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Smirnov trees

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Abstract. We introduce a generalization of Smirnov words in the context of labeled binary trees, which we call Smirnov trees. We study the generating function for ascent-descent statistics on Smirnov trees and establish that it is e-positive, which is akin to the classical case of Smirnov words. Our proof relies on an intricate weight-preserving bijection.

Keywords: ascent-descents, e-positivity, Smirnov trees, Smirnov words

1 Introduction

The study of permutation statistics has been an active area of research since the seminal work of MacMahon [11]. A particular permutation statistic that plays a prominent role in algebraic combinatorics is the descent statistic. The associated integer sequence obtained by counting permutations according to their number of descents is the well-known sequence of Eulerian numbers. Rather than giving an exhaustive list of areas in mathematics where this sequence makes an appearance, we refer the reader to [12] for a beautiful book exposition. Given that permutations may be considered to be linear trees, it is natural to consider generalizations of classical permutation statistics in the context of labeled plane binary trees.

Gessel [7] was the first to study the analogue of the descent statistic for labeled binary trees, and he further pointed out intriguing connections to the enumerative theory of Coxeter arrangements. There has been a flurry of activity towards understanding these connections better, and the reader is referred to [3, 4, 5, 8, 20, 21] for more details. Gessel–Griffin–Tewari [8] investigated these connections from the perspective of symmetric functions; they attached a multivariate generating function tracking ascent-descents over all labeled binary trees and subsequently proved that this generating function expands positively in terms of ribbon Schur functions. This result is part of the motivation for our work. The rest of it stems from recent work of Shareshian–Wachs [14] on chromatic quasisymmetric functions, which we discuss next.

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Stanley [17] introduced chromatic symmetric functions of graphs as a way to generalize chromatic polynomials of graphs. In the case where the graph has n nodes and no edges, the chromatic polynomial is the sum of $\mathbf{x}_w$ where w ranges over all words of length n. Here, and throughout the rest of this extended abstract, set $\mathbf{x}_w := x_{w_1} \cdots x_{w_n}$ if $w = w_1 \dots w_n$, where $\mathbf{x} = \{x_1, x_2, \dots\}$ is an alphabet with commuting indeterminates. It is worth remarking that the refined version tracking the distribution of descents over the set of all words of length n is ribbon Schur-positive. This motivates studying an analogue of the descent statistic for general graphs in tandem with the chromatic symmetric function, and this was done by Shareshian and Wachs in [14], wherein they introduced chromatic quasisymmetric functions. In the special case where the underlying graph is a path on n nodes, the chromatic quasisymmetric function tracks descents in Smirnov words of length n, i.e. words where two adjacent letters are distinct. More importantly, in contrast to the ribbon-positivity in the case where the graph was completely disconnected, the chromatic quasisymmetric function of the path graph is e-positive, i.e. it can be expressed as a non-negative integer linear combination of elementary symmetric functions. This raises the following natural questions:

1. Is there an analogue of Smirnov words and the descent statistic in the context of labeled binary trees?

2. If yes, is the generating function tracking the distribution of descents e-positive?

In this extended abstract, we provide answers to both these questions in the positive by introducing Smirnov trees — labeled rooted plane binary trees with the property that if the parent has the same label as one of its children, then the left child must have a larger label than the right child. While this may not sound like the most natural way to generalize Smirnov words, we remark that we "discovered" rather than "invented" this definition when we were studying the solution of the equation in Theorem 1.1. Denote the set of all Smirnov trees by $\mathcal{T}$. For any labeled binary tree T, let $\mathsf{lasc}(T)$, $\mathsf{ldes}(T)$, $\mathsf{rasc}(T)$ and $\mathsf{rdes}(T)$ denote the number of ascents and descents in the labeling to the left and right, such that rasc and lasc are determined by weak inequalities, whereas rdes and ldes are strict. See Section 2.2 for details. We associate a monomial $\mathbf{x}_T$ with T as follows. For a node $v \in T$ labeled i, let x_v be x_i. Then $\mathbf{x}_T = \prod_{v \in T} x_v$.

