The minimal excludant and colored partitions

Cristina Ballantine*¹ and Mircea Merca†²

¹Department of Mathematics, College of The Holy Cross, Worcester, MA, USA
²Department of Mathematics, University of Craiova, Craiova, Romania and Academy of Romanian Scientists, Bucharest, Romania

Abstract. The minimal excludant of a partition \( \lambda \), \( \text{mex}(\lambda) \), is the smallest positive integer that is not a part of \( \lambda \). For a positive integer \( n \), \( \sigma \text{mex}(n) \) denotes the sum of the minimal excludants of all partitions of \( n \). Recently, Andrews and Newman obtained a new combinatorial interpretation for \( \sigma \text{mex}(n) \). They showed, using generating functions, that \( \sigma \text{mex}(n) \) equals the number of partitions of \( n \) into distinct parts using two colors. We give a purely combinatorial proof of this result and derive its generalization to the sum of least \( r \)-gaps. We introduce several new identities connecting the function \( \sigma \text{mex}(n) \) to the number of partitions with colored parts satisfying certain congruences.

Keywords: Partitions, minimal excludant, least gap in partitions, colored partitions.

1 Introduction

The minimal excludant or mex-function of a set \( S \) of positive integers is the least positive integer not in \( S \). The history of this notion goes back to at least the 1930s when it was applied to combinatorial game theory [9, 8].

Recently, Andrews and Newman [3] considered the mex-function applied to integer partitions. They defined the minimal excludant of a partition \( \lambda \), \( \text{mex}(\lambda) \), as the smallest positive integer that is not a part of \( \lambda \). Then, for each positive integer \( n \), they defined

\[
\sigma \text{mex}(n) := \sum_{\lambda \in \mathcal{P}(n)} \text{mex}(\lambda),
\]

where \( \mathcal{P}(n) \) is the set of all partitions of \( n \). Elsewhere in the literature, the minimal excludant of a partition \( \lambda \) is referred to as the least gap or smallest gap of \( \lambda \). An exact and asymptotic formula for \( \sigma \text{mex}(n) \), as well as its generating function, is given in [7]. In [5] we studied a generalization of \( \sigma \text{mex}(n) \) and its connection to polygonal numbers.

Let \( \mathcal{D}_2(n) \) be the set of partitions of \( n \) into distinct parts using two colors and let \( \mathcal{D}_2(n) = |\mathcal{D}_2(n)| \). We denote the colors of the parts of partitions in \( \mathcal{D}_2(n) \) by 0 and 1. For example, \( \mathcal{D}_2(4) = \{4_0, 4_1, 3_0 + 1_0, 3_0 + 1_1, 3_1 + 1_0, 3_1 + 1_1, 2_1 + 2_0, 2_1 + 1_1 + 1_0, 2_0 + 1_1 + 1_0 \} \), and thus \( \mathcal{D}_2(4) = 9 \). In [3], the authors give two proofs of the following theorem.

*cballant@holycross.edu.
†mircea.merca@proinfo.edu.ro.
Theorem 1.1. Given an integer \( n \geq 0 \), we have \( \sigma \text{mex}(n) = D_2(n) \).

In Section 2, we provide a bijective proof of Theorem 1.1. We make use of the fact that

\[
\sigma \text{mex}(n) = \sum_{j \geq 0} p(n - j(j + 1)/2),
\]

where, as usual, \( p(n) \) denotes the number of partitions of \( n \). A combinatorial proof of (1.1) is given in [5, Theorem 1.1]. The same argument is also described in the second proof of [3, Theorem 1.1]. In fact, the result proven in [5] is a generalization of (1.1) to \( \sigma_r \text{mex}(n) \), the sum of \( r \)-gaps in all partitions of \( n \). The \( r \)-gap of a partition \( \lambda \) is the least positive integer that does not appear \( r \) times as a part of \( \lambda \). In Section 3, we give two generalizations of Theorem 1.1 to \( \sigma_r \text{mex}(n) \).

In [1], the authors considered a restricted mex function. They defined \( M_k(n) \) to be the number of partitions \( \lambda \) of \( n \) with \( \text{mex}(\lambda) = k \) and more parts \( > k \) than parts \( < k \). When \( k = 1 \), \( M_1(n) \) is the number of partitions of \( n \) with smallest part greater than 1. Thus, if \( n > 0 \), we have \( M_1(n) = p(n) - p(n-1) \), and from (1.1), we obtain

\[
\sigma \text{mex}(n) - \sigma \text{mex}(n-1) - \delta(n) = \sum_{j=0}^{\infty} M_1(n - j(j + 1)/2),
\]

where \( \delta \) is the characteristic function of the set of triangular numbers.

