Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #20, 12 pp.
Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Euler Characteristics of Exotic Configuration Spaces

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Abstract. Exponential generating function for Euler characteristics of configuration spaces have remarkably simple representation in terms of the local geometry of the underlying spaces.

Keywords: Euler characteristics, configuration spaces, stratified spaces

1 Introduction

Configuration spaces – the spaces of tuples of distinct, distinguishable points in a topological space – are important in numerous applications, from dynamical systems to computer science. The topology of such configuration spaces is relatively well understood for manifolds, but for spaces with singularities, the recorded knowledge is still quite limited.

In this note we will generalize several existing results dealing with the Euler characteristics of certain (generalized) configuration spaces.

1.1 Precursors

We start with quoting some of those results.

1.1.1 Configuration spaces of simplicial complexes

Let $\mathbb{X}$ be (the geometric realization of) a finite simplicial complex

$$\mathbb{X} = \amalg_\alpha \sigma_\alpha,$$

where σ_α are the (relative)-open simplices of the triangulation.

For a finite set $\boldsymbol{N} = \{1, \ldots, n\}$ of size n, denote by $\mathtt{Conf}(\mathbb{X}, \boldsymbol{N})$ the conventional configuration space of n distinct points, that is

$$\mathtt{Conf}(\mathbb{X}, \boldsymbol{N}) := \mathbb{X}^{\boldsymbol{N}} - \cup_{k \neq l} \Delta_{kl},$$

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where $\Delta_{kl} = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{X}^{\boldsymbol{N}} : x_k = x_l\}$ are the *big* diagonals in $\mathbb{X}^{\boldsymbol{N}}$.

For this setup, [3] proves the following

Theorem 1.1 (S. Gal)**.** *The* exponential generating function *for the sequence of Euler characteristics of the configuration spaces of* $\mathbb{X}$ *is given by*

$$\sum_{n\geq 0} \chi(\mathtt{Conf}(\mathbb{X}, \boldsymbol{N}))\frac{z^n}{n!} = \prod_\alpha \left(1 + (-1)^{d(\alpha)}(1 - \chi(L(\alpha)))z\right)^{(-1)^{d(\alpha)}}, \tag{1.1}$$

where the product is taken over all simplices of $\mathbb{X}$*,* $d(\alpha)$ *is the dimension, and* $L(\alpha)$ *is the* link *of the simplex* $\boldsymbol{\sigma}_\alpha$*.*

(Here the link of a simplex $\boldsymbol{\sigma}_\alpha$ is (the geometric realization of) the simplicial complex formed by all simplices of $\mathbb{X}$ properly containing α, or, equivalently, is the base of the cone obtained by the intersection of piece-wise linearly embedded $\mathbb{X}$ with a small ball in a linear subspace transversally intersecting α at a single point, around that point.)

1.1.2 No-k-equal spaces

Exponential generating function for Euler characteristics appeared also in the following setting.

Take the real line as $\mathbb{X} = \mathbb{R}^1$, and consider the *no-k-equal* configuration spaces,

$$\mathtt{Conf}_k(\mathbb{R}^1, \boldsymbol{N}) = \mathbb{R}^n - \cup_{I=(i_1<\ldots<i_k)}\Delta_I, \tag{1.2}$$

where $\Delta_I = \{x_{i_1} = x_{i_2} = \ldots = x_{i_k}\}$, are k-diagonals (of codimension $(k-1)$), and the union is taken over all such diagonals. In other words, the no-k-equal configuration space is obtained by forbidding all configurations of n points in $\mathbb{R}$ that have k or more points coinciding.

These spaces appeared as an useful testing ground for topological lower bounds of complexity of linear decision trees, and were investigated intensely since their introduction (mostly from the viewpoint of the linear subspace arrangements theory).