Consider the formal power series in $\mathbf{x}$ with coefficients in $\mathbb{Q}[\bar{\rho}, \rho, \bar{\lambda}, \lambda]$,

$$G := G(\bar{\rho}, \rho, \bar{\lambda}, \lambda) = \sum_{T \in \mathcal{T}} \bar{\rho}^{\mathsf{rasc}(T)} \rho^{\mathsf{rdes}(T)} \bar{\lambda}^{\mathsf{lasc}(T)} \lambda^{\mathsf{ldes}(T)} \mathbf{x}_T.$$

It is not immediate that G is a symmetric function in $\mathbf{x}$ with coefficients in $\mathbb{Q}[\bar{\rho}, \rho, \bar{\lambda}, \lambda]$. We establish the preceding fact via the following functional equation satisfied by G, which is also our first main result.

Theorem 1.1. *Let $\mathcal{W}_n$ be the set of Smirnov words of length n. Then we have*

$$G(\bar{\rho},\rho,\bar{\lambda},\lambda) = \sum_{n\geq 1}\sum_{w\in\mathcal{W}_n}(\bar{\rho}\bar{\lambda}G+\bar{\rho}+\bar{\lambda})^{\mathsf{asc}(w)}(\rho\lambda G+\rho+\lambda)^{\mathsf{des}(w)}\mathbf{x}_w.$$

Theorem 1.1 follows from an intricate bijection between $\bigcup_{n\geq 1}\mathcal{W}_n\times(\mathcal{T}\cup\{D,U\})^{n-1}$ and $\mathcal{T}$, and this forms the technical crux of our work. Here D and U stand for "down step" and "up step", and their role will be clear from the proof.

Let $e_n := e_n(\mathbf{x})$ denote the n-th elementary symmetric function, and let $E(z) := \sum_{n\geq 0} e_n z^n$. It is well known [14, Theorem C.4] that the generating function tracking ascents and descents over all Smirnov words is e-positive. This fact in conjunction with Theorem 1.1 implies the following theorem.

Theorem 1.2. *$G(\bar{\rho},\rho,\bar{\lambda},\lambda)$ is e-positive.*

In fact, a combinatorially explicit description for the coefficients in the basis of elementary symmetric functions can be provided. Additionally, we establish the following functional equation satisfied by G.

Theorem 1.3.

$$\frac{(1+\bar{\rho}G)(1+\bar{\lambda}G)}{(1+\rho G)(1+\lambda G)} = \frac{E(\bar{\rho}\bar{\lambda}G+\bar{\rho}+\bar{\lambda})}{E(\rho\lambda G+\rho+\lambda)}.$$

Outline of the extended abstract: In Section 2, we provide the necessary definitions for Smirnov words and Smirnov trees. In Section 3, we outline the proof of Theorem 1.1, how e-positivity follows from the functional equation, and give a proof of Theorem 1.3. The reader can find the missing details in the full version of this extended abstract [10].

2 Background

2.1 Words

For any undefined terms, we refer the reader to [18].

We denote the set of positive integers by $\mathbb{N}$. For $n\in\mathbb{N}$, set $[n] := \{1,\ldots,n\}$. In particular, $[0] = \emptyset$. Let $\mathbb{N}^*$ denote the set of all words in the alphabet $\mathbb{N}$. Given $w = w_1\ldots w_n \in \mathbb{N}^*$, we call n the *length* of w. An index $i\in[n-1]$ is called a *descent* of w if $w_i > w_{i+1}$ and an *ascent* otherwise. We denote the set of descents (respectively ascents) of w by $\mathrm{Des}(w)$ (respectively $\mathrm{Asc}(w)$) and denote $|\mathrm{Des}(w)|$ (respectively $|\mathrm{Asc}(w)|$) by $\mathsf{des}(w)$ (respectively $\mathsf{asc}(w)$).

A word $w = w_1\ldots w_n$ is a *Smirnov word* if adjacent letters are different, i.e. if $w_i \neq w_{i+1}$ for $i\in[n-1]$. The set of all Smirnov words is denoted by $\mathcal{W}$ and the set of all Smirnov words of length n by $\mathcal{W}_n$. Consider the multivariate generating function tracking the distribution of ascents and descents over $\mathcal{W}_n$

$$Q_n(s,t) = \sum_{w \in \mathcal{W}_n} s^{\mathsf{asc}(w)} t^{\mathsf{des}(w)} \mathbf{x}_w,$$

and let $Q(z;s,t) := \sum_{n \geq 0} Q_n(s,t) z^n$.

