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# Eulerian polynomials and excedance statistics via continued fractions

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**Abstract.** It is well known that the permutation peak polynomials and descent polynomials are connected via a quadratic transformation. By rephrasing the latter formula with permutation cycle peaks and excedances we are able to prove a series of general formulas expressing polynomials counting permutations by various excedance statistics in terms of refined Eulerian polynomials.

**Résumé.** Il est bien connu que les polynômes de pic de permutations et les polynômes de descente sont connectés via une transformation quadratic. En reformulant cette dernière formule avec les pics de cycle et les excédances du cycle de permutation, nous pouvons prouver une série de formules générales exprimant des polynômes énumératives des permutations par diverses statistiques d'excédance en termes de polynômes eulériens raffinés.

**Keywords:** Eulerian polynomials, peak polynomials, Gamma-positivity, $q$-Narayana polynomials, continued fractions, Laguerre histories, Françon–Viennot bijection, Foata–Zeilberger bijection.

## 1 Introduction

The Eulerian polynomials $A_n(t)$ can be defined through the continued fraction expansion [16]

$$\sum_{n\geq 0} A_n(t)z^n = 1/1 - 1\cdot z/1 - t\cdot z/1 - 2\cdot z/1 - 2t\cdot z/1 - 3\cdot z/1 - 3t\cdot z/1 - \dots \quad (1.1)$$

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For an $n$-permutation $\sigma := \sigma(1)\sigma(2)\cdots\sigma(n)$ of the word $1\ldots n$, an index $i$ $(1 \le i \le n-1)$ is a *descent* (resp. *excedance*) of $\sigma$ if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). It is well-known [7, 13] that

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{des}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{exc}\,\sigma}, \tag{1.2}$$

where $\mathfrak{S}_n$ is the set of $n$-permutations and $\mathsf{des}\,\sigma$ (resp. $\mathsf{exc}\,\sigma$) denotes the number of descents (resp. excedances) of $\sigma$. The value $\sigma(i)$ $(2 \le i \le n-1)$ is a *peak* of $\sigma$ if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and the *peak polynomials* are defined by

$$P_n^{\mathsf{pk}}(x) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathsf{pk}'\,\sigma} \tag{1.3}$$

where $\mathsf{pk}'\,\sigma$ denotes the number of peaks of $\sigma$. The peak polynomials are related to the Eulerian polynomials by Stembridge's identity, see [2, 18].

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n-1} P_n^{\mathsf{pk}}\left(\frac{4t}{(1+t)^2}\right), \tag{1.4}$$

which can be used to compute the peak polynomials. Obviously (1.4) is equivalent to the so-called $\gamma$-expansion of Eulerian polynomials

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} 2^{2k+1-n}\gamma_{n,k}t^k(1+t)^{n-1-2k}, \tag{1.5}$$

where $\gamma_{n,k}$ is the number of $n$-permutations with $k$ peaks. In the form of (1.5) it is not difficult to see that Stembridge's formula (1.4) is actually equivalent to a formula of Foata and Schüzenberger [7, Théorème 5.6] via Brändén's modified Foata-Strehl action (cf. [2]). In the last two decades, many people studied the refinements of Stembridge's identity, see Brändén [2], Shin and Zeng [14, 15], Zhuang [18], Athanasiadis [1] and the references therein. In particular, Zhuang [18] has proved several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials, extending results of Stembridge, Petersen, and Brändén.

In this paper we shall prove generalizations of Stembridge's formula using excedance statistics by further exploiting the continued fraction technique in [14, 15]. Our main tool is the combinatorial theory of continued fractions due to Flajolet [6] and bijections due to Françon-Viennot, Foata–Zeilberger between permutations and *Laguarre histories*, see [9, 8, 6, 4]. As in [14] this approach uses both linear and cycle statistics on permutations.

This extended abstract is a summary of the recent paper [11]. In Section 2 we introduce the work in [14, 15], and construct a bijection $\psi$, which is an analogue of Foata–Zeilberger's bijection from $\mathfrak{S}_{n+1}$ to $\mathcal{LH}_n$. In Section 3 we present two analogues of (1.4) using excedance statistics for permutations.

# 2 Background and preliminaries

## 2.1 Permutation statistics and two bijections

For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention 0–0, i.e., $\sigma(0) = \sigma(n+1) = 0$, a value $\sigma(i)$ $(1 \le i \le n)$ is called

- a *peak* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;
- a *valley* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double ascent* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double descent* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) > \sigma(i+1)$.

