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Some Algebraic Properties of Lecture Hall Polytopes

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Abstract. In this note, we investigate some of the fundamental algebraic and geometric properties of s-lecture hall simplices and their generalizations. We show that all s-lecture hall order polytopes, which simultaneously generalize s-lecture hall simplices and order polytopes, satisfy a property which implies the integer decomposition property. This answers one conjecture of Hibi, Olsen and Tsuchiya. By relating s-lecture hall polytopes to alcoved polytopes, we then use this property to show that families of s-lecture hall simplices admit a quadratic Gröbner basis with a square-free initial ideal. Consequently, we find that all s-lecture hall simplices for which the first order difference sequence of s is a $0, 1$-sequence have a regular and unimodular triangulation. This answers a second conjecture of Hibi, Olsen and Tsuchiya, and it gives a partial answer to a conjecture of Beck, Braun, Köppe, Savage and Zafeirakopoulos.

Keywords: lecture hall polytope, integer decomposition property, regular unimodular triangulation, Gröbner basis, toric ideal

1 Introduction

Let $s = (s_1, \ldots, s_n)$ be a sequence of positive integers. An *s-lecture hall partition* is a (lattice) point in $\mathbb{Z}^n$ living in the *s-lecture hall cone*

$$C_n^s := \left\{ \lambda \in \mathbb{R}^n : 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \right\}.$$

The s-lecture hall partitions are generalizations of the *lecture hall partitions*, which correspond to the special case where $s = (1, 2, \ldots, n)$. Lecture hall partitions were first studied by Bousquet-Mélou and Eriksson [4] who proved that

$$\sum_{\lambda \in C_n^{(1,2,\ldots,n)} \cap \mathbb{Z}^n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{\prod_{i=1}^n 1 - q^{2i-1}}.$$

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In [11], the *s-lecture hall simplex* is defined to be the lattice polytope

$$P_n^s := \{\lambda \in C_n^s : \lambda_n \leq s_n\}.$$

A d-dimensional lattice polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Z}^n$ whose affine span has dimension d. For a positive integer k, we define $kP := \{kp \in \mathbb{R}^n : p \in P\}$. The generating function

$$1 + \sum_{k>0} |kP \cap \mathbb{Z}^n| x^k = \frac{h_0^* + h_1^* x + \cdots + h_d^* x^d}{(1-x)^{d+1}},$$

is called the *Ehrhart series* of P, and the polynomial $h_0^* + h_1^* x + \cdots + h_d^* x^d$ is called the *(Ehrhart) h^*-polynomial* of P. The h^*-polynomial has only nonnegative integer coefficients, and for the s-lecture hall simplex P_n^s it is called the *s-Eulerian polynomial*. In the case that $s = (1, 2, \ldots, n)$, the s-Eulerian polynomial is the classic n^{th} Eulerian polynomial, which enumerates the permutations of $[n]$ via the descent statistic. One remarkable feature of this generalization is that every s-Eulerian polynomial has only real zeros, and thus they each have a log-concave and unimodal sequence of coefficients [12]. Identifying large families of lattice polytopes with unimodal h^*-polynomials is a popular research topic with natural connections to the algebra and geometry of the toric varieties associated to lattice polytopes. Showing that an h^*-polynomial is real-rooted is a common approach to proving unimodality results in geometric and algebraic combinatorics [3, 6, 12, 13]. However, the applicability of this proof technique to families of h^*-polynomials does not obviously relate to the algebraic structure of the associated toric variety for the underlying polytopes. Consequently, research into the algebraic properties of the s-lecture hall simplices and their generalizations that can be used to verify unimodality of the associated h^*-polynomials is an ongoing and popular topic [2, 1, 5, 7, 9, 10, 11, 12].

In this note, we prove some fundamental algebraic properties of s-lecture hall simplices and their generalizations. We show that all *s-lecture hall order polytopes* [5], a common generalization of s-lecture hall simplices and order polytopes, have the integer decomposition property. This result positively answers a conjecture of [7]. As an application of this result, we then give an explicit description of a quadratic and square-free Gröbner basis for the affine toric ideal of families of s-lecture hall simplices. To do so, we relate s-lecture hall polytopes to *alcoved polytopes* [8]. The identified Gröbner basis is purely lexicographic and can be constructed for any toric ideal associated to an s-lecture hall simplex for which the first order difference sequence of s is a $0,1$-sequence. This answers a second conjecture of [7] in a special case that they noted to be of particular interest, and it provides a partial answer to the conjecture of [2].

2 The Integer Decomposition Property for s-Lecture Hall Order Polytopes

2.1 The algebraic structure of a lattice polytope

There are two important algebraic objects associated to a lattice polytope $P \subset \mathbb{R}^n$. The first is its *toric ideal*, the zero locus of which is the *affine toric variety of* P. The second is the *Ehrhart ring of* P, which is a graded and semistandard semigroup algebra associated to P. The integer decomposition property is precisely the property that tells us when the coordinate ring of the affine toric variety of P coincides with its Ehrhart ring. Hence, it is desirable to know if a family of lattice polytopes admits this property.

