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Regularity theorem for totally nonnegative flag varieties

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Abstract. We show that the totally nonnegative part of a partial flag variety G/P (in the sense of Lusztig) is a regular CW complex, confirming a conjecture of Williams. In particular, the closure of each positroid cell inside the totally nonnegative Grassmannian is homeomorphic to a ball, confirming a conjecture of Postnikov.

Keywords: Total positivity, algebraic group, partial flag variety, Richardson varieties, totally nonnegative Grassmannian, positroid cells.

1 Introduction

Let G be a semisimple algebraic group, split over $\mathbb{R}$, and let $P \subset G$ be a parabolic subgroup. Lusztig [26] introduced the totally nonnegative part of the partial flag variety G/P, denoted $(G/P)_{\geq 0}$, which he called a "remarkable polyhedral subspace". He conjectured and Rietsch proved [32] that $(G/P)_{\geq 0}$ has a decomposition into open cells. We prove the following conjecture of Williams [40]:

Theorem 1.1. *The cell decomposition of $(G/P)_{\geq 0}$ forms a regular CW complex. Thus the closure of each cell is homeomorphic to a closed ball.*

A special case of particular interest is when G/P is the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional linear subspaces of $\mathbb{R}^n$. In this case, $(G/P)_{\geq 0}$ becomes the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$, introduced by Postnikov [29] as the subset of $\mathrm{Gr}(k,n)$ where all Plücker coordinates are nonnegative. He gave a stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ into *positroid cells* according to which Plücker coordinates are zero and which are strictly positive, and conjectured that the closure of each positroid cell is homeomorphic to a closed ball. Postnikov's conjecture follows as a special case of Theorem 1.1:

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Corollary 1.2. *The decomposition of* $\mathrm{Gr}_{\geq 0}(k,n)$ *into positroid cells forms a regular CW complex. Thus the closure of each positroid cell is homeomorphic to a closed ball.*

When $k = 1$, $\mathrm{Gr}_{\geq 0}(1,n)$ is the standard $(n-1)$-dimensional simplex $\Delta_{n-1} \subset \mathbb{P}^{n-1}$, a prototypical example of a regular CW complex.

1.1 History and motivation

A matrix is called *totally nonnegative* if all its minors are nonnegative. The theory of such matrices originated in the 1930's [35, 15]. Later, Lusztig [26] was motivated by a question of Kostant to consider connections between totally nonnegative matrices and his theory of canonical bases for quantum groups [25]. This led him to introduce the totally nonnegative part $G_{\geq 0}$ of a split semisimple G. Inspired by a result of Whitney [39], he defined $G_{\geq 0}$ to be generated by exponentiated Chevalley generators with positive real parameters, and generalized many classical results for $G = \mathrm{SL}_n$ to this setting. He introduced the totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety G/P, and showed [27, Section 4] that $G_{\geq 0}$ and $(G/P)_{\geq 0}$ are contractible.

Fomin and Shapiro [7] realized that Lusztig's work may be used to address a long-standing problem in poset topology. Namely, the Bruhat order of the Weyl group W of G had been shown to be shellable by Björner and Wachs [4], and by general results of Björner [3] it followed that there exists a "synthetic" regular CW complex whose face poset coincides with $(W, \leq)$ (see Figure 1). The motivation of [7] was to answer a natural question due to Bernstein and Björner of whether such a regular CW complex exists "in nature". Let $U \subset G$ be the unipotent radical of the standard Borel subgroup, and let $U_{\geq 0} := G_{\geq 0} \cap U$ be its totally nonnegative part. For $G = \mathrm{SL}_n$, $U_{\geq 0}$ is the semigroup of upper-triangular unipotent matrices with all minors nonnegative. The work of Lusztig [26] implies that $U_{\geq 0}$ has a cell decomposition whose face poset is $(W, \leq)$. The space $U_{\geq 0}$ is not compact, but Fomin and Shapiro [7] conjectured that taking the link of the identity element in $U_{\geq 0}$, which also has $(W, \leq)$ as its face poset, gives the desired regular CW complex. Their conjecture was confirmed by Hersh [19]. Hersh's theorem also follows as a corollary to our proof of Theorem 1.1, see Section 4.1.

Corollary 1.3 ([19])**.** *The link of the identity in* $U_{\geq 0}$ *is a regular CW complex.*

For recent related developments, see [5].