We generalize (1.2) in Section 4 where we give further connections between \( \sigma \text{mex}(n) \) and restricted mex functions or partitions and overpartitions. In Section 5 we present connections with partitions with colored odd parts.

## 2 Combinatorial Proof of Theorem 1.1

To prove the theorem, we adapt Sylvester’s bijective proof of Jacobi’s triple product identity [10]. Given \( \lambda \in D_2(n) \), let \( \lambda^{(i)} \), \( i = 0, 1 \), be the (uncolored) partition whose parts are the parts of color \( i \) in \( \lambda \). Then, \( \lambda^{(1)} \) and \( \lambda^{(2)} \) are partitions with distinct parts.

Example 2.1. If \( \lambda = 4_1 + 3_0 + 3_1 + 2_0 + 1_0 \in D_2(13) \), then \( \lambda^{(0)} = 3 + 2 + 1 \) and \( \lambda^{(1)} = 4 + 3 \).

Denote by \( \eta(j) \) the staircase partition \( \eta(j) = j + (j - 1) + \cdots + 2 + 1 \), with \( \eta(0) = \emptyset \). We write \( \ell(\lambda) \) for the number of parts in partition \( \lambda \). The conjugate of a Ferrers diagram \( \nu \) (not necessarily the diagram of a partition) is obtained by reflecting \( \nu \) across the main diagonal. The sum, \( \alpha + \beta \), of two composition \( \alpha = (a_1,a_2,\ldots) \) and \( \beta = (b_1,b_2,\ldots) \), is the composition whose parts are \( a_i + b_i \) (appropriately using 0 as parts at the end of the shorter composition).
Definition 2.2. Given a diagram of left justified rows of boxes (not necessarily the Ferrers diagram of a partition), the *staircase profile* of the diagram is a zig-zag line starting in the upper left corner of the diagram with a right step and continuing in alternating down and right steps until the end of a row of the diagram is reached.

Example 2.3. Let $\alpha$ be the composition $\alpha = (1, 2, 3, 7, 7, 6, 6, 4, 2)$.

![Figure 1: Staircase profile for $\alpha$ and the conjugate of $\alpha$.](image)

The *shifted Ferrers diagram* of a partition $\lambda$ with distinct parts is the Ferrers diagram (with boxes of unit length) of $\lambda$ with row $i$ shifted $i - 1$ units to the right.

We create a map $\varphi : \bigcup_{j \geq 0} \mathcal{P}(n - j(j + 1)/2) \to \mathcal{D}_2(n)$ as follows. Start with $\lambda \in \mathcal{P}(n - j(j + 1)/2)$ for some $j \geq 0$. Append a diagram with rows of lengths $1, 2, \ldots, j$ (i.e., the diagram of $\eta(j)$ rotated by $90^\circ$ counterclockwise) at the top of the diagram of $\lambda$. We obtain a diagram with $n$ boxes. Draw the staircase profile of the new diagram. Let $\alpha$ be the partition whose parts are the length of the columns to the left of the staircase profile and $\beta$ be the partition whose parts are the length of the rows to the right of the staircase profile. Then $\alpha$ and $\beta$ are partitions with distinct parts. Moreover, $j \leq \ell(\alpha) - \ell(\beta) \leq j + 1$. Color the parts of $\alpha$ with color $j \pmod{2}$ and the parts of $\beta$ with color $(j + 1) \pmod{2}$. Then $\varphi(\lambda)$ is defined as the 2-color partition of $n$ whose parts are the colored parts of $\alpha$ and $\beta$.