For a lattice polytope $P \subset \mathbb{R}^n$, define the *cone over* P to be the convex cone

$$\mathrm{cone}(P) := \mathrm{span}_{\mathbb{R}_{\geq 0}}\{(p,1) \in \mathbb{R}^n \times \mathbb{R} : p \in P\} \subset \mathbb{R}^{n+1}.$$

To any integer point $z = (z_1, \ldots, z_{n+1}) \in \mathbb{Z}^{n+1}$ we associate a Laurent monomial $t^z := t_1^{z_1} t_2^{z_2} \cdots t_{n+1}^{z_{n+1}}$. Let $\{a_1, \ldots, a_m\} := \{(p,1) \in \mathbb{R}^n \times \mathbb{R} : p \in P \cap \mathbb{Z}^n\}$, and let $K[\mathbf{x}] := K[x_1, \ldots, x_m]$ denote the polynomial ring over a field K in m indeterminates. The *toric ideal* of P, denoted $\mathcal{I}_P$, is the kernel of the semigroup algebra homomorphism

$$\Phi : K[\mathbf{x}] \longrightarrow K[t_1, t_2, \ldots, t_{n+1}, t_1^{-1}, t_2^{-1}, \ldots, t_{n+1}^{-1}] \qquad \text{where} \qquad \Phi : x_i \mapsto t^{a_i}.$$

For $k \in \mathbb{Z}_{>0}$ we let $kP := \{kp : p \in P\}$ denote the k^{th} dilate of P, and we let $A(P)_k$ denote the vector space (over K) spanned by the monomials $t_1^{z_1} t_2^{z_2} \cdots t_n^{z_n} t_{n+1}^k$ for $z \in kP \cap \mathbb{Z}^n$. Since P is convex we have that $A(P)_k A(P)_r \subset A(P)_{k+r}$ for all $k, r \in \mathbb{Z}_{>0}$. It follows that the graded algebra

$$A(P) := \bigoplus_{k=0}^{\infty} A(P)_k$$

is finitely generated over $A(P)_0 := K$, and we call it the *Ehrhart Ring of* P. Equivalently, $A(P)$ is the semigroup algebra $K[t^z : z \in \mathrm{cone}(P) \cap \mathbb{Z}^{n+1}]$ with the grading $\deg(t_1^{z_1} \cdots t_{n+1}^{z_{n+1}}) = z_{n+1}$. A polytope $P \subset \mathbb{R}^n$ has the *integer decomposition property*, or is IDP (or is *integrally closed*), if for every positive integer k and every $z \in kP \cap \mathbb{Z}^n$, there exist k points $z^{(1)}, z^{(2)}, \ldots, z^{(k)} \in P \cap \mathbb{Z}^n$ such that $z = \sum_i z^{(i)}$. Since the coordinate ring of the toric ideal $\mathcal{I}_P$ is $K[\mathbf{x}]/\mathcal{I}_P \cong K[t^{a_1}, \ldots, t^{a_m}]$, the polytope P is IDP if and only if this quotient ring is isomorphic to $A(P)$. In this case, the toric algebra of $\mathcal{I}_P$ can be used to recover the Ehrhart theoretical data encoded in $A(P)$. Therefore, it is desirable to understand when combinatorially interesting polytopes are IDP.

2.2 s-Lecture hall order polytopes

A *labeled poset* is a partially ordered set $\mathcal{P}$ on $[n] := \{1, 2, \ldots, n\}$ for some positive integer n; that is, $\mathcal{P} = ([n], \preceq)$ where $\preceq$ denotes the partial order imposed on the ground set $[n]$. In the following, we let $\leq$ denote the usual total order on the integers. We say that $\mathcal{P}$ is *naturally labeled* if it is a labeled poset for which $i \leq j$ whenever $i \preceq j$. Let $s = (s_1, \ldots, s_n)$ be a sequence of positive integers and let $\mathcal{P} = ([n], \preceq)$ be a naturally labeled poset. A $(\mathcal{P}, s)$*-partition* is a map $\lambda : [n] \longrightarrow \mathbb{R}$ such that

$$\frac{\lambda_i}{s_i} \leq \frac{\lambda_j}{s_j} \qquad \text{whenever} \qquad i \prec j,$$

where we let λ_i denote $\lambda(i)$ for all $i \in [n]$. The *s-lecture hall order polytope* associated to $(\mathcal{P}, s)$ is the convex polytope

$$O(\mathcal{P}, s) := \{\lambda \in \mathbb{R}^n : \ \lambda \text{ is a } (\mathcal{P}, s)\text{-partition and } 0 \leq \lambda_i \leq s_i \text{ for all } i \in [n]\}.$$

The s-lecture hall order polytopes were introduced in [5] as a common generalization of the well-known order polytopes and the s-lecture hall simplices. When $s = (1, 1, \ldots, 1)$, then $O(\mathcal{P}, s)$ is the order polytope associated to $\mathcal{P}$, and when $\mathcal{P}$ is the n-chain $O(\mathcal{P}, s) = P_n^s$. In [7], it is conjectured that all s-lecture hall simplices are IDP. We now prove a more general (and stronger) statement.