2.2 Smirnov trees

A *plane binary tree* is a rooted tree in which every node has at most two children, of which at most one is called a *left child* and at most one is called a *right child*. Henceforth, we simply say binary tree instead of plane binary tree. A *labeled plane binary tree* (or simply a *labeled binary tree*) is a binary tree whose nodes have labels drawn from $\mathbb{N}$. We assign a weight to labeled binary trees as follows. An edge between a parent (resp. a left child) with label a and its right child (resp. its parent) with label b is weighted $\bar{\rho}$ (resp. $\bar{\lambda}$) if $a \leq b$ and ρ (resp. λ) if $a > b$. In other words, if the edge is between a parent and the right child, we use ρ (for right), and if it is between a parent and its left child, we use λ (for left). We add the bar if the nodes form a weak ascent when reading from left to right (either diagonally up or down). A node with label a has weight x_a. The weight of a binary tree T, denoted by $\mathsf{wt}(T)$, is the product of the weights of all its edges and nodes.

A labeled binary tree is called *Smirnov* if the following holds:

- if the left (resp. right) child has the same label i as its parent, then the parent must also have a right (resp. left) child with label $< i$ (resp. $> i$).

In other words, if the parent has the same label as its child, the left child must have a larger label than the right child. Smirnov trees inherit a weight function from the one defined for ordinary labeled binary trees. In the case where the binary tree underlying a Smirnov tree is such that every node (except the leaf) has only a right child (we can force this by setting $\lambda = \bar{\lambda} = 0$), the earlier condition guarantees that the label on any node is different from the label on its child. In other words, the word obtained by reading the labels from root to leaf is a Smirnov word. Thus, Smirnov trees generalize Smirnov words.

Denote the set of all Smirnov trees by $\mathcal{T}$, and the set of all Smirnov trees whose root has label c by $\mathcal{T}^c$. If $T \in \mathcal{T}$, define its *principal path* $\mathsf{P}(T)$ as follows: it starts at the root; if the current node has no right child, stop; if the current node has a left child with the same label (and therefore a right child with a smaller label), move down left; otherwise, move down right. The last node on the principal path is called the *principal node* and is denoted $\alpha(T)$; its label is $a(T)$.

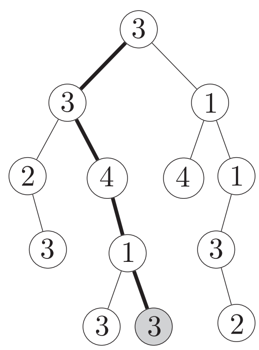


Figure 1: A Smirnov tree.

As an example, let T be the binary tree in Figure 1. There are two nodes with a child with the same label; in both cases, the parent has both children, and the left label is larger than the right label. Therefore T is Smirnov. In particular, $T \in \mathcal{T}^3$. The edges lying on the principal path $\mathsf{P}(T)$ are thickened, and the labels on $\mathsf{P}(T)$ are $3, 3, 4, 1, 3$. So $a(T) = 3$; the principal node $\alpha(T)$ is gray. Furthermore,

$$\mathsf{wt}(T) = \bar{\rho}^4 \rho^3 \bar{\lambda}^2 \lambda^3 \, x_1^3 x_2^2 x_3^6 x_4^2.$$

As mentioned in the introduction, we prove Theorem 1.1 by establishing a weight-preserving bijection between $\mathcal{X} := \bigcup_{n \geq 1} \mathcal{W}_n \times (\mathcal{T} \cup \{D, U\})^{n-1}$ and $\mathcal{T}$. We have already described how to assign weights to elements of the latter, so we proceed to describe weights assigned to elements of the former.

Given a statement P, let $[P]$ equal 1 if P is true and 0 otherwise. If a, b are distinct positive integers and $Y \in \mathcal{T} \cup \{D, U\}$, define

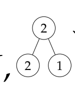
$$\mathsf{f}(a, b, Y) := \begin{cases} \bar{\rho}\bar{\lambda}\, \mathsf{wt}(Y)\, [Y \in \mathcal{T}] + \bar{\rho}\, [Y = D] + \bar{\lambda}\, [Y = U] & \text{if } a < b, \\ \rho\lambda\, \mathsf{wt}(Y)\, [Y \in \mathcal{T}] + \rho\, [Y = D] + \lambda\, [Y = U] & \text{if } a > b. \end{cases}$$