Conversely, start with $\mu \in \mathcal{D}_2(n)$. Let $\ell_i(\mu), i = 0, 1$, be the number of parts of color $i$ in $\mu$ and set $r = \ell_0(\mu) - \ell_1(\mu)$. Let

$$\varepsilon = \begin{cases} 0 & \text{if } r \geq 0 \\ 1 & \text{if } r < 0, \end{cases} \quad \text{and} \quad j = |r| + \frac{(-1)^{|r|+\varepsilon} - 1}{2}.$$  

Remove the top $j$ rows (i.e., the rotated diagram of $\eta(j)$) from the conjugate of the shifted diagram of $\mu^{(\varepsilon)}$ to obtain a composition $\gamma$. Define $\varphi^{-1}(\mu) = \gamma + \mu^{(s)}$ where $s \neq \varepsilon$. Then, $\varphi^{-1}(\mu) \in \mathcal{P}(n - j(j + 1)/2)$. 
Example 2.4. Let \( n = 38, j = 3 \), and let \( \lambda = 7 + 7 + 6 + 6 + 4 + 2 \) be a partition of \( n - j(j + 1)/2 = 32 \). We add the rotated diagram of \( \eta(3) \) to the top of the diagram of \( \lambda \) and draw the staircase profile (see Figure 1). Then \( \alpha = 9 + 8 + 6 + 5 + 3 + 2 \) and \( \beta = 3 + 2 \). Since \( j \) is odd, we have \( \varphi(\lambda) = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0 \).

Conversely, suppose \( \mu = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0 \in \mathcal{D}(38) \). Then \( \ell_0(\mu) = 2 \) and \( \ell_1(\mu) = 6 \). We have \( r = \ell_0(\mu) - \ell_1(\mu) = -4 \) and \( j = 3 \). We remove the first 3 rows from the conjugate of the shifted diagram of \( \mu^{(1)} \) (which is precisely the diagram below the staircase profile in Figure 1) and add the resulting composition \( \gamma \) to \( \mu^{(0)} = (3, 2) \). We obtain \( \varphi^{-1}(\mu) = 7 + 7 + 6 + 6 + 4 + 2 \in \mathcal{P}(32) \).

3 Generalizations of Theorem 1.1 to r-gaps

Recall that the \( r \)-gap of a partition \( \lambda \) is the least positive integer that does not appear \( r \) times as a part of \( \lambda \). In [5], we proved combinatorially that

\[
\sigma_r \text{ mex}(n) = \sum_{j \geq 0} p(n - rj(j + 1)/2). \tag{3.1}
\]

We can employ a transformation similar to that in the combinatorial proof of Theorem 1.1 to prove its generalization to sums of \( r \)-gaps.

Let \( \tilde{D}_3^{(r)}(n) \) be the number of partitions \( \lambda \) of \( n \) into distinct parts using three colors, 0, 1, and 2, such that:

(i) The set of parts of color 2 is either empty or \( \{t(r-1) \mid 1 \leq t \leq j\} \) for some \( j \geq 1 \).

(ii) \( \ell_{j(\text{mod} \ 2)}(\lambda) - \ell_{j+1(\text{mod} \ 2)}(\lambda) \in \{j,j+1\} \), where \( j = 0 \) if \( \lambda^{(2)} = \emptyset \).

Theorem 3.1. Let \( n, r \) be integers with \( r > 0 \) and \( n \geq 0 \). Then \( \sigma_r \text{ mex}(n) = \tilde{D}_3^{(r)}(n) \).

Proof. For a sketch of the proof see [6]. \( \square \)

In [5] we give the generating function for \( \sigma_r \text{ mex}(n) \), namely

\[
\sum_{n=0}^{\infty} \sigma_r \text{ mex}(n)q^n = \frac{(q^{2r};q^{2r})\infty{(q^r;q^{2r})\infty}}{(q;q)\infty{(q^r;q^r)\infty}}, \tag{3.2}
\]

where \( (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \) if \( n > 0 \), \( (a;q)_n = 1 \) if \( n = 0 \), and \( (a;q)\infty = \lim_{n \to \infty} (a;q)_n \).

Denote \( \tilde{D}_2^{(r)}(n) \) the number of partitions \( \lambda \) of \( n \) using two colors, 0 and 1, such that:

(i) \( \lambda^{(0)} \) is a partition into distinct parts divisible by \( r \).

(ii) \( \lambda^{(1)} \) is a partition with parts repeated at most \( 2r - 1 \) times.

The following generalization of Theorem 1.1 is immediate from (3.2).

Theorem 3.2. Let \( n, r \) be integers with \( r > 0 \) and \( n \geq 0 \). Then \( \sigma_r \text{ mex}(n) = \tilde{D}_2^{(r)}(n) \).
4 Identities involving restricted mex-functions

In this section we introduce identities relating $\sigma \text{mex}(n)$ and restricted mex functions for partitions and overpartitions.