A poset $\mathcal{P} = ([n], \preceq_{\mathcal{P}})$ is called a *lattice* if every pair of elements $a, b \in [n]$ has both a least upper bound and a greatest lower bound in $\mathcal{P}$. An element $c \in [n]$ is a *least upper bound* of a and b in $\mathcal{P}$ if $a \preceq_{\mathcal{P}} c$, $b \preceq_{\mathcal{P}} c$ and whenever $d \in [n]$ satisfies $a \preceq_{\mathcal{P}} d$ and $b \preceq_{\mathcal{P}} d$ then $c \preceq_{\mathcal{P}} d$. Analogously, $c \in [n]$ is a *greatest lower bound* of a and b in $\mathcal{P}$ if $a \succeq_{\mathcal{P}} c$, $b \succeq_{\mathcal{P}} c$ and whenever $d \in [n]$ satisfies $a \succeq_{\mathcal{P}} d$ and $b \succeq_{\mathcal{P}} d$ then $c \succeq_{\mathcal{P}} d$. Whenever a least upper bound or greatest lower bound exists, it is unique. So we let $a \vee b$ denote the least upper bound of a and b in $\mathcal{P}$ and $a \wedge b$ denote their greatest lower bound. A lattice $\mathcal{P}$ is called *distributive* if for all triples of elements a, b, c in $\mathcal{P}$ we have that

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Let $\Lambda(\mathcal{P}, s)$ denote the collection of all maps $\lambda : [n] \longrightarrow \mathbb{Z}$ satisfying

$$\frac{\lambda_i}{s_i} \leq \frac{\lambda_j}{s_j} \qquad \text{whenever} \qquad i \preceq j.$$

In general, we will identify a map $p : [n] \longrightarrow \mathbb{R}$ with the point $(p_1, \ldots, p_n) \in \mathbb{R}^n$. Conversely, every point $p \in \mathbb{R}^n$ corresponds to a map $p : [n] \longrightarrow \mathbb{R}$. Note that $\Lambda(\mathcal{P}, s)$ is a distributive sublattice of $\mathbb{Z}^n$, under the usual product ordering. Moreover, $\Lambda(\mathcal{P}, s) = \Lambda(\mathcal{P}, s) + \mathbb{Z}(s_1, \ldots, s_n)$ and $kO(\mathcal{P}, s) \cap \mathbb{Z}^n = \Lambda(\mathcal{P}, s) \cap \prod_{i \in [n]}[0, ks_i]$ for all $k \in \mathbb{Z}_{>0}$. Let $\lambda, \gamma \in O(\mathcal{P}, s) \cap \mathbb{Z}^n$. We write $\lambda \trianglelefteq \gamma$ provided that

1. $\lambda_i \leq \gamma_i$ for all $i \in [n]$, and
2. if $\lambda_i \neq 0$, then $\gamma_i = s_i$.

Theorem 2.1. *Let $\mathcal{P} = ([n], \preceq)$ be a naturally labeled poset and let $s = (s_1, \ldots, s_n)$ be a sequence of positive integers. If $\lambda \in kO(\mathcal{P}, s) \cap \mathbb{Z}^n$ for $k \in \mathbb{Z}_{>0}$, then there are unique elements $\lambda^{(1)}, \ldots, \lambda^{(k)} \in O(\mathcal{P}, s) \cap \mathbb{Z}^n$ such that*

$$\lambda^{(1)} \trianglelefteq \lambda^{(2)} \trianglelefteq \cdots \trianglelefteq \lambda^{(k)} \quad \textit{and} \quad \lambda = \lambda^{(1)} + \lambda^{(2)} + \cdots + \lambda^{(k)}. \tag{2.1}$$

Moreover, if

$$\lambda = \gamma^{(1)} + \gamma^{(2)} + \cdots + \gamma^{(m)}$$

where $m \leq k$ and $\gamma^{(1)}, \ldots, \gamma^{(k)} \in O(\mathcal{P}, s) \cap \mathbb{Z}^n$, then $\lambda^{(1)}(x) \leq \gamma^{(i)}(x) \leq \lambda^{(k)}(x)$ for all $x \in [n]$ and $i \in [k]$.

Proof. We first prove the existence of (2.1) by induction over $k \geq 1$. Suppose $\lambda \in kO(\mathcal{P}, s) \cap \mathbb{Z}^n$, where $k > 1$, and write $\lambda = \lambda \wedge s + (\lambda - s) \vee 0$. Then $\lambda \wedge s \in O(\mathcal{P}, s) \cap \mathbb{Z}^n$ and $(\lambda - s) \vee 0 \in (k-1)O(\mathcal{P}, s) \cap \mathbb{Z}^n$. Let $\lambda^{(k)} = \lambda \wedge s$. By induction, we may write

$$(\lambda - s) \vee 0 = \lambda^{(1)} + \cdots + \lambda^{(k-1)}$$

where $\lambda^{(1)}, \ldots, \lambda^{(k-1)}$ satisfies (2.1). Clearly $\lambda^{(i)} \leq \lambda^{(k)}$ for all $1 \leq i \leq k-1$. Moreover, if $\lambda^{(i)}(x) \neq 0$ for some $1 \leq i \leq k-1$, then $((\lambda - s) \vee 0)(x) \neq 0$. Thus, $\lambda^{(k)}(x) = (\lambda \wedge s)(x) = s(x)$ as desired. This establishes (2.1).

Suppose now that the sequence $\lambda^{(1)}, \ldots, \lambda^{(k)}$ satisfies (2.1). Note $\lambda(x) > s(x)$ if and only if $\lambda^{(i)}(x) > 0$ for at least two distinct i, and this happens if and only if $\lambda^{(k-1)}(x) > 0$ and $\lambda^{(k)}(x) = s(x)$. Hence, $\lambda^{(k)} = \lambda \wedge s$. The uniqueness of $\lambda^{(1)}, \ldots, \lambda^{(k)}$ then follows by induction.

