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Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials

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Abstract. We present several new and compact formulas for the modified and integral form of the Macdonald polynomials, building on the compact "multiline queue" formula for Macdonald polynomials due to Corteel, Mandelshtam and Williams. We also introduce a new quasisymmetric analogue of Macdonald polynomials. These *quasisymmetric Macdonald polynomials* refine the (symmetric) Macdonald polynomials and specialize at $q = t = 0$ to the quasisymmetric Schur polynomials defined by Haglund, Luoto, Mason, and Van Willigenburg.

Résumé. Nous présentons plusieurs formules nouvelles et compactes pour les polynômes de Macdonald modifiés et la forme intégrale des polynômes de Macdonald. Nous utilisons des idées venant des formules combinatoires pour les polynômes de Macdonald en termes de "file d'attente à plusieurs lignes" de Corteel, Mandelshtam et Williams. Nous définissons aussi des polynômes de Macdonald quasisymétriques qui sont un rafinement des polynômes de Macdonald et ont la bonne propriété quand $q = t = 0$ dêtre égaux aux fonctions de Schur quasisymétriques définies par Haglund, Luoto, Mason et Van Willigenburg.

Keywords: Macdonald polynomials, quasisymmetric polynomials, asymmetric simple exclusion process.

1 Introduction

The symmetric *Macdonald polynomials* $P_\lambda(X; q, t)$ [18] are a family of polynomials in $X = \{x_1, x_2, \dots\}$ indexed by partitions, whose coefficients depend on two parameters q

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and t. Macdonald polynomials generalize multiple important families of polynomials, including Schur polynomials and Hall-Littlewood polynomials. They can be defined as the unique monic basis for the ring of symmetric functions that satisfies certain triangularity and orthogonality conditions. The related *nonsymmetric Macdonald polynomials* $E_\mu(X;q,t)$ [17, 18, 4] first appeared shortly after the introduction of Macdonald polynomials as a tool to study Macdonald polynomials. The $E_\mu(X;q,t)$ are indexed by weak compositions and form a basis for the full polynomial ring $\mathbb{Q}[X](q,t)$.

There has been a great deal of work devoted to understanding Macdonald polynomials from a combinatorial point of view. Haglund-Haiman-Loehr [10] gave a combinatorial formula for the *integral forms* $J_\lambda(X;q,t)$, which are scalar multiples of the classical monic forms $P_\lambda(X;q,t)$. They also gave a formula for the nonsymmetric Macdonald polynomials $E_\mu(X;q,t)$ [11], and for the *transformed* or *modified* Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$, which are obtained from $J_\lambda(X;q,t)$ via *plethysm*. Macdonald conjectured and Haiman proved [13], using the geometry of the Hilbert scheme, that the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ have a positive Schur expansion whose coefficients are qt-Kostka polynomials. However, it is still an open problem to give a combinatorial proof of Schur positivity or a manifestly positive formula for the qt-Kostka polynomials.

Recently a beautiful connection has been found between Macdonald polynomials and a model from statistical mechanics called the multispecies *asymmetric simple exclusion process* (ASEP) on a circle. The ASEP is a one-dimensional exactly solvable particle model; Cantini-deGier-Wheeler [3] showed that the partition function of the multispecies ASEP on a circle is equal to a Macdonald polynomial $P_\lambda(x_1,\ldots,x_n;q,t)$ evaluated at $q = 1$ and $x_i = 1$ for all i. Building on this result as well as work of Martin [19], the first, third, and fifth authors recently used *multiline queues* to simultaneously compute the stationary probabilities of the multispecies exclusion process, and give compact formulas for the symmetric Macdonald polynomials P_λ and the nonsymmetric Macdonald polynomials E_λ [7], for any partition λ. These formulas are "compact" in that they have fewer terms than the formulas of Haglund-Haiman-Loehr.