4.1 $\sigma \text{mex}(n)$ and $M_k(n)$

We have the following generalization of (1.2).

**Theorem 4.1.** Let $k, n$ be integers with $k \geq 1$ and $n \geq 0$. Then,

$$(-1)^{k-1} \left( \sum_{j=-(k-1)}^{k} (-1)^j \sigma \text{mex}(n - j(3j - 1)/2) - \delta(n) \right) = \sum_{j=0}^{\infty} M_k(n - j(j + 1)/2).$$

The following infinite family of linear inequalities involving $\sigma \text{mex}$ is immediate.

**Corollary 4.2.** Let $k$ be a positive integer. Given an integer $n \geq 0$, we have

$$(-1)^{k-1} \left( \sum_{j=-(k-1)}^{k} (-1)^j \sigma \text{mex}(n - j(3j - 1)/2) - \delta(n) \right) \geq 0,$$

with strict inequality if $n \geq k(3k + 1)/2$.

**Analytic proof of Theorem 4.1.** In [1], the authors gave the following truncated Euler’s pentagonal number theorem.

$$(-1)^{k-1} \sum_{n=-(k-1)}^{k} (-1)^j q^n (3j - 1)/2 = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q(j)^2 + (k+1)n}{(q; q)_n} \left[ \frac{n-1}{k-1} \right], \quad (4.1)$$

where

$$\left\lfloor \frac{n}{k} \right\rfloor = \begin{cases} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Multiplying both sides of (4.1) by

$$\frac{(q^2, q^2)_\infty}{(q, q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and using (3.2) with $r = 1$ and the generating function for $M_k(n)$ [1],

$$\sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q(j)^2 + (k+1)n}{(q; q)_n} \left[ \frac{n-1}{k-1} \right],$$
we obtain
\[
(-1)^{k-1} \left( \sum_{n=0}^{\infty} \sigma \text{mex}(n)q^n \right) \left( \sum_{n=-(k-1)}^{k} (-1)^{j}q^{n(3n-1)/2} \right) - \sum_{n=0}^{\infty} q^{n(n+1)/2} \right)
\]
\[
\left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} M_k(n)q^n \right).
\]

The proof follows easily using Cauchy’s multiplication of two power series.

Combinatorial proof of Theorem 4.1. The statement of Theorem 4.1 is equivalent to identity (1.2) together with
\[
\sigma \text{mex} \left( n - \frac{k(3k+1)}{2} \right) - \sigma \text{mex} \left( n - \frac{k(3k+5)}{2} - 1 \right)
\]
\[
= \sum_{j=0}^{\infty} \left( M_k(n-j(j+1)/2) + M_{k+1}(n-j(j+1)/2) \right).
\]

Using (1.1), identity (4.2) becomes
\[
\sum_{j=0}^{\infty} \left( p \left( n - \frac{j(j+1)}{2} - \frac{k(3k+1)}{2} \right) - p \left( n - \frac{j(j+1)}{2} - \frac{k(3k+5)}{2} - 1 \right) \right)
\]
\[
= \sum_{j=0}^{\infty} \left( M_k(n-j(j+1)/2) + M_{k+1}(n-j(j+1)/2) \right).
\]

Identity (4.3) was proved combinatorially in [11]. Together with the combinatorial proof of (1.1), this gives a combinatorial proof of Theorem 4.1.

Next, we give a combinatorial interpretation for \( \sum_{t=0}^{\infty} M_k(n-t(t+1)/2) \). For integers \( k, n \) such that \( k \geq 1 \) and \( n \geq 0 \), we denote by \( D_3^{(k)}(n) \) the number of partitions \( \mu \) of \( n \) into distinct parts using three colors and satisfying the following conditions:

(i) \( \mu \) has exactly \( k \) parts of color 2 and, if \( k > 1 \), twice the smallest part of color 2 is greater than largest part of color 2.

(ii) With \( r \) and \( j \) as in the combinatorial proof of Theorem 1.1, the largest part of color \( j \mod 2 \) must equal \( j \) more that the smallest part of color 2.

Proposition 4.3. For integers \( k, n \) such that \( k \geq 1 \) and \( n \geq 0 \), we have
\[
\sum_{t=0}^{\infty} M_k(n-t(t+1)/2) = D_3^{(k)}(n).
\]
The minimal excludant and colored partitions

Proof. See [6]. □

Combining Theorems 1.1 and 4.1, and Proposition 4.3 we obtain the following corollary which, by the discussion above, has both analytic and combinatorial proofs.