This given, consider $(w, S) \in \mathcal{X}$, and let $w := w_1 \ldots w_n$ and $S := S_1 \ldots S_{n-1}$. The weight of (w, S), denoted by $\mathsf{wt}(w, S)$, is defined as follows:

$$\mathsf{wt}(w, S) := \mathbf{x}_w \prod_{1 \leq i \leq n-1} \mathsf{f}(w_i, w_{i+1}, S_i).$$

As an example, consider the Smirnov word $w = 42534242 \in \mathcal{W}_8$ and the sequence

$$S = \left(1, D, 2, D, D, U, \right) \in (\mathcal{T} \cup \{D, U\})^7$$



Then we have

$$\begin{aligned} &\mathsf{f}(w_1, w_2, S_1) = \rho\lambda\, x_1, && \mathsf{f}(w_2, w_3, S_2) = \bar{\rho}, && \mathsf{f}(w_3, w_4, S_3) = \rho\lambda\, x_2, \\ &\mathsf{f}(w_4, w_5, S_4) = \bar{\rho}, && \mathsf{f}(w_5, w_6, S_5) = \rho, && \mathsf{f}(w_6, w_7, S_6) = \bar{\lambda} \\ &\mathsf{f}(w_7, w_8, S_7) = \rho^2 \bar{\lambda}\lambda\, x_1 x_2^2. \end{aligned}$$

It follows that $\mathsf{wt}(w, S) = \bar{\rho}^2 \rho^5 \bar{\lambda}^2 \lambda^3 \, x_1^2 x_2^6 x_3 x_4^3 x_5$.

It is straightforward to see that

$$\sum_{(w,S)\in\mathcal{X}} \mathsf{wt}(w,S) = \sum_{n\geq 1}\sum_{w\in\mathcal{W}_n} (\bar{\rho}\bar{\lambda}G+\bar{\rho}+\bar{\lambda})^{\mathsf{asc}(w)}(\rho\lambda G+\rho+\lambda)^{\mathsf{des}(w)}\mathbf{x}_w. \tag{2.1}$$

Note that the right hand side in equation (2.1) is exactly the right hand side of the equality in Theorem 1.1. This explains why it suffices to exhibit a weight-preserving bijection between $\mathcal{T}$ and $\mathcal{X}$ in order to prove Theorem 1.1.

3 Bijection and applications

3.1 Outline of the bijection

Our weight-preserving bijection between $\mathcal{T}$ and $\mathcal{X}$ uses an intermediate bijection that we call Φ. Consider a triple (T,S,b), where $T\in\mathcal{T}$, $S\in\mathcal{T}\cup\{D,U\}$, and $b\in\mathbb{N}$ is such that $a(T)\neq b$. Recall that $a(T)$ is the label of the principal node of T. We assign a weight to (T,S,b) as follows:

$$\mathsf{wt}(T,S,b) = x_b\,\mathsf{wt}(T)\mathsf{f}(a(T),b,S).$$

Our map Φ, whose sketch we postpone to Section 3.4, satisfies the following.

Theorem 3.1. *The map Φ is a well-defined weight-preserving bijection from the set of triples (T,S,b) in $\mathcal{T}\times(\mathcal{T}\cup\{D,U\})\times\mathbb{N}$ satisfying $a(T)\neq b$ to the set of Smirnov trees with at least 2 nodes, and the principal node of the image has label equal to the third argument. In other words, $\mathsf{wt}(\Phi(T,S,b))=\mathsf{wt}(T,S,b)$ and $a(\Phi(T,S,b))=b$.*

For the moment, we describe how Theorem 1.1 follows from Theorem 3.1, and subsequently we discuss various applications.

Proof of Theorem 1.1. By applying Φ successively, we construct a weight-preserving bijection $\Psi\colon\mathcal{X}\longrightarrow\mathcal{T}$, which proves our main theorem. If $w\in\mathcal{W}_1$, let $\Psi(w,\varnothing)$ be the binary tree with a single node labeled w_1. Here $\varnothing$ denotes the empty sequence. If $n\geq 2$, $w=w_1\cdots w_n\in\mathcal{W}_n$, $S=(S_1,\ldots,S_{n-1})\in(\mathcal{T}\cup\{D,U\})^{n-1}$, define $w'=w_1\cdots w_{n-1}$, $S'=(S_1,\ldots,S_{n-2})$, and

$$\Psi(w,S)=\Phi(\Psi(w',S'),S_{n-1},w_n).$$

We prove by induction on n that for $w\in\mathcal{W}_n$ and $S\in(\mathcal{T}\cup\{D,U\})^{n-1}$, $\Psi(w,S)$ is well defined and $a(\Psi(w,S))=w_n$. For $n=1$, this is obvious, and if it holds for $n-1$, then $a(\Psi(w',S'))=w_{n-1}\neq w_n$, so $\Phi(\Psi(w',S'),S_{n-1},w_n)$ is well defined, and by Theorem 3.1, $a(\Phi(\Psi(w',S'),S_{n-1},w_n))=w_n$.