One of the results of [2] can be formulated as follows:

Theorem 1.2 (A. Björner, L. Lovasz)**.** *The exponential generating function for the Euler characteristics of no-k-equal spaces* $\mathtt{Conf}_k(\mathbb{R}^1, \boldsymbol{N})$ *is given by*

$$\sum_{n\geq 0} \chi(\mathtt{Conf}_k(\mathbb{R}^1, \boldsymbol{N}))\frac{z^n}{n!} = \frac{1}{1 - z + z^2/2 - \ldots + (-z)^{k-1}/(k-1)!}. \tag{1.3}$$

1.2 Colored configuration spaces

The main result of this paper generalizes both theorems above. We work with *colored* configuration spaces.

1.2.1 Colors and Ideals

Fix a finite set $\mathfrak{C}$ of *colors* of size c. The vectors in $\mathbb{N}^{\mathfrak{C}}$ will be referred to as *color counts*.

A subset of color counts $\mathcal{I} \in \mathbb{N}^{\mathfrak{C}}$ is an *ideal* if for any $\mathbf{m} = (m_1, \ldots, m_c) \in I$ and $\mathbf{m}' \leq \mathbf{m}$ (this notation shorthands for $m'_1 \leq m_1, \ldots, m'_c \leq m_c$), the point $\mathbf{m}'$ also lies in $\mathcal{I}$. We will be assuming that all the basis vectors of $\mathbb{N}^{\mathfrak{C}}$ are in $\mathcal{I}$.

We will be considering finite collections of distinct colored points in $\mathbb{X}$ numbered by the elements of $\boldsymbol{N}$. If $I \subset \boldsymbol{N}$, we will denote the corresponding color counts of points in I as $\boldsymbol{c}(I)$.

1.2.2 Ideals and Their Configuration Spaces

Fix an ideal $\mathcal{I}$ (of permissible collisions).

Definition 1.3. For a collection $\boldsymbol{N}$ of colored points, the configuration space $\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N})$ is defined as

$$\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N}) = \mathbb{X}^{\boldsymbol{N}} - \bigcup_{I : \boldsymbol{c}(I) \notin \mathcal{I}} \Delta_I. \tag{1.4}$$

As the configuration spaces for the collections of points with the same color count are natuarally equivalent, we will denote them sometimes as $\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{n})$, with $\mathbf{n} = \boldsymbol{c}(\boldsymbol{N})$,

In words, the configuration space $\mathtt{Conf}_{\mathcal{I}}$ prevents any collection of points in the configuration from coinciding if their color counts are outside of the ideal $\mathcal{I}$ of permissible collisions.

Example 1.4. If $c = 1$, and $\mathcal{I} = \{1\}$, we have the conventional configuration spaces. If $\mathcal{I} = \{1, 2, \ldots, k-1\}$, we have the no-$k$-equal configuration spaces.

The bi-colored ($c = 2$) ideal with $\mathcal{I} = \{(m, 0), (0, m), m = 0, 1, \ldots\}$ forbids any points of different colors to collide, but allows that for any number of points of the same color (the "apartheid" ideal).

Definition 1.5. The *counting function* of the ideal $\mathcal{I}$ is

$$\Phi_{\mathcal{I}}(\boldsymbol{z}) = \sum_{\mathbf{m} \in \mathcal{I}} \frac{\boldsymbol{z}^{\mathbf{m}}}{\mathbf{m}!}; \tag{1.5}$$

here $\boldsymbol{z} = (z_1, \ldots, z_c)$; $\boldsymbol{z}^{\mathbf{m}} = \prod_{k=1}^{c} z_k^{m_k}$ and $\mathbf{m}! = \prod(m_k!)$.

Example 1.6. For no-k-equal configuration space, $\Phi_{\mathcal{I}}(z) = 1 + z + z^2/2 + \ldots + z^{k-1}/(k-1)!$.

For the "apartheid ideal", $\Phi_{\mathcal{I}}(z_1, z_2) = \exp(z_1) + \exp(z_2) - 1$.

1.3 Main result

To formulate our main result we need a few more definitions.

1.3.1 Constructible Functions

Let $\mathbb{X}$ be a compact subanalytic set in $\mathbb{R}^D$, with (finite) Whitney stratification

$$\mathbb{X} = \amalg_\alpha \mathbb{X}_\alpha. \tag{1.6}$$

We will refer to the functions constant on the strata of $\mathbb{X}$ as *constructible* (for a background on constructible functions, see [6]).