In this paper we use the above ideas to continue the search for compact formulas for Macdonald polynomials. Our first two main results are compact formulas for the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ and the integral forms $J_\lambda(X;q,t)$; these new formulas have far fewer terms than other known combinatorial formulas. Our third main result uses the connection with the ASEP on a ring towards a different application: the introduction of a new family of quasisymmetric functions we call *quasisymmetric Macdonald polynomials* $G_\gamma(X;q,t)$. We show that $G_\gamma(X;q,t)$ is indeed a quasisymmetric function and we have a combinatorial formula for the $G_\gamma(X;q,t)$, corresponding to "pieces" of the compact formula for the P_λ from [7]. The Macdonald polynomial $P_\lambda(X;q,t)$ is a sum of these quasisymmetric Macdonald polynomials, and the quasisymmetric function $G_\gamma(X;q,t)$ at $q = t = 0$ specializes to the *quasisymmetric Schur functions* $\mathrm{QS}_\gamma(X)$ introduced by the second and fourth authors, together with Luoto and van Willigenburg [12].

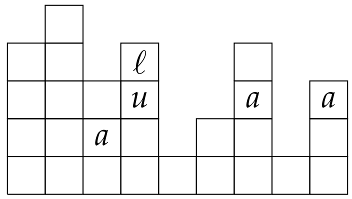

Figure 1: The diagram of the composition $(4, 5, 3, 4, 1, 2, 4, 1, 3)$ and the cells in the leg and the arm of the cell $u = (4, 3)$. Here $\mathrm{leg}(u) = 1$ and $\mathrm{arm}(u) = 3$.

The quasisymmetric Schur functions form a basis for the ring of quasisymmetric functions and until now it has been an open question to find a refinement of the Macdonald polynomials P_λ into quasisymmetric pieces which generalize the quasisymmetric Schur functions.

This paper is organized as follows. In Section 2, we provide the relevant background. Sections 3 and 4 describe our two compact formulas, and Section 5 defines our new quasisymmetric Macdonald polynomials. See the full paper [6] for the proofs of our results.

2 Definitions

We begin by introducing relevant notation and definitions. In our partition/composition diagrams, given in French notation, the columns are labeled from left to right, and the rows are labeled from bottom to top, so that the notation (i, r) refers to the box (or *cell*) in the i^{th} column from the left and the r^{th} row from the bottom. Given a partition/composition α, its diagram $\mathrm{dg}(\alpha)$ is a sequence of columns bottom justified where the i^{th} column has α_i cells. The cell (i, r) refers to the cell in row r of column i. The *leg* of a cell (i, r), denoted $\mathrm{leg}((i, r))$, equals the number of cells in column i above the cell (i, r). Analogously the *arm* of a cell (i, r), denoted $\mathrm{arm}((i, r))$, equals the number of cells in row r to the right of the cell (i, r) contained in a column whose height doesn't exceed α_i, plus the number of cells in row $r - 1$ to the left of the cell $(i, r - 1)$, contained in a column whose height is smaller than α_i. See Figure 1.

In Section 3, we only consider diagrams corresponding to partitions. Given a partition λ, a *filling* $\sigma : \mathrm{dg}(\lambda) \to \mathbb{Z}^+$ is an assignment of positive integers to the cells of $\mathrm{dg}(\lambda)$ and is denoted by σ.[1]

For $s \in \mathrm{dg}(\lambda)$, let $\sigma(s)$ denote the integer assigned to s, i.e. the integer occupying cell s. The numbers appearing in such a filling are called the *entries*. For each filling σ of $\mathrm{dg}(\lambda)$ we associate x, q, and t *weights*. The *x-weight* is defined in a similar fashion to

[1]Note that $\mathrm{dg}(\lambda)$ corresponds to the Ferrers diagram of λ' in French notation.