The set of Peaks (resp. Valleys, double ascents, double descents) of $\sigma$ is denoted by

$$\mathsf{Pk}\sigma \quad (\text{resp. } \mathsf{Val}\sigma,\ \mathsf{Da}\sigma,\ \mathsf{Dd}\sigma).$$

Let $\mathsf{pk}\,\sigma$ (resp. $\mathsf{val}\,\sigma$, $\mathsf{da}\,\sigma$, $\mathsf{dd}\,\sigma$) be the number of peaks (resp. valleys, double ascents, double descents) of $\sigma$. For $i \in [n] := \{1, \dots, n\}$, we introduce the following statistics:

$$\begin{aligned}
(31\text{-}2)_i\,\sigma &= \#\{j : 1 < j < i \text{ and } \sigma(j) < \sigma(i) < \sigma(j-1)\}\\
(2\text{-}31)_i\,\sigma &= \#\{j : i < j < n \text{ and } \sigma(j+1) < \sigma(i) < \sigma(j)\}\\
(2\text{-}13)_i\,\sigma &= \#\{j : i < j < n \text{ and } \sigma(j) < \sigma(i) < \sigma(j+1)\}\\
(13\text{-}2)_i\,\sigma &= \#\{j : 1 < j < i \text{ and } \sigma(j-1) < \sigma(i) < \sigma(j)\}
\end{aligned} \tag{2.1}$$

and define (see (2.33)):

$$(31\text{-}2) = \sum_{i=1}^{n} (31\text{-}2)_i, \quad (2\text{-}31) = \sum_{i=1}^{n} (2\text{-}31)_i, \quad (2\text{-}13) = \sum_{i=1}^{n} (2\text{-}13)_i, \quad (13\text{-}2) = \sum_{i=1}^{n} (13\text{-}2)_i.$$

Now, we consider $\sigma \in \mathfrak{S}_n$ as a bijection $i \mapsto \sigma(i)$ for $i \in [n]$, a value $x = \sigma(i)$ is called a *cyclic peak* if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$; a *cyclic valley* if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$; a *double excedance* if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$; a *double drop* if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$; a *fixed point* if $x = \sigma(x)$. We say that $i \in [n-1]$ is an *ascent* of $\sigma$ if $\sigma(i) < \sigma(i+1)$ and that $i \in [n]$ is a *drop* of $\sigma$ if $\sigma(i) < i$. Let Cpk (resp. Cval, Cda, Cdd, Fix, Asc, Drop) be the set of *cyclic peaks* (resp. *cyclic valleys*, *double excedances*, *double drops*, *fixed points*, *ascents*, *drops*) and denote the corresponding cardinality by $\mathsf{cpk}$ (resp. $\mathsf{cval}$, $\mathsf{cda}$, $\mathsf{cdd}$, $\mathsf{fix}$, $\mathsf{asc}$, $\mathsf{drop}$). Moreover, define

$$\mathsf{cros}_i\,\sigma = \#\{j :\ j < i < \sigma(j) < \sigma(i) \quad \text{or} \quad \sigma(i) < \sigma(j) \le i < j\}, \tag{2.2}$$

$$\mathsf{nest}_i\,\sigma = \#\{j :\ j < i < \sigma(i) < \sigma(j) \quad \text{or} \quad \sigma(j) < \sigma(i) \le i < j\}. \tag{2.3}$$

Let $\mathsf{cros}$[1] $= \sum_{i=1}^{n} \mathsf{cros}_i$ and $\mathsf{icr}\,\sigma = \mathsf{cros}\,\sigma^{-1}$. Define $\mathsf{nest} = \sum_{i=1}^{n} \mathsf{nest}_i$. Note (cf. [10, Remark 2.4]) that

$$\mathsf{nest}\,\sigma^{-1} = \mathsf{nest}\,\sigma \quad \text{for} \quad \sigma \in \mathfrak{S}_n. \tag{2.4}$$

A pair of integers $(i, j)$ is an *inversion* of $\sigma \in \mathfrak{S}_n$ if $i < j$ and $\sigma(i) > \sigma(j)$, and $\sigma(i)$ (resp. $\sigma(j)$) is called *inversion top* (resp. *bottom*). Let $\mathsf{inv}\,\sigma$ be the inverion number of $\sigma$.