1.3.2 Constructible Euler Characteristic

Definition 1.7. If $\mathbb{X}$ is a subanalytic triangulated set in $\mathbb{R}^D$, we will refer to the alternating sum of the numbers of simplices of each dimension as its *constructible Euler characteristic*, denoted $\chi_c(\mathbb{X})$. Equivalently, χ_c is the Euler characteristic computed as alternated sum of ranks of Borel-Moore cohomologies.

It is well known that χ_c is independent of the choice of triangulation, is therefore additive on subanalytic sets (i.e. $\chi_c(A) = \chi_c(A - B) + \chi_c(B)$ for subanalytic $B \subset A$), and matches the standard (homotopy-invariant) Euler characteristic on *compact* subanalytic sets.

One can define the integral of a constructible function with compact support with respect to χ_c as

$$\int f d\chi_c = \sum_s s\chi_c(f^{-1}(s)),$$

where the sum is taken over the (finite) range of f, or, equivalently, as the evaluation of the direct image of the integrand under the mapping of $\mathbb{X}$ to a point – see, e.g. [5].

Definition 1.8. For a constructible function f, its *dual* f° is defined as

$$f^\circ(x) = \int_{\mathbb{X}} \mathbf{1}_{B(x,\epsilon)} f d\chi_c,$$

where $\mathbf{1}_{B(x,\epsilon)}$ is the indicator function of the open Euclidean ball of radius $\epsilon > 0$ around x in $\mathbb{R}^D$, and the integral is with respect to Euler characteristic (the integral stabilizes when ϵ is small enough, and thus is well-defined) .

For such a subanalytic space $\mathbb{X}$ we set $\mathbf{1} = \mathbf{1}_{\mathbb{X}}$ to be the indicator function of $\mathbb{X}$. This is a constructible function, and its dual,

$$\mathbf{1}^\circ(x) = \chi(\mathbb{X} \cap B_\epsilon(x)) \tag{1.7}$$

can be mundanely interpreted as the (constructible) Euler characteristic of the open ball of small enough radius ϵ around a point x intersected with $\mathbb{X}$ (again, this Euler characteristics stabilizes for small ϵ). This, again, is a constructible function, i.e. is constant along the strata of $\mathbb{X}$. It is easy to see that $\mathbf{1}^\circ$ is supported by $\mathbb{X}$.

1.3.3 Main Result

We have the following formula for the (standard, homotopy-invariant) Euler characteristic of the generalized configuration spaces:

Theorem 1.9. *The exponential generating function for the Euler characteristic of $\mathcal{I}$-configuration spaces is given by*

$$\sum_{\mathbf{m}} \chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{m})) \frac{z^{\mathbf{m}}}{\mathbf{m}!} = \prod_{\alpha} \Phi_{\mathcal{I}}(\mathbf{1}^\circ(\alpha) z)^{(-1)^{d(\alpha)} \chi(\alpha)}, \tag{1.8}$$

where the product is taken over all strata of $\mathbb{X}$, $\mathbf{1}^\circ(\alpha)$ is the common value $\mathbf{1}^\circ(x)$ for the points of the stratum α, $d(\alpha)$ is the dimension, and $\chi(\alpha)$ is the Euler characteristic of the stratum $\mathbb{X}_\alpha$.

Equivalently,

Corollary 1.10.

$$\sum_{\mathbf{m}} \chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{m})) \frac{z^{\mathbf{m}}}{\mathbf{m}!} = \prod_{\alpha} \Phi_{\mathcal{I}}(\mathbf{1}^\circ(\alpha) z)^{\chi_c(\alpha)}, \tag{1.9}$$

where $\chi_c(\alpha)$ is the constructible *Euler characteristic of the stratum $\mathbb{X}_\alpha$.*

1.4 Plan

We start with a few examples of applications of the main result. In Section 3 we will present an outline of the proof, drawing on the combinatorial and geometric results excised for space reasons. The full proof is available for now in a preprint form [7].