Meanwhile, Postnikov [29] defined the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, decomposed it into positroid cells, and showed that each positroid cell is homeomorphic to an open ball. Motivated by work of Fomin and Zelevinsky [8] on double Bruhat cells, he conjectured [29, Conjecture 3.6] that this decomposition forms a regular CW complex. It was later realized that the space $\mathrm{Gr}_{\geq 0}(k,n)$ and its cell decomposition coincide with the one studied by Lusztig and Rietsch in the special case that $G/P = \mathrm{Gr}(k,n)$. Williams [40, Section 7] extended Postnikov's conjecture from $\mathrm{Gr}_{\geq 0}(k,n)$ to $(G/P)_{\geq 0}$.

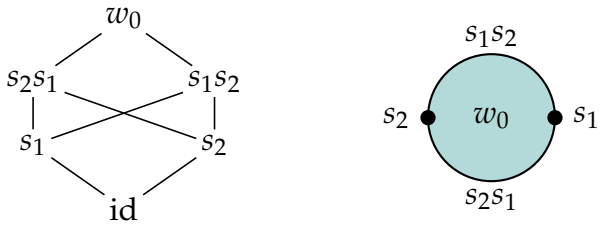


Figure 1: The Bruhat order on $W = S_3$ (after deleting the bottom element) is the face poset of the regular CW complex homeomorphic to a 2-dimensional ball on the right.

There has been much progress towards proving these conjectures. Williams [40] showed that the face poset of $(G/P)_{\geq 0}$ (and hence of $\mathrm{Gr}_{\geq 0}(k,n)$) is graded, thin, and shellable, and therefore by [3] is the face poset of some regular CW complex. Postnikov, Speyer, and Williams [30] showed that $\mathrm{Gr}_{\geq 0}(k,n)$ is a CW complex, and their result was generalized to $(G/P)_{\geq 0}$ by Rietsch and Williams [33]. Rietsch and Williams [34] also showed that the closure of each cell in $(G/P)_{\geq 0}$ is contractible. In previous work [11, 13], we showed that the spaces $\mathrm{Gr}_{\geq 0}(k,n)$ and $(G/P)_{\geq 0}$ are homeomorphic to closed balls, which is the special case of Theorem 1.1 for the top-dimensional cell of $(G/P)_{\geq 0}$. We remark that our proof of Theorem 1.1 uses different methods than those employed in [11, 13], in which we relied on the existence of a vector field on G/P contracting $(G/P)_{\geq 0}$ to a point in its interior. Singularities of lower-dimensional positroid cells give obstructions to the existence of a continuous vector field with analogous properties.

Totally positive spaces have attracted a lot of interest due to their appearances in other contexts such as cluster algebras [9] and the physics of scattering amplitudes [1]. Our original motivation for studying the topology of spaces arising in total positivity was to better understand the *amplituhedra* of Arkani-Hamed and Trnka [2], and more generally the Grassmann polytopes of the third author [22]. A Grassmann polytope is a generalization of a convex polytope in the Grassmannian $\mathrm{Gr}(k,n)$. For example, the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$ is a generalization of a simplex, while amplituhedra generalize cyclic polytopes [38]. The faces of a Grassmann polytope are linear projections of closures of positroid cells, and therefore it is essential to understand the topology of these closures in order to develop a theory of Grassmann polytopes.

1.2 Outline

We provide some background definitions in Section 2. We give a brief overview of our proof of Theorem 1.1 in Section 3. Finally, in Section 4 we consider three examples: the unipotent radical U_n, the complete flag variety Fl_n, and the Grassmannian $\mathrm{Gr}(k,n)$. Further details and full proofs of the results stated here appear in our preprint [12].

2 Background

In this section we review background on regular CW complex and totally nonnegative partial flag varieties. We refer to [16, 24, 26] for further details.

2.1 Regular CW complexes

Let X be a Hausdorff space. We call a finite disjoint union $X = \bigsqcup_{g \in Q} X_g$ a *regular CW complex* if it satisfies the following two properties.

(1) For each $g \in Q$, there exists a homeomorphism from the closure $\overline{X_g}$ to a closed ball $\overline{B}$ which sends X_g to the interior of $\overline{B}$.

(2) For each $g \in Q$, there exists $Q' \subset Q$ such that $\overline{X_g} = \bigsqcup_{f \in Q'} X_f$.

The *face poset* of X is the poset $(Q, \preceq)$, where $f \preceq g$ if and only if $X_f \subset \overline{X_g}$.