semistandard Young tableaux, namely

$$x^{\sigma} = \prod_{s \in \mathrm{dg}(\lambda)} x_{\sigma(s)}.$$

We recall several definitions from [10]. Assume that the diagram of a partition λ has a *basement*, i.e. a zero(th) row of size λ_1 all of whose cells are filled with the entry ∞. Let u, v, and r be positive integers with $u < v$. Given a (diagram of a) partition λ and a filling σ of λ, a *triple* consists of the three cells (if they are present in the diagram) (v, r), (u, r), and $(u, r-1)$. Let $a = \sigma(v, r)$, $b = \sigma(u, r)$, and $c = \sigma(u, r-1)$. We say that the triple is a *counterclockwise inversion triple* if any of the following holds:

$$a < b \leq c \text{ or } c < a < b \text{ or } b \leq c < a.$$

For example, in Figure 2 the entries $(3,3)$, $(1,3)$ and $(1,2)$ form a counterclockwise inversion triple. We say that the triple is a *clockwise inversion triple* if any of the following holds:

$$a > b > c \text{ or } c > a > b \text{ or } b > c > a.$$

Note that since $\sigma(j, 0) = \infty$, if $r = 1$ then $(v, 1)$ and $(u, 1)$ form a (counterclockwise) inversion triple if and only if $\sigma(u, 1) > \sigma(v, 1)$. In this case we say that the triple is *degenerate*. For example, on Figure 2 the entries $(3,1)$ and $(6,1)$ form a degenerate inversion triple. Given a filling σ, let $\mathrm{inv}(\sigma)$ be the total number of (counterclockwise) inversion triples including degenerate triples. Let $\mathrm{coinv}(\sigma)$ be the total number of triples in the filling minus $\mathrm{inv}(\sigma)$. (In other words, $\mathrm{coinv}(\sigma)$ is the number of triples which are not counterclockwise inversion triples.)

Define the *set of descents* of a filling to be

$$\mathrm{Des}(\lambda, \sigma) = \{(u, r) \in \lambda \; : \; \sigma((u, r)) > \sigma((u, r-1))\}.$$

We define the *major index* $\mathrm{maj}(\sigma)$ to be the sum over the legs of the descents of (λ, σ):

$$\mathrm{maj}(\sigma) = \sum_{x \in \mathrm{Des}(\sigma)} (\mathrm{leg}(x) + 1).$$

The tableau σ of $\mathrm{dg}(\lambda)$ in Figure 2 has $\mathrm{inv}(\sigma, \lambda) = 22$ and $\mathrm{maj}(\sigma, \lambda) = 5$. Two cells are *attacking* if their entries are equal and the cells are either in the same row, or they are in adjacent rows, with the rightmost cell in a row strictly below the other cell. A filling is *non-attacking* if it does not contain any attacking pairs of cells.

Given a *weak composition*, i.e. a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers, we let $\mathrm{inc}(\alpha)$ and $\mathrm{dec}(\alpha)$ be the vectors obtained from α by sorting the parts in weakly increasing order, and weakly decreasing order, respectively. Let $\beta(\alpha)$ be the permutation in S_n of maximal length with the property that β applied to the vector α yields $\mathrm{inc}(\alpha)$. Let α^+ be the *strong composition* obtained from α by removing the zeros and let $\ell(\alpha)$ be the number of parts of α^+. For example, if $\alpha = (0, 2, 0, 2, 1, 3)$ then $\mathrm{inc}(\alpha) = (0, 0, 1, 2, 2, 3)$, $\mathrm{dec}(\alpha) = (3, 2, 2, 1, 0, 0)$, $\beta(\alpha) = (3, 1, 5, 4, 2, 6)$, $\alpha^+ = (2, 2, 1, 3)$, and $\ell(\alpha) = 4$.

row 5	6	6	6						
row 4	1	1	1						
row 3	5	5	2						
row 2	5	5	6	6	6				
row 1	9	9	9	1	3	2	2	3	3
row 0	∞	∞	∞	∞	∞	∞	∞	∞	∞

Figure 2: A sorted tableau σ, with $\operatorname{perm}_t(\sigma) = \binom{3}{2,1}_t \binom{2}{1,1}_t \binom{4}{2,2}_t$.

3 Compact formula for modified Macdonald polynomials

Our first main result is a "compact" formula for the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$. Before explaining our result, we first recall the combinatorial formula of Haiman, Haglund, and Loehr [10].