2 Examples

2.1 Gal's formula

If $\mathbb{X}$ is a triangulated space, embedded into some Euclidean space so that it is linear on each simplex. We consider $\mathbb{X}$ as stratified by these simplices. For a point in a simplex of dimension d, the intersection of an open small ball with $\mathbb{X}$ is homeomorphic to $\mathbb{R}^d \times K$, where K is the open cone with the base $L(x)$, the link of $\mathbb{X}$ at x. Hence,

$$\mathbf{1}^\circ(x) = (-1)^d (1 - \chi(L(x))). \tag{2.1}$$

For "conventional" configuration spaces, $\mathcal{I} = \{0, 1\}$, and $\Phi_{\mathcal{I}}(z) = 1 + z$. Substituting these identities into (1.6) leads to

$$\sum_n \chi(\mathtt{Conf}(\mathbb{X}, N)) \frac{z^n}{n!} = \prod_\alpha (1 + (-1)^{d(\alpha)}(1 - \chi(L(\alpha)))z)^{(-1)^{d(\alpha)}}, \tag{2.2}$$

i.e. Gal's formula.

2.2 No-k-equal configuration spaces

For no-k-equal spaces, the counting function is $\Phi_{\mathcal{I}}(z) = 1 + z + \ldots + z^{k-1}/(k-1)! =: e_k(z)$.

If $\mathbb{X}$ is a d-dimensional disk, $\mathbf{1}^\circ = (-1)^d$ in the interior of the disk, and 0 on the boundary. This implies

$$\sum_n \chi(\mathtt{Conf}_k(\mathbb{R}^d, N)) \frac{z^n}{n!} = e_k((-1)^d z)^{(-1)^d}, \tag{2.3}$$

recovering, in particular, the formula (1.3).

2.3 Graphs

If $\mathbb{X}$ is a graph, i.e. geometric realization of a one-dimensional simplicial complex, one has, for any $\Phi_{\mathcal{I}}$,

$$\sum_{\mathbf{n}} \chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{n})) \frac{\mathbf{z}^{\mathbf{n}}}{\mathbf{n}!} = \frac{\prod_{\alpha \in V(\mathbb{X})} \Phi_{\mathcal{I}}((1 - v(\alpha))\mathbf{z})}{\Phi_{\mathcal{I}}^{|E(\mathbb{X})|}(-\mathbf{z})}, \tag{2.4}$$

where the product is taken over the vertices α, and $v(\alpha)$ stands for the degree of α; E is the set of the edges in the graph. (For conventional configuration spaces on graphs, see e.g. [4].)

We remark, in particular, that the leaf vertices do not contribute, and each degree 2 vertex cancels one edge (as it should, to maintain the invariance of $\mathtt{Conf}_{\mathcal{I}}$ with respect to homeomorphisms).

2.4 Manifolds

If $\mathbb{X}$ is a compact manifold of dimension d, one has but one stratum, and

$$\sum_{\mathbf{n}} \chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{n})) \frac{\mathbf{z}^{\mathbf{n}}}{\mathbf{n}!} = \Phi_{\mathcal{I}}((-1)^d \mathbf{z})^{(-1)^d \chi(\mathbb{X})}. \tag{2.5}$$

In particular, for finite ideals $\mathcal{I}$ and even-dimensional manifolds, $\chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{n})) = 0$ for large enough $\mathbf{n}$ (and is $\equiv 0$ for odd-dimensional manifolds).

2.5 Bicolored spaces

Consider now an example of configuration spaces with infinite ideals: the bi-colored apartheid ideal, with $\Phi_{\mathcal{I}} = \exp(z_1) + \exp(z_2) - 1$. Using equation (2.5), we obtain

$$\sum_{n_1,n_2} \chi(\mathtt{Conf}_{\mathcal{I}}(S^2,(n_1,n_2)))\frac{z_1^{n_1} z_2^{n_2}}{n_1!n_2!} = (\exp(z_1) + \exp(z_2) - 1)^2. \tag{2.6}$$

In particular, all Euler characteristics $\chi(\mathtt{Conf}_{\mathcal{I}}(S^2,(n_1,n_2))) = 2$ if $n_1, n_2 \geq 1$.