2.2 Totally nonnegative partial flag varieties

Let $\mathfrak{g}$ denote the Lie algebra of G over $\mathbb{R}$. We fix Chevalley generators $(e_i, f_i)_{i \in I}$ of $\mathfrak{g}$, so that the elements $h_i := [e_i, f_i]$ ($i \in I$) span the Lie algebra of a split real maximal torus T of G. For $i \in I$ and $t \in \mathbb{R}$, we define the elements of G

$$x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i).$$

We also let $\alpha_i^\vee : \mathbb{R}^* \to T$ be the homomorphism of algebraic groups whose tangent map takes $1 \in \mathbb{R}$ to h_i. The $x_i(t)$'s (respectively, $y_i(t)$'s) generate the unipotent radical of a Borel subgroup B (respectively, B_-) of G, with $B \cap B_- = T$. The data $(T, B, B_-, x_i, y_i; i \in I)$ is called a *pinning* for G.

We define the *totally nonnegative part* $G_{\geq 0}$ of G as the semigroup generated by all $x_i(t)$'s, $y_i(t)$'s, and $\alpha_i^\vee(t)$'s with $t > 0$. For a parabolic subgroup $P \supset B$, we define the totally nonnegative part $(G/P)_{\geq 0}$ of G/P as the closure of the image of $G_{\geq 0}$ inside G/P. For examples in the case $G = \mathrm{SL}_n$, see Section 4.

Rietsch [32, 31] established the decomposition

$$(G/P)_{\geq 0} = \bigsqcup_{g \in Q} \Pi_g^{>0} \tag{2.1}$$

of $(G/P)_{\geq 0}$ into open balls $\Pi_g^{>0}$ indexed by the elements g of a certain poset $(Q, \preceq)$, which is the face poset of $(G/P)_{\geq 0}$. When $(G/P)_{\geq 0}$ is the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$, this is the positroid cell decomposition of [29] (see Section 4.3).

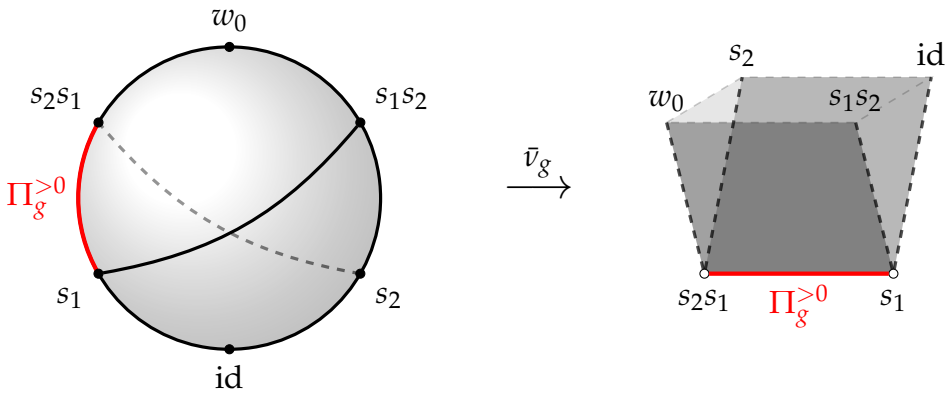


Figure 2: The map $\bar{v}_g$ for the case $G = \mathrm{SL}_3$ and $P = B$ from Example 4.2.

3 Stars, links, and the Fomin–Shapiro atlas

In this section, we outline our proof of Theorem 1.1. Given $g \in Q$, define the *star* of g in $(G/P)_{\geq 0}$ by

$$\mathrm{Star}_g^{\geq 0} := \bigsqcup_{h \succeq g} \Pi_h^{>0}.$$

We also consider the space $\mathrm{Lk}_g^{\geq 0}$ (the *link* of g), stratified as

$$\mathrm{Lk}_g^{\geq 0} = \bigsqcup_{h \succ g} \mathrm{Lk}_{g,h}^{>0}.$$

We show along the way that $\mathrm{Lk}_g^{\geq 0}$ is a regular CW complex homeomorphic to a closed ball.

At the core of our approach is a collection of (stratification-preserving) homeomorphisms

$$\bar{v}_g : \mathrm{Star}_g^{\geq 0} \xrightarrow{\sim} \Pi_g^{>0} \times \mathrm{Cone}(\mathrm{Lk}_g^{\geq 0}), \tag{3.1}$$

one for each $g \in Q$ (see Figure 2). Here $\mathrm{Cone}(A) := (A \times \mathbb{R}_{\geq 0})/(A \times \{0\})$ denotes the *open cone* over A. The homeomorphisms $\bar{v}_g$, along with dilation actions ϑ_g on the cones, are part of the data of what we call a *Fomin–Shapiro atlas*. Our construction is inspired by similar maps introduced in [7] for the unipotent radical $U_{\geq 0}$.