For $\sigma \in \mathfrak{S}_n$ with convention 0–$\infty$, i.e., $\sigma(0) = 0$ and $\sigma(n+1) = \infty$, let $\mathsf{Lpk}$ (resp. $\mathsf{Lval}$, $\mathsf{Lda}$, $\mathsf{Ldd}$) be the set of *peaks* (resp. *valleys*, *double ascents* and *double decents*) and denote the corresponding cardinality by $\mathsf{lpk}$ (resp. $\mathsf{lval}$, $\mathsf{lda}$ and $\mathsf{ldd}$). For $i \in [n]$, the value $\sigma(i)$ is called a *left-to-right maximum* if $\sigma(i) = \max\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ A double ascent $\sigma(i)$ $(i = 1, \ldots, n)$ is called a *foremaximum* of $\sigma$ if it is at the same time a left-to-right maximum Denote the number of foremaxima of $\sigma$ by $\mathsf{fmax}\,\sigma$. Note that for the peak number $\mathsf{pk}'$ in (1.3) we have following identities :

$$\mathsf{pk}' = \mathsf{val} = \mathsf{pk} - 1 \quad \text{and} \quad \mathsf{lval} = \mathsf{lpk}. \tag{2.5}$$

Now we recall two bijections $\Phi$ and $\Psi$ due to Clarke et al. [4] and Shin–Zeng [14], respectively.

## 2.2 The bijection $\Phi$

Let $\sigma = \sigma(1) \ldots \sigma(n) \in \mathfrak{S}_n$, an *inversion top number* (resp. *inversion bottom number*) of a letter $x := \sigma(i)$ in the word $\sigma$ is the number of occurrences of inversions of form $(i, j)$ (resp $(j, i)$). A letter $\sigma(i)$ is a *descent top* (resp. *descent bottom*) if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i-1) > \sigma(i)$). Given a permutation $\sigma$, we first construct two biwords, $\binom{f}{f'}$ and $\binom{g}{g'}$, where $f$ (resp. $g$) is the subword of descent bottoms (resp. nondescent bottoms) in $\sigma$ ordered increasingly, and $f'$ (resp. $g'$) is the permutation of descent tops (resp. nondescent tops) in $\sigma$ such that the inversion bottom (resp. top) number of each letter $x := \sigma(i)$ in $f'$ (resp. $g'$) is $(2\text{-}31)_x\sigma$, and then form the biword $w = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ by concatenating $f$ and $g$, and $f'$ and $g'$, respectively.

Rearranging the columns of $w$, so that the bottom row is in increasing order, we obtain the permutation $\tau = \Phi(\sigma)$ as the top row of the rearranged bi-word.

The following result can be found in [14, Theorem 12] and its proof.

**Lemma 2.1** (Shin–Zeng)**.** *For $\sigma \in \mathfrak{S}_n$, we have*

$$\begin{aligned}
&(2\text{-}31, 31\text{-}2, \mathsf{des}, \mathsf{asc}, \mathsf{lda} - \mathsf{fmax}, \mathsf{ldd}, \mathsf{lval}, \mathsf{lpk}, \mathsf{fmax})\sigma \\
=&(\mathsf{nest}, \mathsf{icr}, \mathsf{drop}, \mathsf{exc} + \mathsf{fix}, \mathsf{cda}, \mathsf{cdd}, \mathsf{cval}, \mathsf{cpk}, \mathsf{fix})\Phi(\sigma) \\
=&(\mathsf{nest}, \mathsf{cros}, \mathsf{exc}, \mathsf{drop} + \mathsf{fix}, \mathsf{cdd}, \mathsf{cda}, \mathsf{cval}, \mathsf{cpk}, \mathsf{fix})(\Phi(\sigma))^{-1},
\end{aligned} \tag{2.6}$$

[1] Our definition of $\mathsf{cros}$ corresponds to $\mathsf{icr}$ in [10].

$$(\mathsf{Lval}, \mathsf{Lpk}, \mathsf{Lda}, \mathsf{Ldd})\sigma = (\mathrm{Cval}, \mathrm{Cpk}, \mathrm{Cda} \cup \mathrm{Fix}, \mathrm{Cdd})\Phi(\sigma), \tag{2.7}$$

$$(2\text{-}31)_i \sigma = \mathsf{nest}_i \Phi(\sigma) \quad \forall i = 1, \ldots, n. \tag{2.8}$$