Theorem 3.1 ([10])**.** *The modified Macdonald polynomial $\widetilde{H}_\lambda(X;q,t)$ is given by*

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma\,:\,\mathrm{dg}(\lambda')\to\mathbb{Z}^+} x^\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)}.$$

(Below we will be using the alternate form of this identity implied by the well-known fact that $\widetilde{H}_\lambda(X;q,t) = \widetilde{H}_{\lambda'}(X;t,q)$.)

While Theorem 3.1 is simple and elegant, it has the disadvantage of having many terms, since it is a sum over *all* fillings of a diagram by positive integers. By contrast, our compact formula (Theorem 3.4) is a sum over far fewer terms – it is a sum over *sorted tableaux*. To define these sorted tableaux, we first define an order on the columns of the fillings.

Definition 3.2. Fix a filling σ of $\mathrm{dg}(\lambda)$ and consider two columns A and B of height j in λ. Let $a_1,\dots,a_j$ and $b_1,\dots,b_j$ be the entries of columns A and B, respectively, read from bottom to top. We say that $A \lhd B$, if either $a_1 < b_1$, or $a_i = b_i$ for $i = 1,2,\dots,h-1$ (for some positive h), and the cells containing b_h, a_h and a_{h-1} do not form an inversion triple.

Definition 3.3. Given a filling σ of $\mathrm{dg}(\lambda)$, we say that σ is a *sorted tableau* if, for all positive integers h, when we read all columns of height h from left to right, the columns appear in weakly increasing order with respect to $\lhd$. We write $\mathrm{ST}(\lambda)$ for the set of all sorted fillings of $\mathrm{dg}(\lambda)$.

Let σ be a sorted tableau. First suppose that the diagram of λ is an $m \times n$ rectangle. The n columns may not all have distinct fillings: suppose that among those n columns, there are j distinct column fillings, with u_1 identical columns of the first filling, u_2 identical columns of the second filling, …, u_j identical columns all containing the jth filling.

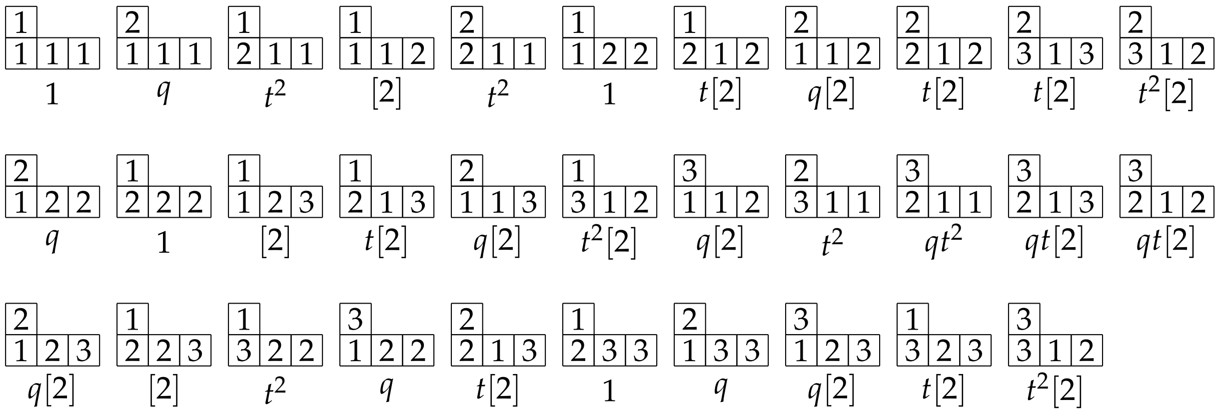

Figure 3: We compute $\widetilde{H}_{(2,1,1)}(x_1, x_2, x_3; q, t)$ by adding the weights of the sorted fillings of $\mathrm{dg}((2,1,1))$. In the figure above, we've listed $t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \mathrm{perm}_t(\sigma)$ below each tableau, but have omitted x^σ to save space. Here $[i]$ denotes $[i]_t$.