As far as the weight-preserving aspect of Ψ is concerned, note that

$$\mathsf{wt}(w,S) = x_{w_n} \cdot \mathsf{wt}(w',S') \cdot \mathsf{f}(w_{n-1},w_n,S_{n-1}).$$

Furthermore,

$$\mathsf{wt}(\Psi(w,S)) = \mathsf{wt}(\Phi(\Psi(w',S'),S_{n-1},w_n)) = x_{w_n} \cdot \mathsf{wt}(\Psi(w',S')) \cdot \mathsf{f}(w_{n-1},w_n,S_{n-1}).$$

In going from the first equality to the second, we have utilized the fact that Φ is a weight-preserving bijection. Induction along with a comparison of the last two equations implies that Ψ is weight preserving. Bijectivity of Ψ also follows from the bijectivity of Φ. □

3.2 e-positivity

To write an explicit expression for $Q_n(s,t)$, we invoke [14, Theorem C.4] which states that the generating function for Smirnov words of length n is e-positive. More precisely, Shareshian and Wachs prove that

$$\sum_{w\in\mathcal{W}_n} t^{\mathsf{des}(w)}\mathbf{x}_w = \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{\substack{k_1,\dots,k_m\geq 2\\ \sum k_i=n+1}} e_{(k_1-1,k_2,\dots,k_m)} t^{m-1}\prod_{i=1}^{m}[k_i-1]_t, \tag{3.1}$$

where $[a]_t = 1+t+\cdots+t^{a-1}$ for $a\in\mathbb{N}$. For example, it is easy to check that $\sum_{w\in\mathcal{W}_3} t^{\mathsf{des}(w)}\mathbf{x}_w = (1+t+t^2)e_3+te_{21}$, which agrees with the formula.

It immediately follows from equation (3.1) that

$$\begin{aligned}
Q_n(s,t) &= s^{n-1}\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{\substack{k_1,\dots,k_m\geq 2\\ \sum k_i=n+1}} e_{(k_1-1,k_2,\dots,k_m)}(t/s)^{m-1}\prod_{i=1}^{m}[k_i-1]_{t/s}\\
&= \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{\substack{k_1,\dots,k_m\geq 2\\ \sum k_i=n+1}} e_{(k_1-1,k_2,\dots,k_m)}s^{m-1}t^{m-1}\prod_{i=1}^{m}(s^{k_i-2}+s^{k_i-3}t+\cdots+t^{k_i-2})\\
&= \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{\substack{k_1,\dots,k_m\geq 2\\ \sum k_i=n+1}} e_{(k_1-1,k_2,\dots,k_m)}(s^{k_1-2}+s^{k_1-3}t+\cdots+t^{k_1-2})\prod_{i=2}^{m}(s^{k_i-1}t+s^{k_i-2}t^2+\cdots+st^{k_i-1})
\end{aligned} \tag{3.2}$$

Our main result Theorem 1.1 states that the generating function for Smirnov trees satisfies the functional equation

$$G = \sum_{n\geq 1}\sum_{w\in\mathcal{W}_n} s^{\mathsf{asc}(w)}t^{\mathsf{des}(w)}\mathbf{x}_w = Q(1;s,t)-1, \tag{3.3}$$

where $s = \bar{\rho}\,\bar{\lambda}\,G + \bar{\rho} + \bar{\lambda}$, $t = \rho\,\lambda\,G + \rho + \lambda$. From equations (3.2) and (3.3), it follows that G is e-positive. It is not hard to come up with combinatorial objects that allow us to describe explicitly the coefficient of e_π in the e-expansion of G for an arbitrary partition π. We refer to [10, Section 3.2] for details.

