch. By Propositions 2.2 and 2.5, we only need to prove that $\mathbf{M}_\mathcal{T}$ is the inverse of $\mathbf{T}_\mathcal{M}$. It is clear from definitions that $\mathbf{T}_\mathcal{M} \circ \mathbf{M}_\mathcal{T} = \mathrm{id}_\mathcal{T}$. To show that $\mathbf{M}_\mathcal{T} \circ \mathbf{T}_\mathcal{M} = \mathrm{id}_\mathcal{M}$, the only case to check is Case (A2) of $\mathbf{T}_\mathcal{M}$. However, by planarity, there is only one way to revert Case (A2) by creating a face, which is exactly Case (A2') in $\mathbf{M}_\mathcal{T}$. □

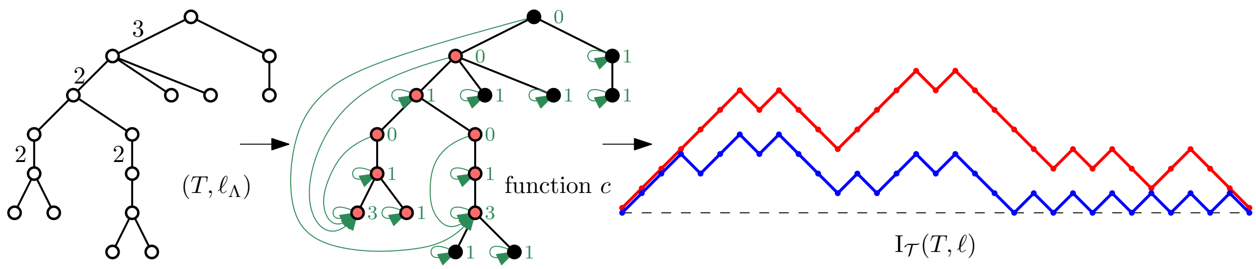

Figure 6: Example of the bijection $\mathbf{I}_\mathcal{T}$ on a degree tree represented by its edge labeling. The middle shows the certificate of each node.

3 Degree trees and new intervals

We now relate degree trees to new intervals, which also explains the conditions of new intervals in terms of trees.

3.1 From degree trees to new intervals

Given $(T, \ell) \in \mathcal{T}_n$, let $\lambda = \Lambda(\ell)$. We define a transformation $\mathbf{I}_\mathcal{T}$ from degree trees to new intervals by constructing a new interval $[P, Q]$ from (T, ℓ). We first introduce a classical bijection between plane trees and Dyck paths. Given the plane tree T, to get a Dyck path Q', we perform a *preorder traversal* (parent first, then subtrees from left to right) of T, and append u (resp. d) to Q' each time we move away from (resp. closer to) the root. This is a bijection. We then take $Q = uQ'd$. For P, we first assign to every node a *certificate*. We process nodes in T, initially all colored black, in the *reverse order* of the preorder of T. At the step for a node v, if v is a leaf, then its certificate is itself. Otherwise, we visit nodes after v in preorder, and color each visited black node by red, stopping at some node w just before the $(\lambda(e)+1)$-st black node, where e is the leftmost descending edge of v. We take w as the certificate of v. When $\lambda(e) = 0$, we take $w = v$. We now define a *certificate function* c on vertices of T similar to those in [7, 8], with $c(w)$ the number of nodes with w as certificate. The path P is given by concatenation of $ud^{c(v)}$ for all nodes v in preorder. We then define $\mathbf{I}_\mathcal{T}(T, \ell) = [P, Q]$. See an example of $\mathbf{I}_\mathcal{T}$ in Figure 6.

To prove that $\mathbf{I}_\mathcal{T}(T, \ell)$ is a new interval, we start by some properties of certificates.

