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Cyclic sieving phenomenon on dominant maximal weights

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Abstract. Dominant maximal weights are significant objects in the representation theory of affine Kac–Moody algebras. We construct a (bi)cyclic sieving phenomenon on the union of dominant maximal weights for highest weight modules over affine Kac–Moody algebras in a way not depending on types, ranks and levels. Exploiting this phenomenon, we derive closed and recursive formulae for the number of dominant maximal weights for every highest weight module and observe level-rank duality on the cardinalities. We also observe interesting interrelations among the recursive formulae of classical affine Kac–Moody algebras.

Keywords: affine Kac–Moody algebra, dominant maximal weight, (bi)cyclic sieving phenomenon

1 Introduction

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $V(\Lambda)$ be the irreducible highest weight module with highest weight $\Lambda \in P^+$, where P^+ denotes the set of dominant integral weights. Due to Kac [6], all weights of $V(\Lambda)$ are given by the disjoint union of δ-strings attached to maximal weights and every maximal weight is conjugate to a unique dominant maximal weight under Weyl group action.

In [6], Kac established lots of fundamental properties concerned with $\mathrm{wt}(\Lambda)$, the set of weights of $V(\Lambda)$, using the orthogonal projection $\bar{\ }: \mathfrak{h}^* \to \mathfrak{h}_0^*$. In particular, he showed that $\max^+(\Lambda)$, the set of dominant maximal weights, is in bijection with $\ell \mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ under this projection, thus it is finite. Here ℓ denotes the level of Λ and Q denotes the

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root lattice of $\mathfrak{g}$. Based on this result, there have been a number of studies on $\max^+(\Lambda)$ (see [5, 13]). However, in the best knowledge of the authors, approachable combinatorial models and cardinality formulae of $\max^+(\Lambda)$ have not been available up to now except for limited cases, which motivates our study.

The *cyclic sieving phenomenon* was introduced by Reiner–Stanton–White in [8]. It was generalized and developed in various aspects including combinatorics and representation theory (see [1, 10] for examples).

The main purpose of this extended abstract is to investigate $\max^+(\Lambda)$ by constructing a combinatorial model for $\max^+(\Lambda)$ and a (bi)cyclic sieving phenomenon on this model in a way not depending on types, ranks and levels. As applications, we derive closed and recursive formulae of $|\max^+(\Lambda)|$ for all affine types, and observe interesting symmetries by considering $\max^+(\Lambda)$ for all ranks and levels. We find out intriguing interrelations among the recursive formulae of various affine Kac–Moody algebras. We also realize that our symmetry for $A_n^{(1)}$ type gets along with the level-rank duality introduced by Frenkel [3].

For details and more results, we refer the reader to [7].

2 Preliminaries

Let $I = \{0, 1, \dots, n\}$ be an index set. An *affine Cartan datum* $(\mathsf{A}, P, \Pi, P^\vee, \Pi^\vee)$ consists of (a) an *affine Cartan matrix* $\mathsf{A} = (\mathsf{a}_{ij})_{i,j \in I}$ of corank 1, (b) the *weight lattice* $P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$, (c) the set of *simple roots* $\Pi = \{\alpha_i \mid i \in I\} \subset P$, (d) the *coweight lattice* $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$, (e) the set of *simple coroots* $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ subject to the condition $\langle h_i, \alpha_j \rangle = \mathsf{a}_{ij}$ and $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $i, j \in I$. Here $\delta = \sum_{i=0}^{n} a_i \alpha_i$ is the *null root*. Let

$$c = a_0^\vee h_0 + a_1^\vee h_1 + \cdots + a_n^\vee h_n$$

be the *canonical central element*. We say that a weight $\Lambda \in P$ is of *level* ℓ if $\langle c, \Lambda \rangle = \ell$. We call the free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the *root lattice*. The elements of $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geqslant 0},\ i \in I\}$ are called *dominant integral weights*. For a nonnegative integer ℓ, we set

$$P_\ell^+ := \{\Lambda \in P^+ \mid \langle c, \Lambda \rangle = \ell\} \quad \text{and} \quad P_{\mathrm{cl},\ell}^+ := P_\ell^+ / \mathbb{Z}\delta.$$

The *affine Kac–Moody algebra* $\mathfrak{g}$ associated with the affine Cartan datum $(\mathsf{A}, P, \Pi, P^\vee, \Pi^\vee)$ is the Lie algebra over $\mathbb{C}$ generated by e_i, f_i $(i \in I)$ and $h \in P^\vee$ with defining relations. Let C be the Cartan matrix associated to a finite simple Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$.

It is well known that the category $\mathcal{O}_{\mathrm{int}}$, consisting of integrable weight $\mathfrak{g}$-modules, is a semisimple tensor category such that every irreducible object is isomorphic to *the highest weight module* $V(\Lambda)$ $(\Lambda \in P^+)$.

Let $\mathrm{wt}(V(\Lambda))$ be the set of weights of $V(\Lambda)$. The elements of $\max_{\mathfrak{g}}(\Lambda) := \{\mu \in \mathrm{wt}(V(\Lambda)) \mid \mu + \delta \notin \mathrm{wt}(V(\Lambda))\}$ is called *maximal weights*. We set

$$\max_{\mathfrak{g}}^{+}(\Lambda) := \max_{\mathfrak{g}}(\Lambda) \cap P^{+}.$$

We sometimes omit the subscript $\mathfrak{g}$ for simplicity. It is well known that $\max(\Lambda) = W \cdot \max^{+}(\Lambda)$, where W is the Weyl group of $\mathfrak{g}$.

Let $\mathfrak{h}_0$ be the vector space spanned by $\{h_i \mid i \in I_0 := I \backslash \{0\}\}$. The *orthogonal projection* $\bar{\ }: \mathfrak{h}^* \to \mathfrak{h}_0^*$ and the set $\mathcal{C}_{\mathrm{af}}$ are introduced in [6].