**Corollary 4.4.** For integers $k, n$ such that $k \geq 1$ and $n \geq 0$, we have

$$(-1)^{\max(0,k-1)} \left( \sum_{j=-\max(0,k-1)}^{k} (-1)^j \sigma \text{mex}(n-j(3j-1)/2) - \delta(n) \right) = D_3^{(k)}(n).$$

Note that, if $k = 0$, the statement of the corollary reduces to Theorem 1.1.

### 4.2 $\sigma \text{mex}(n)$ and overpartitions

Overpartitions are ordinary partitions with the added condition that the first appearance of any part may be overlined. There are eight overpartitions of 3:

$$3, \overline{3}, 2 + 1, \overline{2} + 1, 2 + \overline{1}, \overline{2} + \overline{1}, 1 + 1 + 1, \overline{1} + 1 + 1.$$

As usual, we denote by $p(n)$ the number of overpartitions of $n$. The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}.$$  

We have the following identity relating $\sigma \text{mex}(n), \overline{p}(n)$ and $M_k(n)$.

**Theorem 4.5.** Let $k$ be a positive integer. Given an integer $n \geq 0$, we have

$$(-1)^{k-1} \left( \sum_{j=-(k-1)}^{k} (-1)^j \overline{p}(n-j(3j-1)) - \sigma \text{mex}(n) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} M_k(j) \sigma \text{mex}(n-2j).$$

**Proof.** By (4.1), with $q$ replaced by $q^2$, we obtain

$$\frac{(-1)^{k-1}}{(q^2; q^2)_{\infty}} \left( \sum_{n=-(k-1)}^{k} (-1)^j q^{n(3j-1)} - 1 \right) = \sum_{n=k}^{\infty} M_k(n)q^{2n}. \quad (4.5)$$

Multiplying both sides of (4.5) by the generating function for $\sigma \text{mex}(n)$, we obtain

$$(-1)^{k-1} \left( \sum_{n=0}^{\infty} \overline{p}(n)q^n \right) \left( \sum_{n=-(k-1)}^{k} (-1)^j q^{n(3j-1)} \right) - \sum_{n=0}^{\infty} \sigma \text{mex}(n)q^n
= \left( \sum_{n=0}^{\infty} \sigma \text{mex}(n)q^n \right) \left( \sum_{n=0}^{\infty} M_k(n)q^{2n} \right).$$

The proof follows by equating the coefficients of $q^n$ in this identity. □
The limiting case \( k \to \infty \) of Theorem 4.5 reads as follows.

**Corollary 4.6.** For \( n \geq 0 \), \( \sigma \text{mex}(n) = \sum_{j=-\infty}^{\infty} (-1)^j \overline{p}(n - j(3j - 1)) \).

**Remark 4.7.** Since it is known that \( \overline{p}(n) \) is odd if and only if \( n = 0 \), it follows that \( \sigma \text{mex}(n) \) is odd if and only if \( 12n + 1 \) is a square.

In [2], the authors denoted by \( M_k(n) \) the number of overpartitions of \( n \) in which the first part larger than \( k \) appears at least \( k + 1 \) times. We have the following identity.

**Theorem 4.8.** For integers \( k, n > 0 \), we have

\[
(-1)^k \left( \sigma \text{mex}(n) + 2 \sum_{j=1}^{k} (-1)^j \sigma \text{mex}(n - j^2) - \delta'(n) \right) = \sum_{j=-\infty}^{\infty} (-1)^j M_k(n - j(3j - 1)),
\]

where \( \delta'(n) = (-1)^m \) if \( n = m(3m - 1), m \in \mathbb{Z} \) and \( \delta'(n) = 0 \) otherwise.

**Proof.** The proof, given in [6], follows from a truncated theta series identity [2]. \qed

There is a substantial amount of numerical evidence to conjecture the following inequality.

**Conjecture 4.9.** For \( k, n > 0 \),

\[
\sum_{j=-\infty}^{\infty} (-1)^j M_k(n - j(3j - 1)) \geq 0,
\]

with strict inequality if \( n \geq (k + 1)^2 \).

A combinatorial interpretation for the sum in this conjecture would be interesting.