Suppose next that

$$\lambda = \gamma^{(1)} + \gamma^{(2)} + \cdots + \gamma^{(m)} \in kO(\mathcal{P}, s) \cap \mathbb{Z}^n,$$

where $m \leq k$ and $\gamma^{(1)}, \ldots, \gamma^{(m)} \in O(\mathcal{P}, s) \cap \mathbb{Z}^n$. Then $\gamma^{(i)}(x) \leq \min\{\lambda(x), s(x)\} = \lambda^{(k)}(x)$. If $\gamma^{(i)}(x) < \lambda^{(1)}(x)$ for some $x \in [n]$ and $1 \leq i \leq m$, then $\lambda^{(i)}(x) = s(x)$ for all $2 \leq i \leq k$ (since $\lambda^{(1)}(x) \neq 0$). Hence, $\lambda^{(1)}(x) = \lambda(x) - (k-1)s(x) > 0$ and

$$\lambda(x) - \gamma^{(i)}(x) = \lambda^{(1)}(x) - \gamma^{(i)}(x) + (k-1)s(x) > (k-1)s(x),$$

which is a contradiction since $\lambda - \gamma^{(i)} \in (m-1)O(\mathcal{P}, s) \cap \mathbb{Z}^n$. $\square$

It follows from Theorem 2.1 that all s-lecture hall order polytopes are IDP. In the remainder of this note, we use Theorem 2.1 to identify a regular and unimodular triangulation of some s-lecture hall polytopes by computing a quadratic and square-free Gröbner basis for their associated toric ideals.

3 A Quadratic and Square-Free Gröbner Basis for Some s-Lecture Hall Simplices

Let $P \subset \mathbb{R}^n$ be a lattice polytope and let $\mathcal{A} := \{(p,1) \in \mathbb{R}^n \times \mathbb{R} : p \in P \cap \mathbb{Z}^n\}$. Label $\mathcal{A}$ as $\mathcal{A} = \{a_1, \ldots, a_m\}$, and suppose that $\succ$ is a *term order* on the polynomial ring $K[\mathbf{x}] := K[x_1, \ldots, x_m]$; that is, $\succ$ is a total order on the monomials in $K[\mathbf{x}]$ satisfying

1. $x^a \succ x^b$ implies that $x^a x^c \succ x^b x^c$ for all $c \in \mathbb{Z}^n_{\geq 0}$, and
2. $x^a \succ x^0 = 1$ for all $a \in \mathbb{Z}^n_{>0}$.

Given a polynomial $f = \sum_{a \in \mathbb{Z}^n_{\geq 0}} c_a x^a$ with coefficients $c_a \in K$ we call the set

$$\mathrm{Supp}(f) := \{a \in \mathbb{Z}^n : c_a \neq 0\}$$

the *support* of f. Fixing a term order $\succ$ on the monomials in $K[\mathbf{x}]$, we define the *initial term* of f to be the term $c_a x^a$ for which $x^a \succ x^b$ for every $b \in \mathrm{Supp}(f) \setminus \{a\}$. We denote the initial term of f with respect to the term order $\succ$ by $\mathrm{in}_\succ(f)$. Given an ideal $\mathcal{I} \subset K[\mathbf{x}]$, the *initial ideal of $\mathcal{I}$ with respect to $\succ$* is

$$\mathrm{in}_\succ(\mathcal{I}) := \langle \mathrm{in}_\succ(f) : f \in \mathcal{I} \rangle.$$

A finite set of polynomials $\mathcal{G}_\succ(\mathcal{I}) := \{g_1, \ldots, g_p\}$ is called a *Gröbner basis* of $\mathcal{I}$ with respect to $\succ$ if $\mathrm{in}_\succ(\mathcal{I}) = \langle \mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_p) \rangle$. If $\{\mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_p)\}$ is the unique minimal generating set for $\mathrm{in}_\succ(\mathcal{I})$, then $\mathcal{G}_\succ(\mathcal{I})$ is called *minimal*. A minimal Gröbner basis $\mathcal{G}_\succ(\mathcal{I})$ if further called *reduced* if no non-inital term of any g_i is divisible by some element of $\{\mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_p)\}$. The monomials of $K[\mathbf{x}]$ that are not in $\mathrm{in}_\succ(\mathcal{I})$ are called the *standard monomials* of $\mathrm{in}_\succ(\mathcal{I})$.

Let $P \subset \mathbb{R}^n$ be a lattice polytope and let $\mathcal{A} := \{(p,1) \in \mathbb{R}^n \times \mathbb{R} : p \in P \cap \mathbb{Z}^n\}$. We denote the sublattice of $\mathbb{Z}^{n+1}$ spanned by the lattice points in $\mathcal{A}$ by $\mathbb{Z}\mathcal{A}$. Any sufficiently generic height function $\omega : \mathcal{A} \longrightarrow \mathbb{R}_{\geq 0}$ on the points in $\mathcal{A}$ induces a term order $\succ_\omega$ on $K[\mathbf{x}]$ and yields a corresponding Gröbner basis $\mathcal{G}_{\succ_\omega}(\mathcal{I}_P)$ for the toric ideal $\mathcal{I}_P$ of P. On the other hand, the collection of faces of