Proposition 2.1 ([6, Proposition 12.6])**.** *The map $\mu \longmapsto \overline{\mu}$ defines a bijection from* $\max^{+}(\Lambda)$ *onto $\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ where Λ is of level ℓ.*

Throughout this paper, we denote by $\bullet$ the *dot product* on $\mathbb{Q}^n$. In addition, for $k \in \mathbb{Z}_{>0}$ and $m, m' \in \mathbb{Z}$, we write $m \equiv_k m'$ if $k \mid m - m'$, and $m \not\equiv_k m'$ otherwise.

3 Embedding $\max^{+}(\Lambda)$ into $P^{+}_{\mathrm{cl},\ell}$

In this section, we will assume that Λ is of the form $\sum_{0 \leqslant i \leqslant n} p_i \Lambda_i$ because $\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) = \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda + k\delta} + \overline{Q})$ for all $k \in \mathbb{Z}$.

An equivalence relation $\sim$ on P, defined by $\Lambda \sim \Lambda'$ if and only if $\Lambda - \Lambda' \in Q$, was introduced in [2, Definition 3.1]. We note that it induces an equivalence relation, called the *sieving equivalence relation*, on $P^{+}_{\mathrm{cl},\ell}$ defined as follows: For $\Lambda, \Lambda' \in P^{+}_{\mathrm{cl},\ell}$,

$$\Lambda \sim \Lambda' \quad \text{if and only if} \quad \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) = \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda'} + \overline{Q}). \tag{3.1}$$

Let $\Pi_0 := \{\overline{\alpha}_i \mid i \in I_0\}$ be the set of simple roots of $\mathfrak{g}_0$ and $\varpi := \{\varpi_i \mid i \in I_0\}$ the set of fundamental dominant weights of $\mathfrak{g}_0$. Let $P_0 := \mathbb{Z}\varpi$ be the weight lattice of $\mathfrak{g}_0$ and $Q_0 := \mathbb{Z}\Pi_0$ the root lattice of $\mathfrak{g}_0$. Then P_0/Q_0 is known to be a finite group, called the *fundamental group of* Φ_0 (*the set of roots of* $\mathfrak{g}_0$). Its structure is well known in the literature. For instance, see [4].

We note that there are at most $|P_0/Q_0|$ equivalence classes on $P^{+}_{\mathrm{cl},\ell}$. For each type, we define a set $\mathtt{DR}(P^{+}_{\mathrm{cl},\ell})$, called the set of *distinguished representatives*. Indeed it is designed so that every $\Lambda \in \mathtt{DR}(P^{+}_{\mathrm{cl},\ell})$ is of the form $(\ell-1)\Lambda_0 + \Lambda_i$. For instance,

$$\mathtt{DR}(P^{+}_{\mathrm{cl},\ell}) = \begin{cases} \{(\ell-1)\Lambda_0 + \Lambda_i \mid i = 0, 1, \dots, n\} & \text{if } \mathfrak{g} = A_n^{(1)}, \\ \{(\ell-1)\Lambda_0 + \Lambda_i \mid i = 0, 1, n-1, n\} & \text{if } \mathfrak{g} = D_n^{(1)}. \end{cases}$$

For other types, see [7, Table 2.2].

Lemma 3.1. $\mathtt{DR}(P^{+}_{\mathrm{cl},\ell})$ *is a complete set of pairwise inequivalent representatives of $P^{+}_{\mathrm{cl},\ell}/\sim$, the set of equivalence classes of $P^{+}_{\mathrm{cl},\ell}$ under the sieving equivalence relation. In particular, the number of equivalence classes is given by $|P_0/Q_0|$.*

For $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, set $P^+_{\mathrm{cl},\ell}(\Lambda) := \{\Lambda' \in P^+_{\mathrm{cl},\ell} \mid \Lambda \sim \Lambda'\}$ and consider the map $\iota_\Lambda : \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) \to P^+_{\mathrm{cl},\ell}(\Lambda)$ defined by

$$\sum_{1 \leqslant i \leqslant n} m_i \varpi_i \mapsto m_0 \Lambda_0 + \sum_{1 \leqslant i \leqslant n} m_i \Lambda_i \ \ (m_0 := \ell - \sum_{1 \leqslant i \leqslant n} a_i^\vee m_i).$$

Proposition 3.2. *For $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, the map ι_Λ is a bijection.*

Note that $\left(\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})\right) \cap \left(\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda'} + \overline{Q})\right) = \varnothing$ if $\Lambda \not\sim \Lambda'$, which yields a bijection

$$\bigsqcup_{\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})} \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) \xleftrightarrow{1-1} \bigsqcup_{\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})} P^+_{\mathrm{cl},\ell}(\Lambda) = P^+_{\mathrm{cl},\ell} \,. \tag{3.2}$$

Now, we introduce a simple description of $P^+_{\mathrm{cl},\ell}(\Lambda)$ for each $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$. To do that, we need preparations: Let $\mathsf{N} := |P_0/Q_0|$ and $\hat{\mathsf{N}} := \max\{|g| \mid g \in P_0/Q_0\}$. For a subset $S \subset \mathbb{Z}^n$, set $\mathrm{red}_{\mathsf{N}}(S) := \{\overline{\mathbf{s}} \subset (\mathbb{Z}_{\mathsf{N}})^n \mid \mathbf{s} \in S\}$, where $\overline{\mathbf{s}} = \mathbf{s} + (\mathsf{N}\mathbb{Z})^n$.