The first few terms of the expansion of G are

$$e_1 + (\bar{\rho} + \rho + \bar{\lambda} + \lambda)e_2 + (\bar{\rho}^2 + \bar{\rho}\rho + \rho^2 + 2\bar{\rho}\bar{\lambda} + \rho\bar{\lambda} + \bar{\lambda}^2 + \bar{\rho}\lambda + 2\rho\lambda + \bar{\lambda}\lambda + \lambda^2)e_3 \\ + (\bar{\rho}\rho + \bar{\rho}\bar{\lambda} + \rho\bar{\lambda} + \bar{\rho}\lambda + \rho\lambda + \bar{\lambda}\lambda)e_{21} + \cdots$$

The expansion may be interpreted as an instance of Schur-γ-nonnegativity defined in [15]. See also [2] for more on this theme.

3.3 Another functional equation and exponential specialization

We now give a proof of the functional equation satisfied by G stated in Theorem 1.3, and then relate it to earlier work of Gessel.

Proof of Theorem 1.3. We have

$$Q(z;s,t) = 1 + s^{-1}\left(Q(sz;1,ts^{-1}) - 1\right). \tag{3.4}$$

From [14, Theorem C.3] and (3.4), we obtain

$$Q(1;s,t) - 1 = \frac{E(s) - E(t)}{sE(t) - tE(s)}.$$

Set $s = \bar{\rho}\bar{\lambda}G + \bar{\rho} + \bar{\lambda}$ and $t = \rho\lambda G + \rho + \lambda$ henceforth. By Theorem 1.1, we have $G = Q(1;s,t) - 1$, which in turn implies $(1 + \bar{\rho}G)(1 + \bar{\lambda}G) = 1 + Gs = s(Q(1;s,t) - 1) + 1$ and $(1 + \rho G)(1 + \lambda G) = 1 + Gt = t(Q(1;s,t) - 1) + 1$. Thus we obtain

$$\frac{(1 + \bar{\rho}G)(1 + \bar{\lambda}G)}{(1 + \rho G)(1 + \lambda G)} = \frac{s\frac{E(s) - E(t)}{sE(t) - tE(s)} + 1}{t\frac{E(s) - E(t)}{sE(t) - tE(s)} + 1} = \frac{E(s)}{E(t)}. \tag{3.5}$$

This establishes the claim. □

While the equality in Theorem 1.3 is less transparent than Theorem 1.1, it immediately allows us to establish a result present in unpublished work of Gessel, and then proved in [9] and [5]. We call a labeled binary tree on n nodes *standard* if the labels are all distinct and drawn from $[n]$. Note that a standard labeled binary trees is necessarily Smirnov. Gessel considered the following generating function that tracks the distributions of ascents and descents over standard labeled binary trees:

$$B := B(\bar{\rho}, \rho, \bar{\lambda}, \lambda) = \sum_{n \geq 1} \sum_{T \text{ standard}} \bar{\rho}^{\mathsf{rasc}(T)} \rho^{\mathsf{rdes}(T)} \bar{\lambda}^{\mathsf{lasc}(T)} \lambda^{\mathsf{ldes}(T)} \frac{x^n}{n!}. \tag{3.6}$$

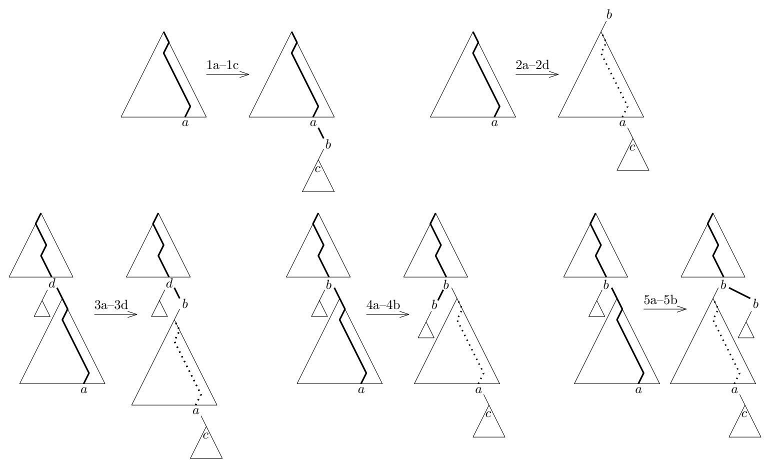


Figure 2: The map Φ.