$$\mathrm{conv}\{(a_i, \omega(a_i)) \in \mathbb{R}^{n+1} : i \in [m]\}$$

that minimize some linear functional in $\mathbb{R}^{n+1}$ with a negative $(n+1)^{st}$ coordinate correspond to the faces of a *regular triangulation* Δ_ω of P given by projecting these faces onto P in $\mathbb{R}^n$. The fundamental correspondence between the regular triangulation Δ_ω and the Gröbner basis $\mathcal{G}_{\succ_\omega}(\mathcal{I}_P)$ states that the square-free standard monomials $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ with respect to $\mathrm{in}_{\succ_\omega}(\mathcal{I}_P)$ correspond to the faces $\mathrm{conv}\{a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}\}$ of Δ_ω [14, Theorem 8.3]. Furthermore, if the $\mathrm{in}_{\succ_\omega}(\mathcal{I}_P)$ is *square-free* (i.e. generated by square-free monomials), then the simplices of Δ_ω have smallest possible volume (i.e. are *unimodular*) with respect to the lattice $\mathbb{Z}\mathcal{A}$. In this case, the regular triangulation Δ_ω is called *unimodular*. If $\mathrm{in}_{\succ_\omega}(\mathcal{I}_P)$ consists only of quadratic monomials, then Δ_ω is *flag*, meaning its minimal non-faces are pairs of points $\{a_i, a_j\}$. When an n-dimensional lattice polytope P is IDP, then $\mathbb{Z}\mathcal{A} = \mathbb{Z}^{n+1}$, and a square-free Gröbner basis for $\mathcal{I}_P$ identifies a regular unimodular triangulation of P with respect to the lattice $\mathbb{Z}^n$.

Our goal in this section is to identify a quadratic Gröbner basis with a square-free initial ideal for the toric ideals of a subcollection of s-lecture hall simplices that includes the lecture hall simplex $P_n^{(1,2,\ldots,n)}$. This is the first explicit description of such a Gröbner basis for the toric ideal of $P_n^{(1,2,\ldots,n)}$. In the remainder of this section, we will assume that $s = (s_1, \ldots, s_n)$ is a weakly increasing sequence of positive integers satisfying $0 \leq s_{i+1} - s_i \leq 1$ for all $i \in [n-1]$; that is, we will assume that the first order difference sequence of s is a $0,1$-sequence.

3.1 s-lecture hall simplices and alcoved polytopes

To produce the desired quadratic and square-free Gröbner basis for the toric ideal of P_n^s we will use the following transformation. For the sequence $s = (s_1, \ldots, s_n)$, set $s_{n+1} := s_n + 1$. Notice that since s is assumed to be weakly increasing then $x_i \leq x_{i+1}$ for all $i \in [n]$ and $x \in P_n^s$. Now consider the unimodular transformation

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \qquad \varphi : x_i \mapsto x_i - x_{i-1}, \quad \text{where } x_0 := 0,$$

and the homogenizing affine transformation

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}; \qquad h : x \mapsto \left(x_1, \ldots, x_n, s_{n+1} - \sum_{i=1}^{n} x_i \right).$$

Then the convex lattice polytope $A_n^s := (h \circ \varphi)(P_n^s)$ is defined by the linear inequalities

$$\begin{gathered} 0 \leq x_1 + \cdots + x_i \leq s_i, \text{ for all } i \in [n], \\ 0 \leq (s_{i+1} - s_i)(x_1 + \cdots + x_i) \leq s_i x_{i+1}, \text{ for all } i \in [n-1], \text{ and} \\ x_1 + \cdots + x_{n+1} = s_{n+1}. \end{gathered}$$

The following lemma notes that the lattice points within A_n^s consist of the lattice points in the *alcoved polytope* [8] defined by the inequalities

$$\begin{aligned} &0 \leq x_1 + \cdots + x_i \leq s_i, \text{ for all } i \in [n], \text{ and} \\ &x_1 + \cdots + x_{n+1} = s_{n+1} \end{aligned}$$

that satisfy a useful combinatorial criterion. Conditions (1) and (2) of the lemma specify that a lattice point in A_n^s must lie in this alcoved polytope, and conditions (3) and (4) constitute the combinatorial criterion we desire.

Lemma 3.1. *Suppose that s is a weakly increasing sequence of positive integers for which the first order difference sequence is a $0,1$-sequence. Then a lattice point $(z_1,\ldots,z_{n+1})$ is in $A_n^s \cap \mathbb{Z}^{n+1}$ if and only if the following conditions hold:*

1. $z_1+\cdots+z_{n+1}=s_{n+1}$,
2. $0 \leq z_1+\cdots+z_i \leq s_i$ *for all* $i \in [n+1]$,
3. *whenever* $s_{i+1}-s_i=0$, *then* $0 \leq z_{i+1}$, *and*
4. *whenever* $s_{i+1}-s_i \neq 0$ *and* $z_k \neq 0$ *for some* $k < i+1$, *then* $z_{i+1} \neq 0$.