3 Proving the Main Result

3.1 Setting the Stage

Fix a finite set $\boldsymbol{N}$ with colors attached to each of the points in $\boldsymbol{N}$; set $\mathbf{n} = \boldsymbol{c}(\boldsymbol{N}) = (n_1, \ldots, n_c)$ to be the vector of color counts of $\boldsymbol{N}$.

3.1.1 Partitions

We will denote the set of partitions of $\boldsymbol{N}$ as $\mathtt{Part}(\boldsymbol{N})$. (A partition is an unordered collection of disjoint subsets of $\boldsymbol{N}$ exhausting it.)

Partitions of $\boldsymbol{N}$, or, more generally, of any finite set S, are partially ordered by refinement: here we write $\sigma \prec \pi$ if σ is a refinement of π, or π is a coarsening of σ, that is if each bloc of σ is contained entirely within a block of π.

The minimal (with respect to this refinement order) partition of S, consisting of singletons, will be denoted as $\mathbf{0}_S$, or just $\mathbf{0}$, when the context is clear.

The *join* of a family of partitions is the minimal common coarsening of the partitions in the family.

The maximal element for the poset $\mathtt{Part}(S)$, denoted as $\mathbf{1}_S$ is the partition with one block, (S).

3.1.2 Diagonals

For each partition $\pi = (\pi_1)(\pi_2)\ldots(\pi_k)$ of $\boldsymbol{N}$ into k blocks, we will form the diagonal

$$\boldsymbol{\Delta}(\pi) = \{\boldsymbol{x} : x_k = x_l \text{ if } k, l \text{ are in the same block of } \pi\}. \tag{3.1}$$

(Clearly, the diagonals form a lattice isomorphic to the lattice of partitions of $\boldsymbol{N}$ with reversed order.)

Definition 3.1. We define the collection $\mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}(\boldsymbol{N})$ of *forbidden partitions* π as such that for at least one block, the color count of this block is not in $\mathcal{I}$.

Then, by definition,

$$\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \mathbf{n}) = \mathbb{X}^N - \mathbf{\Delta}_{\mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}}, \text{ where } \quad \mathbf{\Delta}_{\mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}} = \bigcup_{\pi \in \mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}(N)} \mathbf{\Delta}(\pi). \tag{3.2}$$

3.1.3 Enabling Additivity

One certainly can find the (constructible) Euler characteristic of $\mathtt{Conf}_{\mathcal{I}}$ using (3.2) via additivity and some form of inclusion-exclusion principle. However this constructible χ_c is not equal to χ as $\mathtt{Conf}_{\mathcal{I}}$ is not compact.

To circumvent this problem, we replace the diagonals in (3.2) by their appropriately chosen open vicinities ("fattenings") in $\mathbb{X}^N$. If the compact constructible complement to those vicinities is homotopy equivalent to the corresponding configuration space, we can use additivity of χ_c to compute $\chi(\mathtt{Conf}_{\mathcal{I}})$.

3.1.4 Combinatorics and Geometry

The combinatorial component of the computation is rather standard, and amounts to the manipulations of Möbius function for the poset of partitions of colored sets of points. These computations allow one to reduce the problem to finding the constructible Euler characteristics of fattenings of diagonals.

The geometric part of the proof involves definitions of the fattenings of the diagonals and computations of their Euler characteristics. Details are contained in the full version of this paper; here we will just formulate the relevant results.

3.2 Proof of the Main Theorem

We start with straightening the geometry of $\mathbb{X}$.

3.2.1 Cubical embedding

Namely, it will be convenient to *cubulate* the space $\mathbb{X}$. Let $\mathbb{I} = [0, 1]$ be the unit interval.