Lemma 3.1. *Let (T, ℓ) be a degree tree of size n. For a node $v \in T$, let w be the certificate of v. Then either $w = v$, or w is a descendant of v in the leftmost subtree T_* of v. In the latter case, w is not the last node of T_* in preorder.*

Proof sketch. We proceed by induction on nodes in the reverse preorder. The base case is a leaf, thus trivial. For the induction step, Lemma 1.1 ensures that, at the process step of each node, there are enough black vertices in the leftmost subtree to color. □

Lemma 3.2. *Let (T, ℓ) be a degree tree, and v, v' two distinct nodes in T with w, w' their certificates respectively. Suppose that v precedes v' in the preorder. Then w cannot be strictly between v' and w' in the preorder. Furthermore, if $v' \neq w'$, then $w \neq v'$.*

Proof. We only need to consider the case $v \neq w$ and $v' \neq w'$, as other cases are trivial. In the coloring process, since v precedes v' in the preorder, v' is treated before v. By construction, in the coloring process, after the step for v', the nodes between v' to w' (excluding v' but including w') are all colored red. Therefore, in the process step for v, the visit will not stop strictly between v' and w', nor at v', as such a stop requires a succeeding black node. Hence, w is not strictly between v' and w', and $w \neq v'$. □

Note that in the lemma above, we can have $w = v'$ when $v' = w'$.

Proposition 3.3. *Let $(T, \ell) \in \mathcal{T}_n$. Then $[P, Q] = \mathbf{I}_{\mathcal{T}}(T, \ell)$ is a new interval in $\mathcal{I}_{n+1}$.*

Proof sketch. Q is clearly a Dyck path. P is a Dyck path since a node never comes after its certificate in preorder by Lemma 3.1. Let v_i be the i-th node in preorder in T. $V_Q(i)$ is the size of the subtree T_i induced by v_i. The case $V_Q(i) = 0$ is trivial. Suppose that $V_Q(i) > 0$. By Lemma 3.2, steps in $V_P(i)$ are generated by nodes from after v_i till its certificate, which is in the subtree induced by the first child of v_i if $V_Q(i) > 0$, and this child is the $(i+1)$-st node in preorder. We thus have $V_P(i) \leq V_Q(i+1) < V_Q(i)$. □

We also have the following property of a new interval obtained via $\mathbf{I}_{\mathcal{T}}$.

Proposition 3.4. *For a degree tree (T, ℓ) with $\lambda = \Lambda(\ell)$, let $I = [P, Q] = \mathbf{I}_{\mathcal{T}}(T, \ell)$. For an internal node $v \in T$, let e be the edge linking v to its leftmost child v'. Let P_v be the subpath of P strictly between the up step contributed by v in P and its matching down step. Then the number of rising contacts in P_v as a Dyck path is $\lambda(e)$.*

3.2 From new intervals to degree trees

We now define a transformation $\mathbf{T}_{\mathcal{I}}$ for the reverse direction. Let $I = [P, Q] \in \mathcal{I}_{n+1}$ be a new interval. Since $V_Q(1) = n$, we can write $Q = uQ'd$. We first construct a plane tree T of size n from Q' with the classic bijection described at the beginning of Section 3.1. Now, let $v_1, \ldots, v_{n+1}$ be the nodes of T in preorder. $V_Q(i)$ is the size of the subtree induced by v_i. We now define an edge labeling λ of T. If e is the left-most descending edge of v_i, then we take $\lambda(e)$ the number of rising contacts in P_i, with P_i the subpath of P strictly between the i-th up step and its matching down step. Otherwise, we take $\lambda(e) = 0$. Let $\ell = \Lambda^{-1}(\lambda)$. We define $\mathbf{T}_{\mathcal{I}}(I) = (T, \ell)$. An example of $\mathbf{T}_{\mathcal{I}}$ is given in Figure 7.