Definition 3.3. Let $\mathfrak{g}$ be an affine Kac–Moody algebra. We call a subset $S \subset \mathbb{Z}^n$ a *root-sieving set* if, for all $\mathbf{x} \in P_0$, (1) $\mathbf{x} \in Q_0$ if and only if $\mathbf{s} \bullet [\mathbf{x}]_\varpi \equiv_{\mathsf{N}} 0$ for all $\mathbf{s} \in S$, (2) the set $\mathrm{red}_{\mathsf{N}}(S) \subset (\mathbb{Z}_{\mathsf{N}})^n$ is $\mathbb{Z}_{\hat{\mathsf{N}}}$-linearly independent, and (3) $|\mathrm{red}_{\mathsf{N}}(S)| = |S|$. In this case, the elements of S are called *root-sieving vectors* of S.

Convention 3.4. (1) We choose a special root sieving set, denoted by $\boldsymbol{S}$, as follows:

$$\boldsymbol{S} = \begin{cases} \{\boldsymbol{s} = (1,2,\dots,n)\} & \text{if } \mathfrak{g} = A_n^{(1)}, \\ \{\boldsymbol{s} = (2,0,2,0,\dots,0,2,1,3)\} & \text{if } \mathfrak{g} = D_n^{(1)} (n \equiv_2 1), \\ \{\boldsymbol{s}^{(1)} = (0,0,\dots,0,2,2), \boldsymbol{s}^{(2)} = (2,0,2,0,\dots,2,0,2,0)\} & \text{if } \mathfrak{g} = D_n^{(1)} (n \equiv_2 0). \end{cases}$$

For other types, see [7, Table 2.4].
(2) For a root sieving vector $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$, we denote $(0, s_1, s_2, \dots, s_n)$ by $\widetilde{\mathbf{s}}$.

With the root sieving sets $\boldsymbol{S}$ given in Convention 3.4, we define a new statistics $\mathsf{ev}_{\boldsymbol{s}}$, called the *$\boldsymbol{S}$-evaluation*, by

$$\mathsf{ev}_{\boldsymbol{s}} : P^+_{\mathrm{cl},\ell} \to \mathbb{Z}^k_{\geqslant 0}, \quad \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \mapsto \left(\widetilde{\mathbf{s}}^{(k)} \bullet \mathbf{m}\right)_{k=1 \text{ or } 1,2} \tag{3.3}$$

where $\mathbf{m} = (m_0, m_1, \dots, m_n)$.

Theorem 3.5. *Let $\boldsymbol{S}$ be the set given in Convention 3.4. For any $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we have*

$$P^+_{\mathrm{cl},\ell}(\Lambda) = \left\{\Lambda' \in P^+_{\mathrm{cl},\ell} \,\middle|\, \mathsf{ev}_{\boldsymbol{s}}(\Lambda') \equiv_{\mathsf{N}} \mathsf{ev}_{\boldsymbol{s}}(\Lambda)\right\}. \tag{3.4}$$

Note that Theorem 3.5 does not depend on the choice of a root-sieving set.

4 Sagan's action and generalization

From this section, we will investigate the structure and enumeration of $P^+_{\mathrm{cl},\ell}(\Lambda)$ for all $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$ in a viewpoint of (bi)cyclic sieving phenomena ([8]). In order to do this, we give a suitable (bi)cyclic group action on $P^+_{\mathrm{cl},\ell}$ by generalizing Sagan's action in [9]. For details of Sagan's action and our generalization, see [7, Section 3].

Throughout this section, we assume that d, k are positive integers and ℓ is a non-negative integer. Given a kd-tuple $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbb{Z}^{kd}_{\geqslant 0}$, we set $\mathbf{m}[j;d] := \sum_{0 \leqslant t \leqslant d-1} m_{jd+t}$ for $0 \leqslant j \leqslant k-1$. Also, given a k-tuple $\boldsymbol{\nu} = (\nu_0, \nu_1, \dots, \nu_{k-1}) \in \mathbb{Z}^k_{>0}$, we set

$$\mathbf{M}_\ell(d;\boldsymbol{\nu}) := \left\{ \mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbb{Z}^{kd}_{\geqslant 0} \;\middle|\; \sum_{0 \leqslant j \leqslant k-1} \nu_j \mathbf{m}[j;d] = \ell \right\}.$$

To each $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$ we associate a word $\mathbf{w}(\mathbf{m}; d; \boldsymbol{\nu})$ with entries in $\{0, \nu_0, \nu_1, \dots, \nu_{k-1}\}$ produced by the following algorithm:

Algorithm 4.1. Assume we have a kd-tuple $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$.

(**A**1) Set $\mathbf{w}$ to be the empty word and $j = 0, t = 0$. Go to (**A**2).

(**A**2) Set $\mathbf{w}$ to be the word obtained by concatenating m_{jd+t} ν_j's at the right of $\mathbf{w}$. If $j = k-1$ and $t = d-1$, return $\mathbf{w}$ and terminate the algorithm. Otherwise, go to (**A**3).

(**A**3) Set $\mathbf{w}$ to be the word obtained by concatenating 0 at the right. Go to (**A**4).

(**A**4) If $t \neq d-1$ then set $t = t+1$ and go to (**A**2). If $t = d-1$ set $j = j+1$ and $t = 0$, and go to (**A**2).

Set $\mathcal{W}_\ell(d;\boldsymbol{\nu}) := \{\mathbf{w}(\mathbf{m}; d; \boldsymbol{\nu}) \mid \mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu})\}$. Let $\Psi : \mathbf{M}_\ell(d;\boldsymbol{\nu}) \to \mathcal{W}_\ell(d;\boldsymbol{\nu})$ be a map defined by $\Psi(\mathbf{m}) = \mathbf{w}(\mathbf{m}; d; \boldsymbol{\nu})$.