Proof. Suppose first that $(z_1,\ldots,z_{n+1}) \in A_n^s \cap \mathbb{Z}^{n+1}$. Then certainly conditions (1) and (2) hold. To see that condition (3) holds, suppose that $s_{i+1}-s_i=0$. Then by the defining inequalities for A_n^s, we know that $0 \leq z_{i+1}$. Finally, to see condition (4) holds, suppose that $s_{i+1}-s_i \neq 0$ and that $z_k \neq 0$ for some $k < i+1$. Then since $s_{i+1}-s_i \neq 0$, we know that $s_{i+1}-s_i=1$. So the inequality

$$0 \leq (s_{i+1}-s_i)(z_1+\cdots+z_k+\cdots+z_i) \leq s_i z_{i+1}$$

reduces to

$$0 \leq z_1+\cdots+z_k+\cdots+z_i \leq s_i z_{i+1},$$

and since $z_k \neq 0$, it follows that $z_{i+1} \neq 0$.

Conversely, suppose that $(z_1,\ldots,z_{n+1})$ is a lattice point satisfying the conditions (1), (2), (3), and (4). Then by conditions (1) and (2) it suffices to show that $(z_1,\ldots,z_{n+1})$ satisfies the inequalities

$$0 \leq (s_{i+1}-s_i)(z_1+\cdots+z_i) \leq s_i z_{i+1}$$

for all $i \in [n]$. However, since $(z_1,\ldots,z_{n+1})$ is a lattice point, whenever $z_{i+1} \neq 0$, we know that $z_{i+1} \geq 1$. Thus, the conditions (3) and (4) show that $s_{i+1}-s_i \leq z_{i+1}$ for all $i \in [n]$. Therefore, condition (2) implies that the inequalities

$$0 \leq (s_{i+1}-s_i)(z_1+\cdots+z_i) \leq s_i z_{i+1}$$

hold for all $i \in [n]$. □

Let $A(s) := \{i+1 \in [n+1] : s_i < s_{i+1}\}$ denote the collection of indices $i+1 \in [n+1]$ for which $s_{i+1}-s_i \neq 0$. Notice that a lattice point $z=(z_1,\ldots,z_{n+1}) \in A_n^s \cap \mathbb{Z}^{n+1}$ indexes the multiset

$$\{1^{z_1}, 2^{z_2}, \ldots, (n+1)^{z_{n+1}}\}.$$

We call any such multiset an *s-lecture hall multiset (of order n)*. The notion of multisets and their corresponding lattice points in $\mathbb{Z}^n$ will be used in the coming section. It will be useful to have the following notation.

For $i \in [n]$ and a multiset I of $[n]$ we let $\mathrm{mult}_I(i)$ denote the multiplicity of i in I. Given a collection of multisets $\mathcal{I} = \{I_1, \dots, I_k\}$, we let $\Sigma\mathcal{I}$ denote the multiunion $\bigcup_{I \in \mathcal{I}} I$. For each multiset $I_i \in \mathcal{I}$ we let $x^{(i)} := (\mathrm{mult}_{I_i}(1), \mathrm{mult}_{I_i}(2), \dots, \mathrm{mult}_{I_i}(n)) \in \mathbb{Z}^n$ denote its *multiplicity vector*. The multiplicity vectors $x^{(1)}, \dots, x^{(k)}$ can be ordered *lexicographically*, i.e., for two vectors $x, y \in \mathbb{Z}^n$ we say $x \succ_{\mathrm{lex}} y$ if and only if the leftmost nonzero entry in $x - y$ is positive. Given this, we may reindex the collection $\mathcal{I}$ such that $x^{(1)} \succ_{\mathrm{lex}} x^{(2)} \succ_{\mathrm{lex}} \cdots \succ_{\mathrm{lex}} x^{(k)}$. Moreover, the lexicographic ordering on the lattice points in $\mathbb{Z}^n$ induces a lexicographic ordering on the multisets of $[n]$. That is, for two multisets I_1, I_2 of $[n]$, we say $I_1 \succ_{\mathrm{lex}} I_2$ if and only if $x^{(1)} \succ_{\mathrm{lex}} x^{(2)}$. Furthermore, given two collections of k multisets $\mathcal{I} = \{I_1, \dots, I_k\}$ and $\mathcal{J} = \{J_1, \dots, J_k\}$ of $[n]$, we write $\mathcal{I} \succ \mathcal{J}$ if and only if the $I_k \succ_{\mathrm{lex}} J_k$ for the smallest index k for which $I_k \neq J_k$. A collection $\mathcal{I} = \{I_1, \dots, I_k\}$ of k multisets is said to be *minimal* if $\mathcal{I}' \succ_{\mathrm{lex}} \mathcal{I}$ for any collection $\mathcal{I}'$ of k multisets of $[n]$ satisfying $\Sigma\mathcal{I}' = \Sigma\mathcal{I}$. Equivalently, the collection of vectors $\{x^{(1)}, \dots, x^{(k)}\}$ is called minimal.

3.2 A lexicographic Gröbner basis

We now use the notion of s-lecture hall multisets described in Section 3.1 to describe a quadratic Gröbner basis with a square-free initial ideal for the toric ideal associated to P_n^s. To get started, we do not yet need to speak directly about s-lecture hall multisets, but instead, we need only their corresponding multiplicity vectors in $A_n^s \cap \mathbb{Z}^n$. If $x \in kA_n^s \cap \mathbb{Z}^{n+1}$, let

$$\alpha_r(x) = \min\{i \in [n+1] : x_i \geq r, \text{ and } x_j \geq r \text{ for all } j > i \text{ such that } j \in A(s)\}.$$

If $x = (x_1, \dots, x_{n+1}) \neq 0$, let $\ell(x) = \min\{i : x_i \neq 0\}$.