Proposition 3.5. *Let $I = [P, Q] \in \mathcal{I}_{n+1}$, then $(T, \ell) = \mathbf{T}_{\mathcal{I}}(I)$ is a degree tree of size n.*

Proof sketch. Let $\lambda = \Lambda(\ell)$. Suppose that the j-th node v_j of T in preorder is not a leaf. By the conditions of new intervals, the steps counted in $V_P(j)$ are all in the leftmost subtree of v_j. We thus check that (T, ℓ) satisfies the conditions of degree trees. □

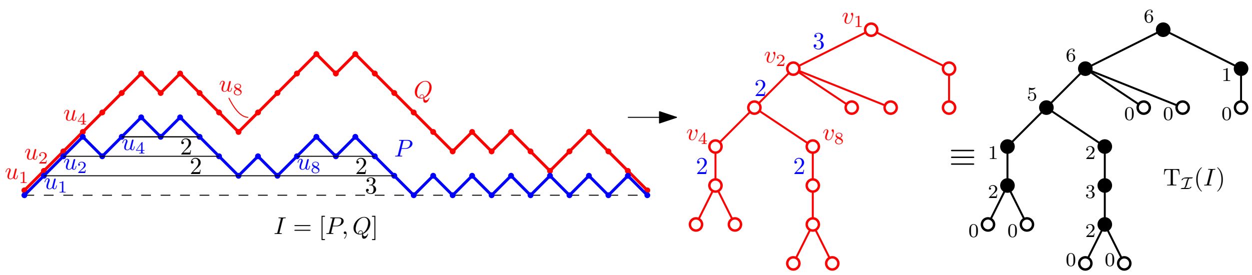

Figure 7: Example of the bijection $\mathbf{T}_{\mathcal{I}}$ on a new interval $I = [P, Q]$

Proposition 3.6. *Given* $I = [P, Q] \in \mathcal{I}_{n+1}$*, let* $(T, \ell) = \mathbf{T}_{\mathcal{I}}(I)$*. We have*

$$\mathrm{c}_{00}(I) = \mathrm{lnode}(T, \ell), \quad \mathrm{c}_{01}(I) = \mathrm{znode}(T, \ell), \quad \mathrm{c}_{11}(I) = \mathrm{pnode}(T, \ell).$$

Proof. Let v_i be the i-th node of T in preorder. By the definition of $\mathbf{T}_{\mathcal{I}}$, the node v_i is a leaf if and only if $\mathrm{Type}(Q)_i = 0$. Hence, $\mathrm{c}_{00}(I) = \mathrm{lnode}(T, \ell)$. Moreover, if v_i is an internal node, then $\mathrm{Type}(P)_i = 0$ if and only if $\lambda(e_i) = 0$, where e_i is the leftmost descending edge of v_i, and $\lambda = \Lambda(\ell)$. We thus conclude for the other equalities. □

Using Proposition 3.4, we check that $\mathbf{I}_{\mathcal{T}}$ and $\mathbf{T}_{\mathcal{I}}$ are bijections.

Proposition 3.7. $\mathbf{I}_{\mathcal{T}}$ *is a bijection from* $\mathcal{T}_n$ *to* $\mathcal{I}_{n+1}$ *for any* $n \geq 0$*, with* $\mathbf{T}_{\mathcal{I}}$ *its inverse.*

4 Symmetries and structure

With the bijections in Sections 2 and 3, we construct the bijections in our main result.

Proof of Theorem 0.1. We take $\mathbf{I}_{\mathcal{M}} = \mathbf{I}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{M}}$ and $\mathbf{M}_{\mathcal{I}} = \mathbf{M}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{I}}$, valid by Propositions 2.6 and 3.7. The equalities of statistics come from Propositions 2.3 and 3.6. □

Corollary 4.1. *The generating functions* $F_{\mathcal{I}}$ *and* $F_{\mathcal{M}}$ *are related by* $tF_{\mathcal{M}} = wF_{\mathcal{I}}$*. In particular, the series* $wF_{\mathcal{I}}$ *is symmetric in* u, v, w*.*

Proof. The equality is a direct translation of Theorem 0.1 in generating functions. The symmetry of $wF_{\mathcal{I}}$ comes from that of $F_{\mathcal{M}}$. □

As mentioned before, the symmetry in $\mathrm{c}_{00}, \mathrm{c}_{01}, \mathrm{c}_{11}$ was already known to Chapoton and Fusy, and a proof was outlined in [9], using recursive decompositions of new intervals [3, Lemma 7.1] and bipartite planar maps. Our bijective proof can be seen as a direct version of that proof, as degree trees are canonical descriptions of both decompositions.

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