Lemma 4.2. *The map Ψ is a bijection.*

Now we define a $C_d = \langle \sigma_d \rangle$-action on $\mathcal{W}_\ell(d;\boldsymbol{\nu})$. First, we break $\mathbf{w} = w_1 w_2 \dots w_u$ into subwords of length d as many as possible as follows:

$$\mathbf{w} = w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{td+1} \cdots w_u,$$

where $t = \lfloor u/d \rfloor$ and $w^j = w_{(j-1)d+1} w_{(j-1)d+2} \cdots w_{jd}$ for $1 \leqslant j \leqslant t$. Note that σ_d acts on each subword w^j by cyclic shift, i.e., $\sigma_d \cdot w^j := w_{jd} w_{(j-1)d+1} w_{(j-1)d+2} \cdots w_{jd-1}$. Assume that j_0 is the smallest integer such that $\sigma_d \cdot w^j_0 \neq w^j_0$. Then we set

$$\sigma_d \blacksquare \mathbf{w} := w^1 \mid w^2 \mid \cdots \mid w^{j_0-1} \mid \sigma_d \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{td+1} \cdots w_u. \tag{4.1}$$

If there is no such j_0, we set $\sigma_d \blacksquare \mathbf{w} := \mathbf{w}$.

Theorem 4.3. *For any* $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_{k-1}) \in \mathbb{Z}^k_{>0}$*, the action defined as above is indeed a* C_d*-action on* $\mathcal{W}_\ell(d;\boldsymbol{\nu})$*.*

Now we define a C_d-action on $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ by transporting the C_d-action $\blacksquare$ on $\mathcal{W}_\ell(d;\boldsymbol{\nu})$ via the bijection Ψ, that is,

$$\sigma_d \blacksquare \mathbf{m} := \Psi^{-1}(\sigma_d \blacksquare \Psi(\mathbf{m})) \quad \text{for all } \mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu}). \tag{4.2}$$

Remark 4.4. Suppose that C_d acts on $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ as in (4.2). Then, for any $r \in \mathbb{Z}_{>0}$, $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ is also equipped with a C_{rd}-action $\blacksquare_d$ given by

$$\sigma_{rd} \blacksquare_d \mathbf{m} := \sigma_d \blacksquare \mathbf{m}. \tag{4.3}$$

Let us generalize the above setting a little further. Let $d, k, k', r \in \mathbb{Z} > 0$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. For $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_{k-1}) \in \mathbb{Z}^k_{>0}$ and $\boldsymbol{\nu}' = (\nu'_0, \nu'_1, \ldots, \nu'_{k'-1}) \in \mathbb{Z}^{k'}_{>0}$, set

$$\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}') := \left\{ \mathbf{m} \in \mathbb{Z}^{krd+k'd}_{\geqslant 0} \;\middle|\; \sum_{0 \leqslant j \leqslant k-1} \nu_j \mathbf{m}[j; rd] + \sum_{0 \leqslant j \leqslant k'-1} \nu'_j \mathbf{m}[kr + j; d] = \ell \right\}. \tag{4.4}$$

Using the actions given in (4.2) and (4.3), we define a new C_{rd}-action, denoted by $\blacksquare_{rd,d}$, on $\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')$ as follows: Given $\mathbf{m} \in \mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')$, we break it into $\mathbf{m}_{\leqslant krd-1} := (m_0, m_1, \ldots, m_{krd-1}) \in \mathbf{M}_l(rd; \boldsymbol{\nu})$ and $\mathbf{m}_{\geqslant krd} := (m_{kd}, m_{kd+1}, \ldots, m_{krd+k'd-1}) \in \mathbf{M}_{l'}(d; \boldsymbol{\nu}')$, where $\ell = l + l'$. Now, we define

$$\sigma_{rd} \blacksquare_{rd,d} \mathbf{m} := \begin{cases} (\sigma_{rd} \blacksquare \mathbf{m}_{\leqslant krd-1}) * \mathbf{m}_{\geqslant krd} & \text{if } \sigma_{rd} \blacksquare \mathbf{m}_{\leqslant krd-1} \neq \mathbf{m}_{\leqslant krd-1}, \\ \mathbf{m}_{\leqslant krd-1} * (\sigma_{rd} \blacksquare_d \mathbf{m}_{\geqslant krd}) & \text{otherwise}, \end{cases} \tag{4.5}$$

where $\mathbf{m} * \mathbf{m}'$ is the tuple obtained by concatenating $\mathbf{m}$ and $\mathbf{m}'$.

5 (Bi)cyclic sieving phenomena on $P^+_{\mathrm{cl},\ell}$

The *cyclic sieving phenomenon* was introduced by Reiner–Stanton–White in [8]. Let X be a finite set, with an action of a cyclic group C of order m. Elements within a C-orbit share the same stabilizer subgroup, whose cardinality is called the *stabilizer-order* for the orbit. Let $X(q)$ be a polynomial in q with nonnegative integer coefficients. For $d \in \mathbb{Z}_{>0}$, let ω_d be a dth primitive root of the unity. We say that $(X, C, X(q))$ exhibits the cyclic sieving phenomenon if, for all $c \in C$, we have

$$|X^c| = X(\omega_{o(c)}),$$

where $o(c)$ is the order of c and X^c is the fixed point set under the action of c. Note that this condition is equivalent to the following congruence:

$$X(q) \equiv \sum_{0 \leqslant i \leqslant m-1} b_i q^i \pmod{q^m - 1},$$

where b_i counts the number of C-orbits on X for which the stabilizer-order divides i.

A generalization of the cyclic sieving phenomenon, called the bicyclic sieving phenomenon, was introduced in [1, Section 3]. Let X be a finite set with a permutation action of a finite *bicyclic group*, that is, a product $C_m \times C_{m'}$ for some $m, m' \in \mathbb{Z}_{>0}$. Fix embeddings $\omega : C_m \to \mathbb{C}^\times$ and $\omega' : C_{m'} \to \mathbb{C}^\times$ into the complex roots of unity. Let $X(q_1, q_2) \in \mathbb{Z}_{\geqslant 0}[q_1, q_2]$. We say that the triple $(X, C_m \times C_{m'}, X(q_1, q_2))$ exhibits the *bicyclic sieving phenomenon* if for all $(c, c') \in C_m \times C_{m'}$, we have

$$X(\omega(c), \omega'(c')) = \left|\left\{x \in X \;\middle|\; (c, c')\, x = x\right\}\right|.$$

This condition is equivalent to the following congruence:

$$X(q_1, q_2) \equiv \sum_{0 \leqslant j_1 < m,\ 0 \leqslant j_2 < m'} b(j_1, j_2) q_1^{j_1} q_2^{j_2} \pmod{q_1^m - 1, q_2^{m'} - 1}$$

where $b(j_1, j_2)$ is the number of orbits of $C_m \times C_{m'}$ on X satisfying certain conditions (see [1, Proposition 3.1]).