### 4.3 \( \sigma \text{mex}(n) \) and partitions into distinct parts

To keep notation uniform, let \( D_1(n) \) be the number of partitions of \( n \) into distinct parts. Set \( D_1(x) = 0 \) if \( x \) is not a positive integer. For proof of the next theorem see [6].

**Theorem 4.10.** For any integer \( n \geq 0 \), we have

\[
\sum_{j=0}^{\infty} (-1)^{(j+1)/2} \sigma \text{mex}(n - j(j + 1)/2) = \sum_{j=0}^{\infty} D_1 \left( \frac{n - j(j + 1)/2}{2} \right).
\]

Let \( D_2^+(n) \) be the number of partitions of \( n \) with distinct parts using two colors such that: (i) parts of color 0 form a gap-free partition (staircase) and (ii) only even parts can have color 1. Then, we have the following identity of Watson type [4] which gives a combinatorial interpretation for the right hand side of (4.6). For its proof see [6].
Proposition 4.11. For $n \geq 0$,
\[ \sum_{j=0}^{\infty} D_1 \left( \frac{n - j(j + 1)/2}{2} \right) = D_2^*(n). \]

In [2], the authors denoted by $MP_k(n)$ the number of partitions of $n$ in which the first part larger than $2k - 1$ is odd and appears exactly $k$ times. All other odd parts appear at most once. We have the following truncated form of Theorem 4.10.

Theorem 4.12. For integers $n, k > 0$,
\[ (-1)^{k-1} \left( \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} \sigma \text{mex}(n - j(j + 1)/2) - D_2^*(n) \right) = \sum_{j=0}^{n} MP_k(j) D_2^*(n - j). \]

Proof. The proof, given in [6], follows from the truncated theta series identity of [2].

A combinatorial interpretation for $\sum_{j=0}^{n} MP_k(j) D_2^*(n - j)$ would be very welcome.

The following corollary of Theorem 4.12 is immediate.

Corollary 4.13. For integers $n, k > 0$,
\[ (-1)^{k-1} \left( \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} \sigma \text{mex}(n - j(j + 1)/2) - D_2^*(n) \right) \geq 0, \]
with strict inequality if $n \geq k(2k + 1)$.

A second corollary involves the function $\text{pod}(n)$, the number of partitions of $n$ in which odd parts are not repeated, i.e.,

Corollary 4.14. For $n \geq 0$, $\sigma \text{mex}(n) = \sum_{j=0}^{n} \text{pod}(j) D_2^*(n - j)$.

5 $\sigma \text{mex}(n)$ and partitions with colored odd parts

In this section we present several identities relating $\sigma \text{mex}(n)$ with the number of partitions of $n$ in which odd parts are colored in with $j$ colors, $j = 2, 3, 4$. Elsewhere in the literature, colored partitions are referred to as vector partitions. Due to space restrictions, we will present the proofs of all theorems in this section in a future article.
5.1 Three colors for the odd parts

Let \( C_3(n) \) be the number of partitions of \( n \) using 3 colors for the odd parts and let \( C'_3(n) \) be the number of partitions of \( n \) into parts not congruent to 2 mod 4 using 3 colors for the odd parts. The generating functions for \( C_3(n) \) and \( C'_3(n) \) are respectively

\[
\sum_{n=0}^{\infty} C_3(n)q^n = \frac{1}{(q^2;q^2)_\infty(q;q^2)_\infty^3} \quad \text{and} \quad \sum_{n=0}^{\infty} C'_3(n)q^n = \frac{1}{(q^4;q^4)_\infty(q;q^2)_\infty^3}.
\]

Using the truncated Euler's pentagonal number theorem [1], we prove the following identity which relates \( C_3(n) \) and the function \( M_k(n) \) defined in Section 1.

**Theorem 5.1.** Let \( k \) be a positive integer. Given an integer \( n \geq 0 \), we have

\[
(-1)^{k-1} \left( \sum_{j=-k}^{k} (-1)^j C_3(n - j(3j - 1)/2) - \sigma \text{mex}(n) \right) = \sum_{j=0}^{n} \sigma \text{mex}(j) M_k(n - j).
\]

A combinatorial interpretation of \( \sum_{j=0}^{n} \sigma \text{mex}(j) M_k(n - j) \) would be appealing.

The limiting case \( k \to \infty \) of Theorem 5.1 gives the following decomposition of \( \sigma \text{mex}(n) \).