Lemma 3.2. *Suppose $x^{(1)} \succeq_{\mathrm{lex}} x^{(2)} \succeq_{\mathrm{lex}} \cdots \succeq_{\mathrm{lex}} x^{(m)}$ are integer points in $A_n^s \cap \mathbb{Z}^{n+1}$, and let $y = x^{(1)} + x^{(2)} + \cdots + x^{(m)}$. If $x^{(1)} \succeq_{\mathrm{lex}} x^{(2)} \succeq_{\mathrm{lex}} \cdots \succeq_{\mathrm{lex}} x^{(m)}$ are pairwise minimal, then*

$$\ell(x^{(i)}) = \alpha_i(y), \quad \textit{for all } 1 \leq i \leq m.$$

Proof. The proof is by induction over i for $1 \leq i \leq m$. Note that $\alpha_1(y)$ is the first nonzero coordinate of y. Since the $x^{(i)}$ are ordered lexicographically, we have $\ell(x^{(1)}) = \alpha_1(y)$ as claimed.

Suppose now that the claim is true for all indices less than or equal to $i \geq 1$, but that it is not true for $i+1$. Then $a := \alpha_{i+1}(y) < \ell(x^{(i+1)}) =: b$. Let c be the largest integer in $\{j : a \leq j < b, y_j \geq i+1\}$. Note that $x_j^{(k)} = 0$ for all $k \geq i+1$, since the $x^{(k)}$ are ordered lexicographically. Since $y_c \geq i+1$, there is a k satisfying $1 \leq k \leq i$ such that $x_c^{(k)} \geq 2$. Let $d > c$ be the smallest index for which $x_d^{(k)} > 0$ and either $d \notin A(s)$ or $x_d^{(k)} \geq 2$. Then the pair $\{x^{(k)} + e_d - e_c, x^{(i)} + e_c - e_d\}$ is smaller than $\{x^{(k)}, x^{(i)}\}$, which contradicts pairwise minimality. $\square$

Theorem 3.3. *If $x^{(1)} \succeq_{\text{lex}} x^{(2)} \succeq_{\text{lex}} \cdots \succeq_{\text{lex}} x^{(k)}$ are pairwise minimal, then the collection $\{x^{(1)}, \ldots, x^{(k)}\}$ is minimal.*

Proof. Let $y = x^{(1)} + x^{(2)} + \cdots + x^{(k)}$. We prove that $x^{(1)} \succeq_{\text{lex}} x^{(2)} \succeq_{\text{lex}} \cdots \succeq_{\text{lex}} x^{(k)}$ are uniquely determined given y. The proof is by induction on $k \geq 2$.

Suppose first that $\alpha_i(y) > \alpha_j(y)$ for some $i < j$. Let m be the last index for which $\alpha_m(y) > \alpha_{m+1}(y)$. Let $u = x^{(1)} + \cdots + x^{(m)}$ and $v = x^{(m+1)} + \cdots + x^{(k)}$. We prove that u and v are uniquely determined. We claim that if $j \geq \alpha_{m+1}(y)$ and $j \in A(s)$, then

$$v_j = \min\left(y_j - m, s_j(k-m) - \sum_{i=1}^{j-1} v_i\right), \tag{3.1}$$

and if $j \geq \alpha_{m+1}(y)$ and $j \notin A(s)$, then

$$v_j = \min\left(y_j, s_j(k-m) - \sum_{i=1}^{j-1} v_i\right). \tag{3.2}$$

Assume $j \geq \alpha_{m+1}(y)$ and $j \in A(s)$. Then the jth coordinate of each $x^{(i)}$, $i \leq m$, is positive, since $j \in A(s)$. Hence, $v_j = y_j - u_j \leq y_j - m$. Moreover, $v_j \leq s_j(k-m) - \sum_{i=1}^{j-1} v_i$ by the defining inequalties of A_n^s and the definition of s-lecture hall partitions. Thus, if (3.1) fails, then $v_j < y_j - m$ and $\sum_{i=1}^{j} v_i < s_j(k-m)$. So we conclude there are indices i, ℓ such that $i \leq m$ and $\ell \geq m+1$ such that $x_j^{(i)} > 1$ and $\sum_{i=1}^{j} x_i^{(\ell)} < s_j$. Let $p > j$ be the smallest index such that $x_p^{(\ell)} > 1$ or $x_p^{(\ell)} = 1$ and $p \notin A(s)$. Then the pair $\{x^{(i)} - e_j + e_p, x^{(\ell)} + e_j - e_p\}$ is smaller than $\{x^{(i)}, x^{(\ell)}\}$, contradicting pairwise minimality. Thus, (3.1) follows, and the case when $j \geq \alpha_{m+1}(y)$ and $j \notin A(s)$ follows similarly.