Proposition 3.2. *There exists a refinement of the stratification of $\mathbb{X}$ which is homeomorphic to a subcomplex of the natural cubical complex of the unit cube $\mathbb{I}^D$ in some Euclidean space.*

From now on, we will assume that $\mathbb{X}$ is a cubical subcomplex of I^D, equipped with the metric induced from the sup norm on $\mathbb{R}^D$.

3.2.2 Deformation Retract

For $\boldsymbol{x} \in \mathbb{X}^{\boldsymbol{N}}$, we define the partition $\pi(\boldsymbol{x}, \epsilon)$ as the partition into the classes of equivalence defined by the transitive closure of the relation

$$x_k \sim x_l \Leftrightarrow |x_k - x_l| < \epsilon.$$

Definition 3.3. We define (ϵ-)fattening of $\boldsymbol{\Delta}^\epsilon(\pi)$ as the set of configurations $\boldsymbol{x} \in \mathbb{X}^{\boldsymbol{N}}$: $\pi \preceq \pi(\boldsymbol{x}, \epsilon)$, and

$$\boldsymbol{\Delta}^\epsilon(\mathcal{I}) := \bigcup_{\pi \in \mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}(\boldsymbol{N})} \boldsymbol{\Delta}^\epsilon(\pi).$$

Clearly, $\boldsymbol{\Delta}^\epsilon(\mathcal{I})$ is an open vicinity of $\boldsymbol{\Delta}_{\mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}}$.

Consequently,

$$\mathtt{Conf}^\epsilon_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N}) = \mathbb{X}^{\boldsymbol{N}} - \boldsymbol{\Delta}^\epsilon(\mathcal{I}) \tag{3.3}$$

is a compact subset of $\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N})$.

Proposition 3.4. *Under the assumptions of Section 3.2.1, $\mathtt{Conf}^\epsilon_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N})$ is a deformation retract of $\mathtt{Conf}_{\mathcal{I}}(\mathbb{X}, \boldsymbol{N})$.*

3.2.3 Inclusion-Exclusion Formulae

Proposition 3.5. *There exists a function $c_{\mathcal{I}}$ of partitions of $\boldsymbol{N}$ such that*

$$\sum_{\sigma \preccurlyeq \pi} c_{\mathcal{I}}(\sigma) = \begin{cases} 0 & \text{if } \pi \in \mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}(\boldsymbol{N}), \\ 1 & \text{otherwise} \end{cases} \tag{3.4}$$

For single blocks, this function depends only on the color content $c_{\mathcal{I}}(\boldsymbol{N}) = c_{\mathcal{I}}(\boldsymbol{c}(\boldsymbol{N}))$, and is multiplicative, in the sense that

$$c_{\mathcal{I}}((\pi_1) \ldots (\pi_k)) = \prod_l c_{\mathcal{I}}(\boldsymbol{c}(\pi_l)). \tag{3.5}$$

Lemma 3.6. *For sufficiently small ϵ, the intersection lattice generated by the sets $\boldsymbol{\Delta}^\epsilon(\pi), \pi \in \mathtt{Part}(\boldsymbol{N})$ is isomorphic to the partition lattice on $\boldsymbol{N}$: for any family of partitions $\{\pi_\lambda\}, \lambda = 1, \ldots, \Lambda$, the fattening of their join $\pi = \pi_1 \wedge \ldots \wedge \pi_\Lambda$ equals the intersection of the fattenings of the partitions in the family:*

$$\boldsymbol{\Delta}^\epsilon(\pi) = \bigcap \boldsymbol{\Delta}_\epsilon(\pi_k). \tag{3.6}$$

Proof. This follows directly from the definition of fattenings $\boldsymbol{\Delta}^\epsilon(\pi)$. □

We remark that this implies that $\boldsymbol{\Delta}^\epsilon(\pi) \subset \boldsymbol{\Delta}^\epsilon(\pi')$ for any refinement π' or π.

Proposition 3.5 and Lemma 3.6 allow us to represent the (topological) Euler characteristics in terms of the (constructible) Euler characteristics of the diagonal fattenings.