Define

$$\mathrm{perm}_t(\sigma) = \binom{n}{u_1, \ldots, u_j}_t.$$

Suppose σ is a sorted tableau which is a concatenation of rectangular sorted tableaux $\sigma_1, \ldots, \sigma_\ell$, all of different heights. Define $\mathrm{perm}_t(\sigma) = \prod_{i=1}^{\ell} \mathrm{perm}_t(\sigma_i)$. (See Figure 2.)

Our main result in this section is a compact formula for $\widetilde{H}_{\lambda'}$.

Theorem 3.4. *The modified Macdonald polynomial $\widetilde{H}_{\lambda'}(X; q, t)$ equals*

$$\widetilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \mathrm{ST}(\lambda)} x^\sigma t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \mathrm{perm}_t(\sigma),$$

where the sum is over all sorted fillings of $\mathrm{dg}(\lambda)$.

Example 3.5. We use the theorem to compute $\widetilde{H}_{2,1,1}(x_1, x_2, x_3; q, t)$ in Figure 3.

To prove Theorem 3.4, we define *inversion flip operators* $\mathcal{T}_i$ which act on fillings. These operators fix the maj statistic and change the inv statistic by one; in other words, they change the number of counterclockwise inversion triples by one. Our operators are a generalization of the *inversion flip* move introduced by Loehr and Niese [16] to prove two-column recursions for Macdonald polynomials.

Starting from a sorted tableau σ, we generate a family $\mathcal{F}(\sigma)$ of fillings by sequentially applying the operators $\mathcal{T}_i$. We then show that

$$\sum_{\tau \in \mathcal{F}(\sigma)} x^\tau t^{\mathrm{inv}(\tau)} q^{\mathrm{maj}(\tau)} = x^\sigma t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \mathrm{perm}_t(\sigma)$$

and that $\bigsqcup_{\sigma \in \mathrm{ST}(\lambda)} \{\mathcal{F}(\sigma)\} = \bigcup_{\tau : \lambda \to \mathbb{Z}^+} \tau$, thus completing the proof.

4 A compact formula for integral Macdonald polynomials

In this section we provide a compact formula for the integral form Macdonald polynomials $J_\mu(X; q; t)$. We use notational conventions from [9, Appendix A]; in particular, the reader should consult that source for the definition of type A and type B inversion and coinversion triples. Recall that for any composition α, $\mathrm{dg}(\alpha)$ refers to the diagram with α_i squares in column i. We first recall the formula for $J_\mu(X; q, t)$ from [10].

Theorem 4.1 ([10])**.** *The integral form Macdonald polynomial is given by*

$$J_\mu(X; q, t) = (1-t)^{\ell(\mu)} \sum_{\textit{nonattacking fillings } \sigma \textit{ of } \mathrm{dg}(\mu)} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \tag{4.1}$$
$$\times \prod_{\substack{s \in \mathrm{dg}(\mu) \\ s \textit{ not in row } 1 \\ \sigma(s) = \sigma(\mathrm{South}(s))}} (1 - q^{leg(s)+1} t^{\mathrm{arm}(s)+1}) \prod_{\substack{s \in \mathrm{dg}(\mu) \\ s \textit{ not in row } 1 \\ \sigma(s) \neq \sigma(\mathrm{South}(s))}} (1-t),$$

where $\ell(\mu)$ is the number of parts of μ, and for a square s not in row 1, $\mathrm{South}(s)$ *denotes the square directly below s in the same column as s. Here the sum is over all nonattacking filings of* $\mathrm{dg}(\mu)$ *(there is no basement in these fillings).*

In [10] the authors note that the right-hand-side of (4.1) actually yields a correct formula for J_μ if we replace μ everywhere by α, where α is any weak composition of n into n parts satisfying $\mathrm{dec}(\alpha) = \mu$. (See [9, Appendix A] for details about how to extend the notion of coinv to increasing column heights using type B triples.) In fact, the most efficient formula for J_μ seems to be the case where $\alpha = \mathrm{inc}(\mu)$.