If $\alpha_i(y) = \alpha_j(y) = a$ for all i, j, we claim that the first nonzero coordinate of the $x^{(i)}$ differ by at most one. Indeed if the first nonzero coordinate of $x^{(i)}$ and $x^{(j)}$, $i < j$, differ by at least two, then let b be the smallest integer greater than a for which the entry in $x^{(i)}$ is either greater than one or equal to one and not in $A(s)$. Then the pair $\{x^{(i)} - e_a + e_b, x^{(j)} + e_a - e_b\}$ is smaller than $\{x^{(i)}, x^{(j)}\}$, which contradicts pairwise minimality. If the first nonzero entry of all $x^{(i)}$ is equal, we may delete this entry for each $x^{(i)}$ and repeat our argument. Hence, we reduce to the case when either $\alpha_i(y) = \alpha_j(y) = a$ for all i, j and some first coordinates differ, or $\alpha_i(y) > \alpha_j(y)$. The latter case is dealt with above. For the former case, let m be the index for which the first coordinates of $x^{(m)}$ and $x^{(m+1)}$ differ. Let $u = x^{(1)} + \cdots + x^{(m)}$ and $v = x^{(m+1)} + \cdots + x^{(k)}$ and argue as above. $\square$

Theorem 3.3 allows us to compute the desired Gröbner basis. Let

$$K[\mathbf{x}] := K[x_I : I \text{ is a } s\text{-lecture hall multiset}]$$

be a polynomial ring over a field K in the indeterminants x_I. Given a collection of s-lecture hall multisets $\mathcal{I} = \{I_1, \ldots, I_r\}$, we denote the monomial $x_{I_1} \cdots x_{I_r}$ by $x^{\mathcal{I}}$. In the following, we denote the toric ideal $\mathcal{I}_{A_n^s}$ in $K[\mathbf{x}]$ simply by $\mathcal{I}_n^s$. For a collection of k s-lecture hall multisets $\{I_1, \ldots, I_k\}$, we let $\{I_1^-, \ldots, I_k^-\}$ denote the minimal collection of k s-lecture hall multisets satisfying $\Sigma\{I_1^-, \ldots, I_k^-\} = \Sigma\{I_1, \ldots, I_k\}$.

Theorem 3.4. *There exists a term order $\succ$ on $K[\mathbf{x}]$ such that the marked set of binomials*

$$G := \{\underline{x_I x_J} - x_{I^-} x_{J^-} : I \text{ and } J \text{ are } s\text{-lecture hall multisets}\}$$

is a reduced Gröbner basis for $\mathcal{I}_n^s$ with respect to $\succ$. The initial ideal $\mathrm{in}_\succ \mathcal{I}_n^s$ is generated by the underlined terms, all of which are square-free.

Proof. For a collection of s-lecture hall multisets $\mathcal{I} = \{I_1, \ldots, I_r\}$, the relation $x_{I_1} \cdots x_{I_r} - x_{I_1^-} \cdots x_{I_r^-}$ lies in the ideal $\mathcal{I}_n^s$. This is because the multiunion over each collection of multisets is the same and $I_1^-, \ldots, I_k^-$ are all s-lecture hall multisets. The binomials in G define a reduction relation on $k[\mathbf{x}]$ for which the underlined term is treated as the leading term of the binomials. We say a monomial is in *normal* form with respect to a reduction relation if it is the remainder upon division with respect to the given set of polynomials and their specified leading terms [14, Chapter 3]. It follows from Theorem 3.3 that if $\mathcal{I}$ is not minimal, then there exists some pair $\{I_i, I_j\} \subset \mathcal{I}$ for which $\{I_i, I_j\}$ is not minimal. So a monomial $x^{\mathcal{I}}$ for $\mathcal{I} = \{I_1, \ldots, I_r\}$ is in normal form with respect to the reduction relation defined by G if and only if $\mathcal{I}$ is minimal. Notice also that the reduction modulo G is Noetherian; i.e., every sequence of reductions modulo G terminates. This is because reduction of the monomial $x_{I_1} \cdots x_{I_r}$ by $x_{I_i} x_{I_j} - x_{I_i^-} x_{I_j^-}$ amounts to replacing the multiset $\mathcal{I} = \{I_1, \ldots, I_r\}$ with the multiset $\mathcal{I}' := \mathcal{I} \backslash \{I_i, I_j\} \cup \{I_i^-, I_j^-\}$. Since $\mathcal{I}'$ is lexicographically smaller than $\mathcal{I}$, reduction modulo G is Noetherian. So by applying [14, Theorem 3.12] we find that G is a coherently marked collection of binomials. Thus, it is a Gröbner basis for $\mathcal{I}_n^s$ with respect to some term order $\succ$ on $K[\mathbf{x}]$ and its initial ideal is generated by the underlined terms. It follows readily that the monomials in the initial ideal with respect to this term order are precisely the non-minimal monomials. Thus, G is a quadratic and reduced Gröbner basis for $\mathcal{I}_n^s$ with a square-free initial ideal. □

The following corollary extends the results of [2] and [7]. In particular, it provides a partial answer to [7, Conjecture 5.2] in a special case that they noted to be of particular interest; namely, when the first order difference sequence of s is a $0,1$-sequence.

Corollary 3.5. *Let s be a weakly increasing sequence of positive integers whose first order difference sequence of s is a $0,1$-sequence. There exists a regular, flag, and unimodular triangulation of P_n^s.*

Proof. By Theorem 3.4 we know that the toric ideal of the polytope A_n^s has a quadratic Gröbner basis for some term order that has a square-free initial ideal. It follows from

[14, Theorem 8.8] and [14, Corollary 8.9] that A_n^s has a regular, flag, and unimodular triangulation whose minimal non-faces are indexed by the lexicographically non-minimal sets of s-lecture hall multisets. Since A_n^s is unimodularly equivalent to P_n^s, we conclude that P_n^s has a regular, flag, and unimodular triangulation. □

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