We also introduce the abstract notion of a *totally nonnegative space*, which captures several known combinatorial and geometric properties of $(G/P)_{\geq 0}$ used in our proof. This includes the shellability of Q due to Williams [40], and some topological results [31, 21] on Richardson varieties. We prove that every totally nonnegative space that admits a Fomin–Shapiro atlas is a regular CW complex. Our argument proceeds by induction on the dimension of $\mathrm{Lk}_{g,h}^{>0}$, and depends on a delicate interplay between objects in smooth

and topological categories. We use crucially that the maps (3.1) in a Fomin–Shapiro atlas are restrictions of smooth maps. On the topological level, we rely on the generalized Poincaré conjecture [36, 10, 28] combined with some general results on poset topology. We formulate our results in the abstract language of totally nonnegative spaces since we expect that they can be applied in other contexts, such as to totally nonnegative Kac–Moody flag varieties, totally nonnegative double Bruhat cells [8], spaces of electrical networks [23], spaces of boundary correlations of planar Ising models [14], amplituhedra [2], and the totally nonnegative part of the wonderful compactification [17].

The bulk of the proof of Theorem 1.1 is devoted to the construction of the Fomin–Shapiro atlas. For each $g \in Q$ we give an isomorphism $\bar{\varphi}_u$ between an open dense subset $\mathcal{O}_g \subset G/P$ and a certain subset of the *affine flag variety* $\mathcal{G}/\mathcal{B}$ of the loop group $\mathcal{G}$ associated with G. The map $\bar{\varphi}_u$, which we call an *affine Bruhat atlas*, sends the projected Richardson stratification [21] of G/P to the affine Richardson stratification of its image inside $\mathcal{G}/\mathcal{B}$. The hardest part of the proof consists of showing that the subset $\mathcal{O}_g \subset G/P$ contains $\mathrm{Star}_g^{\geq 0}$.

Remark 3.1. The map $\bar{\varphi}_u$ generalizes the map of Snider [37] from $\mathrm{Gr}(k,n)$ to all G/P. A different approach to give such a generalization is due to He, Knutson, and Lu [18], which led them to the notion of a *Bruhat atlas*. See [6] for the definition. Huang [20] has independently constructed a map similar to our $\bar{\varphi}_u$.

4 Examples

In this section we discuss three examples (in type A) of regular CW complexes which are addressed by Theorem 1.1. We fix $n \geq 1$, and let $[n] := \{1, 2, \ldots, n\}$. For $0 \leq k \leq n$, let $\binom{[n]}{k}$ denote the set of all k-element subsets of n. We set $G := \mathrm{SL}_n$. In the setup of Section 2.2, we may take $I := [n-1]$, where $x_i(t)$, $y_i(t)$, and $\alpha_i^\vee(t)$ are obtained from the $n \times n$ identity matrix by placing, respectively,

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \text{ in rows and columns } i, i+1.$$

Then B and B_- are the subgroups of G of upper- and lower-triangular matrices, respectively, and $G_{\geq 0}$ is the subset of G of matrices whose minors are all nonnegative.

The Weyl group is the symmetric group S_n. We let $w_0 := (i \mapsto n+1-i)$ denote the longest permutation in S_n. For $w \in S_n$, we let $\dot{w} \in G$ denote the signed permutation matrix which contains a ± 1 in row $w(k)$ and column k for each $k \in [n]$, where the signs are chosen so that the submatrix with rows $\{w(1), \ldots, w(k)\}$ and columns $[k]$ has

determinant 1. For example,

$$\dot{s}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and for } w = s_1 s_2, \text{ we have } \dot{w} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

4.1 Unipotent radical U_n

Let U_n denote the subgroup of B of all unipotent matrices, and $U_n^{\geq 0} := G_{\geq 0} \cap U_n$ its totally nonnegative part. We have the decomposition into totally positive Bruhat cells [26]

$$U_n^{\geq 0} = \bigsqcup_{w \in S_n} \Pi_w^{>0}, \quad \text{where} \quad \Pi_w^{>0} := U_n^{\geq 0} \cap B_- \dot{w} B_-.$$

The closure relation on cells is given by the strong Bruhat order.