One unpleasant feature of all these formulas for J_μ is that for the special case $\mu = 1^n$,

$$J_{1^n}(X; q, t) = x_1 x_2 \cdots x_n (1-t)(1-t^2) \cdots (1-t^n), \tag{4.2}$$

while the formula from (4.1) reduces to

$$x_1 x_2 \cdots x_n (1-t)^n \sum_{\sigma \in S_n} t^{\mathrm{coinv}(\sigma)}, \tag{4.3}$$

a sum of $n!$ terms. In this section we show how the identity

$$P_\mu(X; q, t) = \sum_{\alpha:\ \mathrm{dec}(\alpha) = \mu} E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X; q, t), \tag{4.4}$$

yields a corresponding formula for J_μ which, when applied to the case $\mu = 1^n$, has only one term - identity (4.2). We mention that (4.4) is closely related to Lenart's formula for P_μ [15] (which he proved under the additional assumption that μ has distinct parts).

				3
	5	1	2	7
4	5	3	1	6

Figure 4: An ordered nonattacking filling of $\mathrm{dg}((0,0,1,2,2,2,3))$ with $\mathrm{maj} = 3$ and $\mathrm{coinv} = 7$.

Definition 4.2. Let α be a weak composition of n into n parts such that $\mathrm{inc}(\alpha) = \alpha$. We say a nonattacking filling σ of $\mathrm{dg}(\alpha)$ (with or without a basement) is *ordered* if in the bottom row of $\mathrm{dg}(\alpha)$, entries of σ below columns of the same height are strictly decreasing when read left to right. We call $\mathrm{OF}(\alpha)$ the set of ordered, nonattacking fillings of $\mathrm{dg}(\alpha)$.

Figure 4 shows an ordered nonattacking filling. The 7 coinversion triples for the filling in Figure 4 are $(1,6,7), (3,6,7), (5,6,7), (6,7,9), (1,2,3), (1,2,9), (3,5,7)$.

Recall that Macdonald's definition of the integral form $J_\mu(X;q,t)$ is

$$J_\mu(X;q,t) = P_\mu(X;q,t)\mathrm{PR1}(\mu), \tag{4.5}$$

where

$$\mathrm{PR1}(\mu) = \prod_{s\in \mathrm{dg}(\mu')} (1 - q^{\mathrm{arm}(s)}t^{\mathrm{leg}(s)+1}) = \prod_{s\in \mathrm{dg}(\mu)} (1 - q^{\mathrm{leg}(s)}t^{\mathrm{arm}(s)+1}). \tag{4.6}$$

For a composition α, define $m_i(\alpha)$ to be the number of times i occurs in α for $i \geq 1$.

Proposition 4.3. *For α a weak composition, define*

$$PR2(\alpha) = \prod_{i\geq 1}(t;t)_{m_i} \prod_{\substack{s\in \mathrm{dg}(\mathrm{inc}(\alpha)) \\ s \text{ not in the bottom row}}} (1 - q^{leg(s)+1}t^{\mathrm{arm}(s)+1}), \tag{4.7}$$

where $m_i = m_i(\alpha)$, and the statistics arm *and* leg *are defined as in Section 2. Then if μ is any partition, $PR1(\mu) = PR2(\mathrm{inc}(\mu))$.*

Definition 4.4. Given a composition α of n into n parts, we define the *integral form* version of $E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$ as

$$\mathcal{E}^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t) = PR2(\mathrm{inc}(\alpha))E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t). \tag{4.8}$$

Recall the following combinatorial formula for $E^{\sigma}_{\alpha}(X;q,t)$ in [1];

$$E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t) = \sum_{\sigma} x^\sigma \mathrm{wt}(\sigma), \tag{4.9}$$

where the sum is over all nonattacking fillings σ of $\mathrm{dg}(\alpha)$ with basement $\beta(\alpha)$. The weight of such a filling is

$$\mathrm{wt}(\sigma) = q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_{s:\ \sigma(s) \neq \sigma(\mathrm{South}(s))} \frac{1-t}{1-q^{\mathrm{leg(s)}+1} t^{\mathrm{arm(s)}+1}}. \tag{4.10}$$