Proposition 3.7.
$$\chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X},\mathbf{n})) = \sum_{\pi\in\mathtt{Part}(\mathbf{n})} c_{\mathcal{I}}(\pi)\chi_c(\mathbf{\Delta}^\epsilon(\pi)). \tag{3.7}$$

Proof. For $x \in \mathbb{X}^N$ denote by $\pi(x)$ the coarsest partition π for which $x \in N^\epsilon(\pi)$ (it exists by Lemma 3.6). Then
$$\sum_{\sigma\in\mathtt{Part}(\mathbf{n})} c_{\mathcal{I}}(\sigma)\mathbf{1}_{\mathbf{\Delta}^\epsilon(\sigma)}(x) = \sum_{\sigma\preceq\pi(x)} c_{\mathcal{I}}(\sigma) \tag{3.8}$$
which is 1 exactly when $x \in \mathbb{X}^N - \bigcup_{\pi\in\mathtt{Part}^{\mathtt{F}}_{\mathcal{I}}(N)} \mathbf{\Delta}^\epsilon(\pi)$. Invoking the additivity of the constructible Euler characteristics completes the proof. □

3.2.4 Computations

We will need also the multiplicativity of the Euler characteristics of $\mathbf{\Delta}^\epsilon(\pi)$:

Lemma 3.8. *If $(\pi_1)\dots(\pi_p)$ are the blocks of the partition π, then*
$$\chi_c(\mathbf{\Delta}^\epsilon(\pi)) = \prod_l \chi_c(\mathbf{\Delta}^\epsilon(\pi_l)).$$
(Here the notation $\mathbf{\Delta}^\epsilon(\pi_l)$ is used to denote the fattening of the main diagonal in $\mathbb{X}^{\pi_i}$ - remark that it depends only on the size of the block π_l.)

Proof. The fact that a subset $I = (k_1,\dots,k_p)$ of $\mathbf{N}$ belongs to a part of $\pi(\boldsymbol{x},\epsilon)$ depends only on the points $x_{k_1},\dots,x_{k_p}$ implies that the set $\mathbf{\Delta}_\epsilon(\pi)$ is a product of its projections to $\mathbb{X}^{\pi_k}$, over all parts of $\pi = (\pi_1)\cdots(\pi_p)$. □

3.2.5 Partitions and Exponents

The following Lemma is standard:

Lemma 3.9. *Let $\kappa : \mathbb{N}^{\mathfrak{C}} \to \mathbb{C}$ be an arbitrary valuation such that $\kappa(\mathbf{0}) = 0$. Then*
$$\exp\left(\sum_{\mathbf{m}} \frac{z^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m})\right) = \sum_{\mathbf{n}} \frac{z^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi\in\mathtt{Part}(\mathbf{n})\\ \pi=(\pi_1)\dots(\pi_k)}} \prod_{l=1}^{k} \kappa(\mathbf{n}(\pi_l)).$$

Proof. See discussion of partitional composites of structures in [1]. □

Using Proposition 3.7 we can transform the exponential generating function as
$$F(z) = \sum_{\mathbf{n}} \frac{z^{\mathbf{n}}}{\mathbf{n}!}\chi(\mathtt{Conf}_{\mathcal{I}}(\mathbb{X},\mathbf{n})) = \sum_{\mathbf{n}} \frac{z^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi\in\mathtt{Part}(\mathbf{n})\\ \pi=(\pi_1)\dots(\pi_k)}} \prod_{l=1}^{k} c_{\mathcal{I}}(\mathbf{n}(\pi_l))\chi_c(\mathbf{\Delta}^\epsilon(\pi_l)). \tag{3.9}$$

Lemma 3.10.

$$F(z) = \exp\left(\sum_{\mathbf{m}>0} \frac{z^{\mathbf{m}}}{\mathbf{m}!} c_{\mathcal{I}}(\mathbf{m})\chi_c(\mathbf{\Delta}^{\epsilon}(|\mathbf{m}|))\right). \tag{3.10}$$