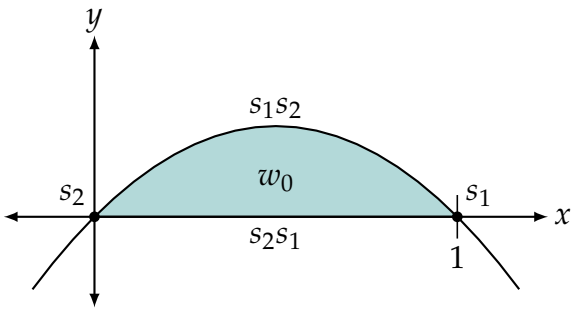
We can take the link of the identity in $U_n^{\geq 0}$ to be the subset of matrices whose $n-1$ entries immediately above the diagonal sum to 1. It has a similar cell decomposition indexed by permutations $w \in S_n$ with $w \neq \mathrm{id}$. The natural inclusion $U_n \hookrightarrow G/B_-$ allows us to identify this link with one appearing in the proof of Theorem 1.1 for G/B_-, which implies that it is a regular CW complex homeomorphic to a closed ball. This result was conjectured by Fomin and Shapiro [7] and proved by Hersh [19], all in general Lie type (cf. Corollary 1.3).

Example 4.1. Let $n = 3$. Then

$$U_3^{\geq 0} = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, xz - y \geq 0 \right\}.$$

The permutation $w_0 = s_1 s_2 s_1 = 321$ indexes the cell where $x, y, z, xz - y > 0$, while $w = s_1 s_2 = 231$ indexes the cell where $x, y, z > 0$ and $xz - y = 0$.

The link of the identity of $U_3^{\geq 0}$ consists of all matrices in $U_3^{\geq 0}$ with $x + z = 1$. We can plot this region in the xy-plane:



We see that this agrees with the regular CW complex of Figure 1.

4.2 Complete flag variety Fl_n

Taking $P = B$, we can identify G/B with the space Fl_n of *complete flags* in $\mathbb{R}^n$, i.e. the space of tuples $(V_1, \dots, V_{n-1})$, where V_k is a subspace of $\mathbb{R}^n$ of dimension k ($1 \le k \le n-1$) and $V_1 \subset \cdots \subset V_{n-1}$. (Here V_k is the subspace spanned by the first k columns of $x \in G/B$.) The totally nonnegative part $\mathrm{Fl}_n^{\ge 0}$ consists of all complete flags which can be represented by an element $x \in G$ with nonnegative initial minors:

$$\det(x_{I,[k]}) \ge 0 \quad \text{for all } 1 \le k \le n-1 \text{ and } I \in \tbinom{[n]}{k}.$$

We have the decomposition (2.1) into totally positive Richardson cells

$$\mathrm{Fl}_n^{\ge 0} = \bigsqcup_{v \le w \text{ in } S_n} \Pi_{(v,w)}^{>0}, \quad \text{where} \quad \Pi_{(v,w)}^{>0} := \mathrm{Fl}_n^{\ge 0} \cap B_- \dot{v} B \cap B \dot{w} B.$$

The dimension of $\Pi_{(v,w)}^{>0}$ is $\ell(w) - \ell(v)$. The closure relation on cells is given by containment of intervals in the strong Bruhat order:

$$(v,w) \preceq (v',w') \quad \Leftrightarrow \quad v' \le v \le w \le w'.$$

Example 4.2. Let $n = 3$. Then $\mathrm{Fl}_3^{\ge 0}$ gives a cell decomposition of a 3-dimensional ball, see Figure 2 (left). Let us illustrate the homeomorphism (3.1) for $g := (s_1, s_2 s_1)$. Here $\Pi_g^{>0}$ is an open line segment, and $\mathrm{Star}_g^{\ge 0}$ consists of 4 cells: a line segment $\Pi_g^{>0} = \Pi_{(s_1, s_2 s_1)}^{>0}$, two open square faces $\Pi_{(s_1, w_0)}^{>0}$ and $\Pi_{(\mathrm{id}, s_2 s_1)}^{>0}$, and an open 3-dimensional ball $\Pi_{(\mathrm{id}, w_0)}^{>0}$. This union is indeed homeomorphic to $\Pi_g^{>0} \times \mathrm{Cone}(\mathrm{Lk}_g^{\ge 0})$ shown in Figure 2 (right). Here $\mathrm{Lk}_g^{\ge 0}$ is a closed line segment whose endpoints are $\mathrm{Lk}_{g,(s_1,w_0)}^{>0}$ and $\mathrm{Lk}_{g,(\mathrm{id},s_2 s_1)}^{>0}$, and whose interior is $\mathrm{Lk}_{g,(\mathrm{id},w_0)}^{>0}$.