Now, let us introduce the triple for the (bi)cyclic sieving phenomenon on $P^+_{\mathrm{cl},\ell}$. First, we let $X := P^+_{\mathrm{cl},\ell} = \bigsqcup_{\Lambda \in \mathrm{DR}(P^+_{\mathrm{cl},\ell})} P^+_{\mathrm{cl},\ell}(\Lambda)$.

To define a (bi)cyclic group action, we note that the symmetric group $\mathfrak{S}_{[0,n]}$ over the set $\{0, 1, \ldots, n\}$ acts on $P^+_{\mathrm{cl},\ell}$ by permuting indices of coefficients, that is,

$$\sigma \cdot \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i = \sum_{0 \leqslant i \leqslant n} m_{\sigma(i)} \Lambda_i \quad \text{for } \sigma \in \mathfrak{S}_{[0,n]}.$$

We also note that if $\mathfrak{g} \neq D_n^{(1)} (n \equiv_2 0)$ then $P_0/Q_0 \simeq C_{\mathsf{N}}$, where C_{N} is a cyclic group of order N. For $\mathfrak{g} \neq D_n^{(1)} (n \equiv_2 0)$, we take an appropriate $\sigma \in \mathfrak{S}_{[0,n]}$ of order N. For instance, we let

$$\sigma = \begin{cases} (0, 1, \ldots, n) & \text{if } \mathfrak{g} = A_n^{(1)}, \\ (0, n, 1, n-1)(2, 3)(4, 5) \cdots (n-3, n-2) & \text{if } \mathfrak{g} = D_n^{(1)} (n \equiv_2 1). \end{cases}$$

Indeed $\mathsf{N} = n + 1$ if $\mathfrak{g} = A_n^{(1)}$ and $\mathsf{N} = 4$ if $\mathfrak{g} = D_n^{(1)} (n \equiv_2 1)$. For other types, see [7, Table 4.1]. Now, we define a $C_{\mathsf{N}} = \langle \sigma_{\mathsf{N}} \rangle$-action on $P^+_{\mathrm{cl},\ell}$ by

$$\sigma_{\mathsf{N}} \cdot \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i := \sum_{0 \leqslant i \leqslant n} m_{\sigma(i)} \Lambda_i \quad \text{for any } \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \in P^+_{\mathrm{cl},\ell}. \tag{5.1}$$

For $\mathfrak{g} = D_n^{(1)}(n \equiv_2 0)$, we let

$$\sigma^{(1)} = (0,n)(1,n-1) \in \mathfrak{S}_{[0,n]} \quad \text{and} \quad \sigma^{(2)} = (0,1)(2,3)\cdots(n-4,n-3)(n-1,n) \in \mathfrak{S}_{[0,n]}.$$

Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ commute to each other in $\mathfrak{S}_{[0,n]}$, so $\langle \sigma^{(1)}, \sigma^{(2)} \rangle \simeq C_2 \times C_2$. We define a $C_2 \times C_2 = \langle \sigma_2 \rangle \times \langle \sigma_2 \rangle$-action on $P^+_{\mathrm{cl},\ell}$ by (e denotes the identity of C_2)

$$(\sigma_2, e) \cdot \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i := \sum_{0 \leqslant i \leqslant n} m_{\sigma^{(1)}(i)} \Lambda_i \quad \text{and} \quad (e, \sigma_2) \cdot \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i := \sum_{0 \leqslant i \leqslant n} m_{\sigma^{(2)}(i)} \Lambda_i. \tag{5.2}$$

Finally, to define $X(q)$ or $X(q_1, q_2)$, let us consider its generating function as follows: For $\mathfrak{g} \neq D_n^{(1)}(n \equiv_2 0)$, let $\tilde{\mathfrak{s}} = (s_0, s_1, s_2, \dots, s_n)$. For $\ell \in \mathbb{Z}_{\geqslant 0}$, we define $X(q) := P^+_{\mathrm{cl},\ell}(q)$ by

$$\sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q) t^\ell := \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q^{s_i} t^{a_i^\vee}}. \tag{5.3}$$

where s is the root-sieving vector given in Convention 3.4. Then we have

$$P^+_{\mathrm{cl},\ell}(q) = \sum_{i \geqslant 0} \left| \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathrm{ev}_{\mathsf{s}}(\Lambda) = i \right\} \right| q^i \equiv \sum_{\Lambda \in \mathrm{DR}(P^+_{\mathrm{cl},\ell})} \left| P^+_{\mathrm{cl},\ell}(\Lambda) \right| q^{\mathrm{ev}_{\mathsf{s}}(\Lambda)} \pmod{q^{\mathsf{N}} - 1}.$$

For $\mathfrak{g} = D_n^{(1)}(n \equiv_2 0)$, we let $\mathfrak{s}^{(1)} := \frac{1}{2}\tilde{\mathfrak{s}}^{(1)}$ and $\mathfrak{s}^{(2)} := \frac{1}{2}\tilde{\mathfrak{s}}^{(2)}$ and define $X(q_1, q_2) := P^+_{\mathrm{cl},\ell}(q_1, q_2)$ for $\ell \in \mathbb{Z}_{\geqslant 0}$ by

$$\sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q_1, q_2) t^\ell := \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q_1^{\mathfrak{s}_i^{(1)}} q_2^{\mathfrak{s}_i^{(2)}} t^{a_i^\vee}}.$$