**Corollary 5.2.** For \( n \geq 0 \), we have

\[
\sigma \text{mex}(n) = \sum_{j=-\infty}^{\infty} (-1)^j C_3(n - j(3j - 1)/2).
\]

Using the truncated theta series identity of [2], we prove the following identity which relates \( C'_3(n) \) and the function \( MP_k(n) \) of Section 4.3.

**Theorem 5.3.** Let \( k \) be a positive integer. Given an integer \( n \geq 0 \), we have

\[
(-1)^{k-1} \left( \sum_{j=0}^{2k-1} (-1)^{j(1/2)} C'_3(n - j(j + 1)/2) - \sigma \text{mex}(n) \right) = \sum_{j=0}^{n} \sigma \text{mex}(j) MP_k(n - j).
\]

**Corollary 5.4.** For \( n \geq 0 \), \( \sigma \text{mex}(n) = \sum_{j=0}^{\infty} (-1)^{j(1/2)} C'_3(n - j(j + 1)/2) \).

5.2 Four colors for the odd parts

Let \( C_4(n) \) be the number of partitions of \( n \) using 4 colors for the odd parts. The generating function for \( C_4(n) \) is

\[
\sum_{n=0}^{\infty} C_4(n)q^n = \frac{1}{(q^2;q^2)_\infty(q^2;q^2)_\infty^4}.
\]

Then, \( C_4(n) \) and the function \( M_k(n) \) of Section 4.2 are related by the next theorem and its corollary.
Theorem 5.5. Let $k$ be a positive integer. Given an integer $n \geq 0$, we have

$$(-1)^k \left( C_4(n) + 2 \sum_{j=1}^{k} (-1)^j C_4(n - j^2) - \sigma \text{ mex}(n) \right) = \sum_{j=0}^{n} C_4(j) M_k(n - j).$$

Corollary 5.6. For $n \geq 0$, $\sigma \text{ mex}(n) = C_4(n) + 2 \sum_{j=1}^{\infty} (-1)^j C_4(n - j^2)$.

Note that the partition functions $\sigma \text{ mex}(n)$ and $C_4(n)$ have the same parity.

5.3 Two colors for parts $\not\equiv 0 \mod 4$

Let $C_2(n)$ be the number of partitions of $n$ using two colors for the parts not congruent to 0 mod 4. The generating function for $C_2(n)$ is

$$\sum_{n=0}^{\infty} C_2(n) q^n = \frac{(q^4; q^4)_{\infty}}{(q; q)_2^\infty}.$$

The following identity relating $C_2(n)$ and $M_k(n)$ follows from the truncated theta identity of [2].

Theorem 5.7. Let $k$ be a positive integer. Given an integer $n \geq 0$, we have

$$(-1)^k \left( C_2(n) + 2 \sum_{j=1}^{k} (-1)^j C_2(n - 2j^2) - \sigma \text{ mex}(n) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} M_k(j) \sigma \text{ mex}(n - 2j).$$

Corollary 5.8. For $n \geq 0$, $\sigma \text{ mex}(n) = C_2(n) + 2 \sum_{j=1}^{\infty} (-1)^j C_2(n - 2j^2)$.

We see that the partition functions $\sigma \text{ mex}(n)$ and $C_2(n)$ have the same parity.

5.4 Two colors for the odd parts in partitions into parts $\not\equiv 4 \mod 8$

We denote by $C_2^*(n)$ the number of partitions of $n$ into parts not congruent to 4 mod 8 using two colors for the odd parts. The generating function for $C_2^*(n)$ is given by

$$\sum_{n=0}^{\infty} C_2^*(n) q^n = \frac{1}{(q^2, q^6, q^8; q^8)_{\infty}(q; q^2)_2^\infty}.$$

The proof of following theorem relating $C_2^*(n)$ and $MP_k$ again uses results from [2].
Theorem 5.9. Let $k$ be a positive integer. Given an integer $n \geq 0$, we have

\[ (-1)^{k-1} \left( \sum_{j=0}^{2k-1} (-1)^{j(j+1)/2}C_2^*(n-j(j+1)) - \sigma \text{mex}(n) \right) = \sum_{j=0}^{[n/2]} MP_k(j)\sigma \text{mex}(n-2j). \]

Corollary 5.10. For $n \geq 0$, $\sigma \text{mex}(n) = \sum_{j=0}^{\infty} (-1)^{j(j+1)/2}C_2^*(n-j(j+1))$.

References