Consider the homomorphism ex from the ring of symmetric functions to $\mathbb{Q}[[x]]$ defined by setting $\mathrm{ex}(e_n) = \frac{x^n}{n!}$ for $n \in \mathbb{N}$. A key property of ex is the following. Given a symmetric function f, we have $[x_1 \cdots x_n]f = [\frac{x^n}{n!}]\mathrm{ex}(f)$. It follows that $\mathrm{ex}(G) = B$, and we obtain the following corollary.

Corollary 3.2.

$$\frac{(1+\bar{\rho}B)(1+\bar{\lambda}B)}{(1+\rho B)(1+\lambda B)} = e^{((\bar{\lambda}\bar{\rho}-\lambda\rho)B+\bar{\rho}+\bar{\lambda}-\rho-\lambda)x}.$$

3.4 Sketch of the bijection Φ

The crucial ingredient of the proof of Theorem 1.1 is the map

$$\Phi\colon \{(T,S,b) \in \mathcal{T} \times (\mathcal{T} \cup \{D,U\}) \times \mathbb{N}\colon a(T) \neq b\} \longrightarrow \mathcal{T},$$

whose full definition (with examples) is in [10, Section 4], and is illustrated in Figure 2. The definition has 15 rules, split into five groups. Here we just present the simplest group with three rules.

Take a Smirnov tree T; S, which is either a Smirnov tree or a down step D (up steps are covered in other rules); and an integer b that is different from the label of the principal node of T. Write $\alpha = \alpha(T)$, $a = a(T)$, $P = \mathsf{P}(T)$. By definition, $a \neq b$, and α has no right child. If $S \in \mathcal{T}$, let c be the label of its root.

The first group of rules is used if (1a) $S = D$ or (1b) $S \in \mathcal{T}^c$ & $a, c < b$ or (1c) $S \in \mathcal{T}^c$ & $a, c > b$. In these cases, $\Phi(T,S,b)$ is the tree we obtain if we add a right child β with label b to α. If $S = D$, β has no children, and if $S \in \mathcal{T}^c$, β has S as its left subtree.

We take the same tree T for all examples, with the principal node $\alpha = \alpha(T)$ being the left child of the right child of the root, and $a = a(T) = 3$ (again, the principal node is gray). By definition, b cannot equal 3. If $S = D$, we simply add a right child to α, with label b. The weight of T is multiplied by x_b and, if $a < b$ (resp. $a > b$), by $\bar{\lambda}$ (resp. λ); the edge that contributes this weight is red. The situation is similar if S is a tree with root label c, and b is either larger or smaller than both a and c. In that case, add a right child with label b to α, and this right child has S as its left subtree (light gray). The weight of T is multiplied by x_b, $\mathrm{wt}(S)$, and $\bar{\lambda}\,\bar{\rho}$ (resp. $\lambda\,\rho$) if $a < b$ (resp. $a > b$); the edges that contribute these weights are red. Note that in all three cases, the principal path is extended by one right step, and $a(\Phi(T,S,b)) = b$. See Figure 3.

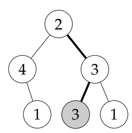
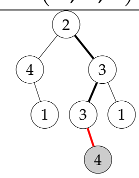
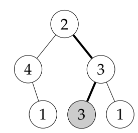
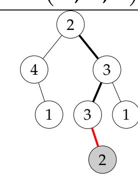
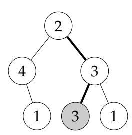
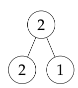
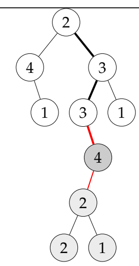
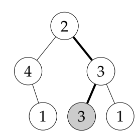
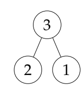
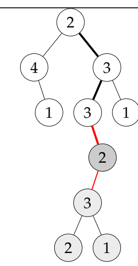
T	S	a	b	c	rule	$\Phi(T,S,b)$	T	S	a	b	c	rule	$\Phi(T,S,b)$
	D	3	4		1a			D	3	2		1a	
		3	4	2	1b				3	2	3	1c	

Figure 3: An illustration of rule 1.

4 Final remarks

1. As mentioned in the introduction, Smirnov words can be interpreted as proper colorings of path graphs. Smirnov trees, on the other hand, are a labeled tree-analogue of Smirnov words. This raises the natural question of constructing labeled (binary) tree analogues for graphs other than path graphs, and defining ascent-descent statistics on these tree analogues that relate to the Shareshian–Wachs chromatic quasisymmetric function. Another potential avenue is to consider the directed or cyclic analogue of the aforementioned question given recent work of Ellzey–Wachs [6] and Alexandersson–Panova [1].