For $t = 1, 2$, let $\mathrm{ev}_{\mathfrak{s}^{(t)}} : P^+_{\mathrm{cl},\ell} \to \mathbb{Z}_{\geqslant 0}$ be a map defined by $\mathrm{ev}_{\mathfrak{s}^{(t)}} \left(\sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \right) = \mathfrak{s}^{(t)} \bullet \mathbf{m}$. Then we have

$$P^+_{\mathrm{cl},\ell}(q_1, q_2) \equiv \sum_{\Lambda \in \mathrm{DR}(P^+_{\mathrm{cl},\ell})} \left| P^+_{\mathrm{cl},\ell}(\Lambda) \right| q_1^{\mathrm{ev}_{\mathfrak{s}^{(1)}}(\Lambda)} q_2^{\mathrm{ev}_{\mathfrak{s}^{(2)}}(\Lambda)} \pmod{q_1^2 - 1, q_2^2 - 1}. \tag{5.4}$$

Theorem 5.1. *For $\mathfrak{g} \neq D_n^{(1)}(n \equiv_2 0)$, the triple $\left(P^+_{\mathrm{cl},\ell}, C_{\mathsf{N}}, P^+_{\mathrm{cl},\ell}(q) \right)$ exhibits the cyclic sieving phenomenon under the C_{N}-action given in* (5.1).

For $\mathfrak{g} = A_n^{(1)}$, Theorem 5.1 follows from the classical result [8, Theorem 1.1 (a)] of cyclic sieving phenomena. For other types, it can be proved by using the action defined in (4.5). We here only deal with $D_n^{(1)}(n \equiv_2 1)$ type since the method of proof for each type is essentially same.

For $m \in \mathbb{Z}_{\geqslant 0}$ and $k \in \mathbb{Z}_{>0}$, we denote by m^k the sequence $m, m, \dots, m$ consisting of k m's. Moreover, we let $\eta := \frac{n-3}{2}$ and

$$P^+_{\mathrm{cl},\ell}(q) \equiv \sum_{0 \leqslant i \leqslant 3} b_i q^i \pmod{q^4 - 1}.$$

Lemma 5.2. *Under the C_4-action on $P^+_{\mathrm{cl},\ell}$ given in (5.1) and the C_4-action on $\mathbf{M}_\ell(4,2;(1),(2^\eta))$ given in (4.5), we have*

$$P^+_{\mathrm{cl},\ell}(q)|_{q=i} = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_4}\right| = \left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| = b_0 - b_2 \quad \text{and} \quad b_1 = b_3.$$

Lemma 5.3. *Under the C_4-action on $P^+_{\mathrm{cl},\ell}$ given in (5.1) and the C_2-action on $\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))$ given in (4.5), we have*

$$P^+_{\mathrm{cl},\ell}(q)|_{q=-1} = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = \left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| = b_0 - b_1 + b_2 - b_3.$$

Proof of Theorem 5.1 for $\mathfrak{g} = D_n^{(1)}$ $(n \equiv_2 1)$. Let ζ_4 be a 4th primitive root of unity. We will see that

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^j}\right| = P^+_{\mathrm{cl},\ell}(\zeta_4^j) \quad \text{for } j = 0,1,2,3.$$

When $j = 0$, since $\left|\left(P^+_{\mathrm{cl},\ell}\right)\right| = P^+_{\mathrm{cl},\ell}(1)$, it is trivial. For the case $j \in \{1,3\}$, note that $\left(P^+_{\mathrm{cl},\ell}\right)^{C_4} = \left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^j}$ and $P^+_{\mathrm{cl},\ell}(\zeta_4^j) = b_0 + b_1\zeta_4^j - b_2 - b_3\zeta_4^j$. Thus, by Lemma 5.2, we have

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_4}\right| = \left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| = b_0 - b_2 = P^+_{\mathrm{cl},\ell}(\zeta_4^j).$$

For the case $j = 2$, note that $P^+_{\mathrm{cl},\ell}(-1) = b_0 - b_1 + b_2 - b_3$. Thus, by Lemma 5.3, we have

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = \left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| = b_0 - b_1 + b_2 - b_3 = P^+_{\mathrm{cl},\ell}(-1). \qquad \square$$

For $\mathfrak{g} = D_n^{(1)}$ $(n \equiv_2 0)$, we obtain the bicyclic sieving phenomenon in a similar way as above.

Theorem 5.4. *For $\mathfrak{g} = D_n^{(1)}$ $(n \equiv_2 0)$, the triple $\left(P^+_{\mathrm{cl},\ell}\,, C_2 \times C_2\,, P^+_{\mathrm{cl},\ell}(q_1,q_2)\right)$ exhibits the bicyclic sieving phenomenon, under the $C_2 \times C_2$-action given in (5.2).*

6 Formulae on the number of maximal dominant weights

Using the (bi)cyclic sieving phenomenon, we derive closed formulae for $|\max^+(\Lambda)|$ in terms of binomial coefficients. In particular, for affine Kac–Moody algebras of classical type, we show that our closed formulae for $|\max^+(\Lambda)|$ can be written as a sum of binomial coefficients. For instance, when $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we have

$$\left|\max^+_{A_n^{(1)}}(\Lambda)\right| = \sum_{d|(n+1,\ell,i)} \frac{d}{(n+1)+\ell} \sum_{d'|(\frac{n+1}{d},\frac{\ell}{d})} \mu(d') \binom{((n+1)+\ell)/dd'}{\ell/dd'}, \tag{6.1}$$

$$\left|\max^+_{B_n^{(1)}}(\Lambda)\right| = \frac{1}{2}\left(P^+_{\mathrm{cl},\ell}(1) + (-1)^{\delta_{in}} P^+_{\mathrm{cl},\ell}(-1)\right) = \binom{n + \left\lfloor \frac{\ell-\delta_{in}}{2} \right\rfloor}{n} + \binom{n + \left\lfloor \frac{\ell-1-\delta_{in}}{2} \right\rfloor}{n}.$$

Here μ denotes the classical Möbius function (see [7, Table 2.2, Table 6.2] for more details).