4.3 Grassmannian $\mathrm{Gr}(k,n)$

Fix $1 \le k \le n-1$, and take P to be the subset of G of all matrices whose lower-left $(n-k) \times k$ block is zero. Then we can identify G/P with the *Grassmannian* $\mathrm{Gr}(k,n)$, i.e. the space of k-dimensional subspaces of $\mathbb{R}^n$. (Here $x \in G/P$ corresponds to the k-dimensional subspace spanned by the first k columns of x.) The totally nonnegative part $\mathrm{Gr}_{\ge 0}(k,n)$ consists of all subspaces which can be represented by an $n \times k$ matrix whose $k \times k$ minors (known as *Plücker coordinates*) are all nonnegative.

In the case of $\mathrm{Gr}_{\ge 0}(k,n)$, we can describe the decomposition (2.1) in terms of *positroid cells* [29]. Namely, let $\mathrm{Bound}(k,n)$ denote the set of *bounded affine permutations*, i.e. bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(i+n) = f(i) + n \text{ and } i \le f(i) \le i+n \text{ for all } i \in \mathbb{Z}, \quad \text{and} \quad \sum_{i=1}^{n} (f(i) - i) = kn.$$

We write f in *window notation* as $[f(1), \ldots, f(n)]$. Given an element of $\mathrm{Gr}(k,n)$ represented by an $n \times k$ matrix M, we associate an element $f \in \mathrm{Bound}(k,n)$ as follows: for each $i \in [n]$, we set $f(i) \geq i$ to be minimum such that row $f(i)$ of M is in the span of rows $i, i+1, \ldots, f(i)-1$ (where indices are taken modulo n). Then the positroid cell $\Pi_f^{>0}$ is defined to be the set of all elements of $\mathrm{Gr}_{\geq 0}(k,n)$ associated to f, and

$$\mathrm{Gr}_{\geq 0}(k,n) = \bigsqcup_{f \in \mathrm{Bound}(k,n)} \Pi_f^{>0}.$$

The closure relation on cells is given by the dual of the Bruhat order on $\mathrm{Bound}(k,n)$.

Example 4.3. Let $(k,n) := (2,4)$, and take $g := [2,4,5,7] \in \mathrm{Bound}(2,4)$. Then $\Pi_g^{>0}$ consists of all elements which can be represented by a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ x_1 & 0 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix} \quad \text{with } x_1, x_3, x_4 > 0.$$

Now let us describe the map $\bar{v}_g$ from (3.1) and the dilation action ϑ_g. We have $\mathrm{Star}_g^{\geq 0} = \Pi_g^{>0} \sqcup \Pi_{[3,4,5,6]}^{>0}$, and $\mathrm{Lk}_g^{\geq 0}$ is a point, so $\mathrm{Cone}(\mathrm{Lk}_g^{\geq 0}) \simeq \mathbb{R}$.

First we must fix a subset $I \in \binom{[4]}{2}$ whose Plücker coordinate does not vanish on $\Pi_g^{>0}$. Here we may take any $I \neq \{1,2\}$; let us take $I := \{1,4\}$. This allows us to define the embedding $\bar{\varphi}_I$ of an open dense subset of $\mathrm{Gr}(2,4)$ into the affine flag variety:

$$\bar{\varphi}_I \left(\begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{z} & 0 & 0 & 0 \\ -x_1 & 1 & 0 & \frac{x_2}{z} \\ -x_3 & 0 & 1 & \frac{x_4}{z} \\ 0 & 0 & 0 & \frac{1}{z} \end{bmatrix}, \text{ where } z \text{ is the formal loop parameter.}$$

After performing some calculations in the affine flag variety and pulling back the result to $\mathrm{Gr}(2,4)$, we find that

$$\bar{v}_g \left(\begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & 0 \\ \frac{x_1 x_4 - x_2 x_3}{x_4} & 0 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix}, \frac{x_2}{x_4} \right).$$

The first component gives a projection $\mathrm{Star}_g^{\geq 0} \to \Pi_g^{>0}$. The dilation action ϑ_g is given by

$$t \cdot \begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ x_1 + (t-1)\frac{x_2 x_3}{x_4} & t x_2 \\ x_3 & x_4 \\ 0 & 1 \end{bmatrix} \quad \text{for } t > 0.$$

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