2. From Section 3.3, the exponential specialization of G tracks ascents-descents in standard labeled binary trees. In particular, this implies $[x_1 \dots x_n]G(1,1,1,1) = n! \times \mathrm{Cat}_n$ and $[x_1 \dots x_n]G(1,1,1,0) = (n+1)^{n-1}$, where Cat_n is the nth Catalan number. It is unclear to us how to derive these equalities starting from our e-positive expansion for G.

3. The h-positivity of ωG (here ω is the standard involution on the algebra of symmetric functions) raises the question of constructing a natural permutation representation that realizes ωG as its Frobenius characteristic. Stanley [16, Proposition 7.7], using a recurrence due to Procesi [13], established that $\omega G(\bar\rho, \rho, 0, 0)$ can be realized as the generating function of the Frobenius characteristic of the representation of S_n on the cohomology of the toric variety associated with the Coxeter complex of type A_{n-1}. Stembridge [19] also constructed a symmetric group representation which realizes $\omega G(\bar\rho, \rho, 0, 0)$ as the Frobenius characteristic via a bijection from permutations to what he calls codes.

It would be interesting to generalize these results to Smirnov trees. The following are the characters that such permutation representations would have. For example, there are 288 standard trees on five nodes that have weight $\bar\rho^2\bar\lambda\lambda$ (i.e., that have two right ascents, one left ascent and one left descent), and, according to line 7 of the last table, 94 of them should be fixed under the action of a transposition.

	111	21	3
$\bar\rho^2, \rho^2, \bar\lambda^2, \lambda^2$	1	1	1
$\bar\rho\rho, \bar\rho\lambda, \rho\bar\lambda, \bar\lambda\lambda$	4	2	1
$\bar\rho\bar\lambda, \rho\lambda$	5	3	2

	1111	211	22	31	4
$\bar\rho^3, \rho^3, \bar\lambda^3, \lambda^3$	1	1	1	1	1
$\bar\rho^2\rho, \bar\rho^2\lambda, \bar\rho\rho^2, \bar\rho\lambda^2, \rho^2\bar\lambda, \rho\bar\lambda^2, \bar\lambda^2\lambda, \bar\lambda\lambda^2$	11	5	3	2	1
$\bar\rho^2\bar\lambda, \bar\rho\bar\lambda^2, \rho^2\lambda, \rho\lambda^2$	17	9	5	5	3
$\bar\rho\rho\bar\lambda, \bar\rho\rho\lambda, \bar\rho\bar\lambda\lambda, \rho\bar\lambda\lambda$	44	16	8	5	2

	11111	2111	221	311	32	41	5
$\bar\rho^4, \rho^4, \bar\lambda^4, \lambda^4$	1	1	1	1	1	1	1
$\bar\rho^3\rho, \bar\rho^3\lambda, \bar\rho\rho^3, \bar\rho\lambda^3, \rho^3\bar\lambda, \rho\bar\lambda^3, \bar\lambda^3\lambda, \bar\lambda\lambda^3$	26	12	6	5	3	2	1
$\bar\rho^3\bar\lambda, \bar\rho\bar\lambda^3, \rho^3\lambda, \rho\lambda^3$	49	25	13	13	7	7	4
$\bar\rho^2\rho^2, \bar\rho^2\lambda^2, \rho^2\bar\lambda^2, \bar\lambda^2\lambda^2$	66	22	10	6	4	2	1
$\bar\rho^2\bar\lambda^2, \rho^2\lambda^2$	146	60	26	26	12	12	6
$\bar\rho^2\rho\lambda, \bar\rho\rho^2\bar\lambda, \bar\rho\bar\lambda\lambda^2, \rho\bar\lambda^2\lambda$	237	73	29	18	10	5	2
$\bar\rho^2\rho\bar\lambda, \bar\rho^2\bar\lambda\lambda, \bar\rho\rho^2\lambda, \bar\rho\rho\bar\lambda^2, \bar\rho\rho\lambda^2, \bar\rho\bar\lambda^2\lambda, \rho^2\bar\lambda\lambda, \rho\bar\lambda\lambda^2$	288	94	36	27	13	8	3
$\bar\rho\rho\bar\lambda\lambda$	824	228	80	50	24	12	4

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