Proof. Denoting

$$\kappa(\mathbf{m}) := c_{\mathcal{I}}(\mathbf{m})\chi_c(\mathbf{\Delta}^{\epsilon}(|\mathbf{m}|)),$$

and using Lemma 3.9 as

$$\exp\left(\sum_{\mathbf{m}>0} \frac{z^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m})\right) = \sum_{\mathbf{n}} \frac{z^{\mathbf{n}}}{\mathbf{n}!} \sum_{\substack{\pi \in \mathtt{Part}(\mathbf{n}) \\ \pi=(\pi_1)\ldots(\pi_k)}} \prod_{l=1}^{k} \kappa(\mathbf{n}(\pi_l))$$

we arrive at the claim. □

3.2.6 Euler Characteristic of Fattenings

In view of Lemma 3.8, let us turn to $\chi(\mathbf{\Delta}^{\epsilon}(S))$, where $S \subset \mathbf{N}$ has size $|S| =: s$.

Proposition 3.11. *The constructible Euler characteristic of $\mathbf{\Delta}^{\epsilon}(S)$ is*

$$\chi_c(\mathbf{\Delta}^{\epsilon}(S)) = \sum_{\text{faces } \sigma \text{ of } \mathbb{X}} (-1)^{\dim}(\sigma)\chi_c(B_{\sigma})^s = \sum_{\text{faces } \sigma \text{ of } \mathbb{X}} \chi_c(\sigma)\chi_c(B_{\sigma})^s,$$

where the sum is taken over all open cubes σ of the cubulation of $\mathbb{X}$, and B_{σ} is the intersection of small enough ball centered at a point of σ with $\mathbb{X}$.

Corollary 3.12. *The Euler characteristic of $\mathbf{\Delta}^{\epsilon}(S)$ is*

$$\chi(\mathbf{\Delta}^{\epsilon}(S)) = \sum_{\text{strata } \mathbb{X}_{\beta} \text{ of } \mathbb{X}} (-1)^{\dim}(\mathbb{X}_{\beta})\mathbf{1}^{\circ}(\mathbb{X}_{\beta})^s, \tag{3.11}$$

where the sum is taken over all strata of $\mathbb{X}$.

Proof. Follows from Proposition 3.11 and the immediate fact that along a stratum, the constructible Euler characteristic of the intersection of a small ball with the subanalytic space $\mathbb{X}$ does not depend on the simplex of a triangulation compatible with the stratification. □

3.2.7 Final Strokes

Substituting (3.11) into (3.10) we obtain

$$\sum_{\mathbf{m}} \frac{z^{\mathbf{m}}}{\mathbf{m}!}\kappa(\mathbf{m}) = \sum_{\mathbf{m}} c_{\mathcal{I}}(\mathbf{m})\frac{z^{\mathbf{m}}}{\mathbf{m}!}\left(\sum_{\text{strata } \mathbb{X}_{\beta}} (-1)^{\dim}(\mathbb{X}_{\beta})\mathbf{1}^{\circ}(\mathbb{X}_{\beta})^{|\mathbf{m}|}\right), \tag{3.12}$$

which after swapping the order of summations becomes

$$\sum_{\text{strata } \mathbb{X}_\beta} (-1)^{\dim}(\mathbb{X}_\beta) \left(\sum_{\mathbf{m}} c_{\mathcal{I}}(\mathbf{m}) \frac{(\mathbf{1}^\circ(\mathbb{X}_\beta) z)^{\mathbf{m}}}{\mathbf{m}!} \right). \tag{3.13}$$

Proposition 3.13. *Under the assumptions on the ideal $\mathcal{I}$,*

$$\sum_{\mathbf{m}>\mathbf{0}} c_{\mathcal{I}}(\mathbf{m}) \frac{z^{\mathbf{m}}}{\mathbf{m}!} = \log \left(1 + \sum_{\mathbf{n}\in\mathcal{I}} \frac{z^{\mathbf{n}}}{\mathbf{n}!} \right). \tag{3.14}$$

Finishing the proof of the Main Theorem: Plugging (3.14) into (3.10), we obtain Corollary 1.10. □

Acknowledgements

I am thankful to M. Farber who told me about S. Gal's result.

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