For the following, we set $m_i = m_i(\alpha)$. It follows from the above formula that $\mathcal{E}^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$ is $\prod_i (t;t)_{m_i}$ times an element of $\mathbb{Z}[x_1,\ldots,x_n,q,t]$. To see this, note that every nonattacking filling σ of $\mathrm{inc}(\alpha)$ has the property that each entry in the bottom row is equal to the entry in the basement directly below it, and hence doesn't contribute anything to the product in (4.10) defining $\mathrm{wt}(\sigma)$, while if any entry above the bottom row satisfies $\sigma(s) \neq \sigma(\mathrm{South}(s))$, then the associated factor $(1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1})$ is the exact term in the coefficient of $E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$ from (4.8) above corresponding to square s. In fact this argument shows that

$$\mathcal{E}^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t) = \prod_i (t;t)_{m_i} \sum_{\substack{\sigma \in \mathrm{OF}(\mathrm{inc}(\alpha)) \\ \text{with basement } \beta(\alpha)}} x^{\sigma} q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \tag{4.11}$$
$$\times \prod_{\substack{s \in \mathrm{dg}(\mathrm{inc}(\alpha)),\ s \text{ not in row } 1 \\ \sigma(s) = \sigma(\mathrm{South}(s))}} (1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1}) \prod_{\substack{s \in \mathrm{dg}(\mathrm{inc}(\alpha)),\ s \text{ not in row } 1 \\ \sigma(s) \neq \sigma(\mathrm{South}(s))}} (1-t),$$

Corollary 4.5. *The formula for J_μ has the following more compact version.*

$$J_\mu(X;q,t) = \prod_i (t;t)_{m_i} \sum_{\sigma \in \mathrm{OF}(\mu)} x^{\sigma} q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \tag{4.12}$$
$$\times \prod_{\substack{s \in \mathrm{dg}(\mu),\ s \text{ not in row } 1 \\ \sigma(s) = \sigma(\mathrm{South}(s))}} (1-q^{leg(s)+1}t^{\mathrm{arm}(s)+1}) \prod_{\substack{s \in \mathrm{dg}(\mu),\ s \text{ not in row } 1 \\ \sigma(s) \neq \sigma(\mathrm{South}(s))}} (1-t),$$

where $m_i = m_i(\mu)$.

Proof. Start by multiplying both sides of (4.4) by $\mathrm{PR2}(\mathrm{inc}(\mu))$. The left-hand-side then becomes $J_\mu(X;q,t)$ by (4.5) and Proposition 4.3. The summand on the right hand side becomes $\mathcal{E}^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$, which by (4.11) equals the portion of (4.12) which has bottom row determined by $\beta(\alpha)$. □

Note that (4.12) implies the (as far as we know) new fact that $J_\mu(X;q,t)$ is $\prod_i (t;t)_{m_i}$ times an element of $\mathbb{Z}[x_1,\ldots,x_n,q,t]$.

5 A quasisymmetric Macdonald polynomial

Recall that the ring of *quasisymmetric functions* is a graded ring which contains within it the ring of symmetric functions. The ring of quasisymmetric functions has multiple distinguished bases, indexed by (strong) compositions. One such basis is the *quasisymmetric Schur functions* $\mathrm{QS}_\gamma(X)$ introduced by the second and fourth authors, together with Luoto and van Willigenburg [12]. The authors showed that $\mathrm{QS}_\gamma(X)$ is quasisymmetric, and that each (symmetric) Schur function $s_\lambda(X)$ is a positive sum of quasisymmetric Schur functions. In light of this, and the fact that Macdonald polynomials expand positively in terms of Schur polynomials, it is natural to ask if there is a notion of a *quasisymmetric Macdonald polynomial* $G_\gamma(X;q,t)$ such that:

(A) The symmetric Macdonald polynomial $P_\lambda(X;q,t)$ is a positive sum of quasisymmetric Macdonald polynomials;

(B) $G_\gamma(X;q,t)$ is quasisymmetric;

(C) $G_\gamma(X;0,0)$ is the quasisymmetric Schur function $\mathrm{QS}_\gamma(X)$.