From our closed formulae, we find recursive formulae for $|\max^+(\Lambda)|$ which can be described as triangular arrays. For instance, let us consider the case $\mathfrak{g} = A^{(2)}_{2n}$. Define $T_0^{A^{(2)}_{\text{even}}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by $T_0^{A^{(2)}_{\text{even}}}(n,i) = 1$ $(n \geqslant 0, i = 0,1)$, $T_0^{A^{(2)}_{\text{even}}}(0,\ell) = 1$ $(\ell \geqslant 2)$ and

$$T_0^{A^{(2)}_{\text{even}}}(n,\ell) = T_0^{A^{(2)}_{\text{even}}}(n,\ell-2) + T_0^{A^{(2)}_{\text{even}}}(n-1,\ell). \tag{6.2}$$

Then for all $n \in \mathbb{Z}_{\geqslant 2}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{A^{(2)}_{\text{even}}}(n,\ell) = |\max^+_{A^{(2)}_{2n}}(\ell\Lambda_0)| = \binom{n+\lfloor \frac{\ell}{2} \rfloor}{n}$. In particular, as a triangular array, $T_0^{A^{(2)}_{\text{even}}}$ can be described as follows:

$$\begin{array}{lcccccccccccccccr}
n=0 & & & & & & & & 1 & & & & & & & & \ell=0\\
n=1 & & & & & & & 1 & & 1 & & & & & & & \ell=1\\
n=2 & & & & & & ① & & 1 & & 1 & & & & & & \ell=2\\
n=3 & & & & & 1 & & 1 & & ② & & 1 & & & & & \ell=3\\
n=4 & & & & 1 & & 1 & & ③ & & 2 & & 1 & & & & \ell=4\\
n=5 & & & 1 & & 1 & & 4 & & 3 & & 3 & & 1 & & & \ell=5\\
n=6 & & 1 & & 1 & & 5 & & \boxed{4} & & 6 & & 3 & & 1 & & \ell=6\\
n=7 & 1 & & 1 & & 6 & & 5 & & \boxed{10} & & 6 & & 4 & & 1 & \ell=7\\
 & \unicode{x22F0} & & \unicode{x22F0} & & \unicode{x22F0} & & \unicode{x22F0} & \vdots & & \ddots & & \ddots & & \ddots & \ddots & .
\end{array}$$

It is interesting to note that this array coincides with Pascal's triangle with duplicated diagonals, i.e., $T_0^{A^{(2)}_{\text{even}}}(n,2\ell) = T_0^{A^{(2)}_{\text{even}}}(n,2\ell+1) = \binom{n+\ell}{\ell}$ for $n, \ell \geqslant 0$, which appears in [11, A065941] in a totally different context from ours. The recursive condition (6.2) says if we add the ① and the ② then we get ③.

For another example, let us consider the case $\mathfrak{g} = B^{(1)}_n$. Define $T_0^{B^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by $T_0^{B^{(1)}}(n,0) = 1$ $(n \geqslant 1)$, $T_0^{B^{(1)}}(n,1) = 2$ $(n \geqslant 1)$, $T_0^{B^{(1)}}(0,\ell) = 2$ $(\ell \geqslant 0)$ and

$$T_0^{B^{(1)}}(n,\ell) = T_0^{B^{(1)}}(n,\ell-2) + T_0^{B^{(1)}}(n-1,\ell) \ (n \geqslant 1,\ \ell \geqslant 2).$$

Then for all $n \in \mathbb{Z}_{\geqslant 3}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{B^{(1)}}(n,\ell) = |\max^+_{B^{(1)}_n}(\ell\Lambda_0)|$. In particular, as a triangular array, $T_0^{B^{(1)}}$ can be described as follows:

$$\begin{array}{lcccccccccccccccr}
n=0 & & & & & & & & 2 & & & & & & & & \ell=0\\
n=1 & & & & & & & 1 & & 2 & & & & & & & \ell=1\\
n=2 & & & & & & 1 & & 2 & & 2 & & & & & & \ell=2\\
n=3 & & & & & 1 & & 2 & & 3 & & 2 & & & & & \ell=3\\
n=4 & & & & 1 & & 2 & & 4 & & 4 & & 2 & & & & \ell=4\\
n=5 & & & 1 & & 2 & & 5 & & 6 & & 5 & & 2 & & & \ell=5\\
n=6 & & 1 & & 2 & & 6 & & 8 & & 9 & & 6 & & 2 & & \ell=6\\
n=7 & 1 & & 2 & & 7 & & 10 & & \boxed{14} & & 12 & & 7 & & 2 & \ell=7\\
 & \unicode{x22F0} & & \unicode{x22F0} & & \unicode{x22F0} & & \unicode{x22F0} & \vdots & & \ddots & & \ddots & & \ddots & \ddots & .
\end{array}$$

This array is the triangular array obtained by removing the left boundary from the triangular array in [11, A129714] whose row sums are the Fibonacci numbers.

We note that $T_0^{B^{(1)}}(n,\ell)$ can be obtained from $T_0^{A^{(2)}_{\text{even}}}$ as follows:

$$T_0^{B^{(1)}}(n,\ell) = T_0^{A^{(2)}_{\text{even}}}(n,\ell) + T_0^{A^{(2)}_{\text{even}}}(n,\ell-1) \quad \text{for } n,\ell \geqslant 1.$$

For instance, if we add $\boxed{4}$, $\boxed{10}$ in $T_0^{A^{(2)}_{\text{even}}}$ then we get $\boxed{14}$ in $T_0^{B^{(1)}}(n,\ell)$.