(D) $G_\gamma(X;q,t)$ has a combinatorial formula along the lines of the "HHL" formula for the E_α [11], or its compact version from [7].

We show in this section that the answer to this question is yes.

Given a permutation $\tau \in S_n$, let $E^\tau_\alpha(X;q,t)$ be the permuted-basement nonsymmetric Macdonald polynomial defined in [8] and studied in [1, 7], and let $F_\alpha(X;q,t) = E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$. For any partition λ of n, from [7] we have that

$$P_\lambda(X;q,t) = \sum_{\alpha:\ \mathrm{dec}(\alpha)=\lambda} F_\alpha(X;q,t), \tag{5.1}$$

where the sum is over all weak compositions α whose positive parts are a rearrangement of the parts of λ.

Note that if $\mathrm{id} = (1,2,\ldots,n)$ and $w_0 = (n,n-1,\ldots,1)$ are the identity permutation and permutation of maximal length in S_n, respectively, then $E^{\mathrm{id}}_\alpha(X;0,0)$ is the Demazure atom and $E^{w_0}_\alpha(X;0,0)$ the Demazure character. (In the common notation for Demazure characters, i.e. key polynomials, one reverses the vector α, i.e. the key polynomial corresponding to α would be $E^{w_0}_{(\alpha_n,\ldots,\alpha_1)}(X;0,0)$.)

Motivated by (5.1), we have the following definition and theorem.

Theorem 5.1. *We define the* quasisymmetric Macdonald polynomial $G_\gamma(X;q,t)$ *to be*

$$G_\gamma(X;q,t) = \sum_{\alpha:\ \alpha^+=\gamma} F_\alpha(X;q,t) \tag{5.2}$$

$$= \sum_{\alpha:\ \alpha^+=\gamma} E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t), \tag{5.3}$$

where the sum is over all weak compositions α for which $\alpha^+ = \gamma$. Then $G_\gamma(X;q,t)$ satisfies properties (A), (B), (C), and (D).

The fact that $G_\gamma(X;q,t)$ satisfies (A) follows from (5.1). There are several combinatorial proofs that $G_\gamma(X;q,t)$ is quasisymmetric and hence satisfies (B). One proof uses Equation (4.9) and a notion of *packed* nonattacking fillings. Another proof uses the multiline queues from [7]. To see that $G_\gamma(X;q,t)$ satisfies (C), recall that for γ a strong composition of n, QS_γ is defined by the equation

$$\mathrm{QS}_\gamma(X) = \sum_{\alpha:\ \alpha^+=\gamma} E^{\mathrm{id}}_\alpha(X;0,0). \tag{5.4}$$

So to verify (C), it suffices to show that $F_\alpha(X;0,0) = E^{\mathrm{id}}_\alpha(X;0,0)$. We actually show the stronger statement that

$$F_\alpha(X;0,t) = E^{\mathrm{id}}_\alpha(X;0,t), \tag{5.5}$$

where $E^{\mathrm{id}}_\alpha(X;0,t)$ is the Demazure t-atom. To prove this, one can use induction, together with the action of the Hecke operators. In particular, by [5], $T_iF_\alpha = F_{s_i\alpha}$ if $\alpha_i > \alpha_{i+1}$; compare with [1, Corollary 26].

It would be interesting to find a connection between the quasisymmetric Macdonald polynomials $G_\gamma(X;q,t)$ that we introduce in this paper, and other objects in the literature. We note that our G_γ are different from the duals of the noncommutative symmetric function analogues of Macdonald polynomials introduced in [2]; we also do not see a connection to the noncommutative Hall-Littlewood polynomials studied in [14].

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