Remark 6.1. Interestingly, all triangular arrays for classical affine type except for untwisted affine C-type can be constructed by *boundary conditions* and the triangular array of twisted affine even A-type. The triangular arrays for affine C-type ([11, A034851]) can be constructed by boundary conditions and Pascal triangle [7, Appendix A].

On the other hand, we obtain an interesting duality between level and rank from our closed and recursive formulae. For instance, in case where $\mathfrak{g} = A_n^{(1)}$, the closed formula in (6.1) implies that for $n \geqslant 1$ and $\ell > 1$, if $(n+1,\ell,i) = (\ell,n+1,j)$ for some $0 \leqslant i,j \leqslant \min(n,\ell)$, then

$$\left|\max\nolimits^+_{A_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_i)\right| = \left|\max\nolimits^+_{A_{\ell-1}^{(1)}}(n\Lambda_0 + \Lambda_j)\right|, \tag{6.3}$$

i.e., exchanging $n+1$ with ℓ preserves the number of dominant maximal weights.

Let us deal with the relation between our duality and Frenkel's duality in [3]. For a residue i modulo n, let $\Lambda_i^{(n)}$ denote the ith fundamental weight of $A_n^{(1)}$. For $\Lambda = \sum_{i=0}^n m_i\Lambda_i^{(n)} \in P^+_{\text{cl},\ell}$, let Λ' be the dominant integral weight of $A_{\ell-1}^{(1)}$ defined by

$$\Lambda' = \sum_{i=0}^{n} \Lambda^{(\ell-1)}_{m_i+m_{i+1}+\cdots+m_n} \in P^+_{\text{cl},n+1}.$$

With this setting, Frenkel found the following beautiful duality between the q-specialized characters of $V(\Lambda)$ and $V(\Lambda')$:

$$\dim_q(V(\Lambda)) \prod_{k=0}^{\infty} \frac{1}{1-q^{(n+1)k}} = \dim_q(V(\Lambda')) \prod_{k=0}^{\infty} \frac{1}{1-q^{\ell k}}.$$

Here $\dim_q(V)$ denotes the q-specialized character of V (see [3, Theorem 2.3] or [12, Section 4.4]). Based on (6.3), we have

$$\left|\max\nolimits^+_{A_n^{(1)}}(\Lambda)\right| = \left|\max\nolimits^+_{A_{\ell-1}^{(1)}}(\Lambda')\right|.$$

In case where $\mathfrak{g} = B_n^{(1)}$, the closed formula in (6.1) implies the following symmetries:

(1) For $n \geqslant 3$, $\ell \geqslant 7$ and $\ell \equiv_2 1$, we have

$$\left|\max\nolimits^+_{B_n^{(1)}}(\ell\Lambda_0)\right| = \left|\max\nolimits^+_{B_{(\ell-1)/2}^{(1)}}((2n+1)\Lambda_0)\right|,$$

i.e., exchanging n with $(\ell-1)/2$ preserves the number of dominant maximal weights.

(2) For $n \geqslant 3$, $\ell \geqslant 8$ and $\ell \equiv_2 0$, we have

$$\left|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_n)\right| = \left|\max^+_{B_{\ell/2-1}^{(1)}}((2n+1)\Lambda_0+\Lambda_{\ell/2-1})\right|,$$

i.e., exchanging n with $\ell/2-1$ preserves the number of dominant maximal weights.

References

[1] H. Barcelo, V. Reiner, and D. Stanton. "Bimahonian distributions". *J. Lond. Math. Soc. (2)* **77**.3 (2008), pp. 627–646. Link.

[2] O. Barshevsky, M. Fayers, and M.Schaps. "A non-recursive criterion for weights of a highest-weight module for an affine Lie algebra". *Israel J. Math.* **197**.1 (2013), pp. 237–261. Link.

[3] I. Frenkel. "Representations of affine Lie algebras, Hecke modular forms and Korteweg-de Vries type equations". *Lie algebras and related topics (New Brunswick, N.J., 1981)*. Vol. 933. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 71–110. Link.

[4] J. Humphreys. *Introduction to Lie algebras and representation theory*. Vol. 9. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1978.

[5] R. Jayne and K. Misra. "On multiplicities of maximal weights of $\widehat{sl}(n)$-modules". *Algebr. Represent. Theory* **17**.4 (2014), pp. 1303–1321. Link.

[6] V. Kac. *Infinite-dimensional Lie algebras*. 3rd ed. Cambridge University Press, Cambridge, 1990.

[7] Y.-H. Kim, S. j. Oh, and Y.-T. Oh. "Cyclic sieving phenomenon on dominant maximal weights over affine Kac-Moody algebras". 2019. arXiv:1909.07010.

[8] V. Reiner, D. Stanton, and D. White. "The cyclic sieving phenomenon". *J. Combin. Theory Ser. A* **108**.1 (2004), pp. 17–50. Link.

[9] B. Sagan. "Congruence properties of q-analogs". *Adv. Math.* **95**.1 (1992), pp. 127–143. Link.

[10] B. Sagan. "The cyclic sieving phenomenon: a survey". London Math. Soc. Lecture Note Ser. **392** (2011), pp. 183–233. Link.

[11] N. J. A. Sloane. "The online encyclopedia of integer sequences". Link.

[12] P. Tingley. "Three combinatorial models for $\widehat{\mathrm{sl}}_n$ crystals, with applications to cylindric plane partitions". *Int. Math. Res. Not. IMRN* 2 (2008), Art. ID rnm143, 40. Link.

[13] S. Tsuchioka and M. Watanabe. "Pattern avoidance seen in multiplicities of maximal weights of affine Lie algebra representations". *Proc. Amer. Math. Soc.* **146**.1 (2018), pp. 15–28. Link.