## 2.3 The bijection $\Psi$

Given a permutation $\sigma \in \mathfrak{S}_n$, let

$$\hat{\sigma} = \begin{pmatrix} 1 & 2 & \ldots & n & n+1 \\ \sigma(1)+1 & \sigma(2)+1 & \ldots & \sigma(n)+1 & 1 \end{pmatrix}, \tag{2.9}$$

and $\tau := \Phi(\hat{\sigma}) \in \mathfrak{S}_{n+1}$. Since the last element of $\hat{\sigma}$ is 1, the first element of $\tau$ should be $n+1$. Define the bijection $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\Psi(\sigma) := \tau(2) \ldots \tau(n+1) \in \mathfrak{S}_n. \tag{2.10}$$

## 2.4 The star variation

For $\sigma = \sigma(1) \cdots \sigma(n) \in \mathfrak{S}_n$, we define its *star compagnon* $\sigma^*$ as a permutation of $\{0, \ldots, n\}$ by

$$\sigma^* = \begin{pmatrix} 0 & 1 & 2 & \ldots & n \\ n & \sigma(1)-1 & \sigma(2)-1 & \ldots & \sigma(n)-1 \end{pmatrix}. \tag{2.11}$$

We define the following sets of cyclic star statistics for $\sigma$:

$$\mathrm{Cpk}^*\,\sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) < i > \sigma^*(i)\}, \tag{2.12}$$

$$\mathrm{Cval}^*\,\sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) > i < \sigma^*(i)\}, \tag{2.13}$$

$$\mathrm{Cda}^*\,\sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) < i < \sigma^*(i)\}, \tag{2.14}$$

$$\mathrm{Cdd}^*\,\sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) > i > \sigma^*(i)\}, \tag{2.15}$$

$$\mathrm{Fix}^*\,\sigma = \{i \in [n-1] : i = \sigma^*(i)\}, \tag{2.16}$$

$$\mathrm{Wex}^*\sigma = \{i \in [n-1] : i \leq \sigma^*(i)\} (= \mathsf{exc}\,\sigma), \tag{2.17}$$

$$\mathsf{Drop}^*\sigma = \{i \in [n] : i > \sigma^*(i)\}. \tag{2.18}$$

The corresponding cardinalties are denoted by $\mathsf{cpk}^*$, $\mathsf{cval}^*$, $\mathsf{cda}^*$, $\mathsf{cdd}^*$, $\mathsf{fix}^*$, $\mathsf{wex}^*$ and $\mathsf{drop}^*$, respectively.

By (2.12), (2.15) and (2.18), we have $\mathsf{drop}^* - 1 = \mathsf{cdd}^* + \mathsf{cpk}^*$.

**Theorem 2.2.** *For $\sigma \in \mathfrak{S}_n$, we have*

$$(\mathsf{Val}, \mathsf{Pk} \setminus \{n\}, \mathsf{Da}, \mathsf{Dd})\sigma = (\mathrm{Cval}^*, \mathrm{Cpk}^*, \mathrm{Cda}^* \cup \mathrm{Fix}^*, \mathrm{Cdd}^*)\Psi(\sigma) \tag{2.19}$$

*and*

$$((2\text{-}13)_i,\ (31\text{-}2)_i)\sigma = (\mathsf{nest}_i,\ \mathsf{cros}_i)\Psi(\sigma) \quad \textit{for} \quad i \in [n]. \tag{2.20}$$

Since $\mathsf{asc} = \mathsf{val} + \mathsf{da}$, $\mathsf{des} = \mathsf{pk} + \mathsf{dd} - 1$, $\mathsf{wex}^* = \mathsf{cval}^* + \mathsf{cda}^* + \mathsf{fix}^*$, $\mathsf{drop}^* - 1 = \mathsf{cdd}^* + \mathsf{cpk}^*$, we get the following result in [14, Theorem 12].

**Corollary 2.3** (Shin–Zeng). *For $\sigma \in \mathfrak{S}_n$ we have*

$$\begin{aligned}&(\text{2-13}, \text{31-2}, \mathsf{des}, \mathsf{asc}, \mathsf{da}, \mathsf{dd}, \mathsf{val})\sigma \\ &= (\mathsf{nest}, \mathsf{cros}, \mathsf{drop}^* - 1, \mathsf{wex}^*, \mathsf{cda}^* + \mathsf{fix}^*, \mathsf{cdd}^*, \mathsf{cval}^*)\Psi(\sigma).\end{aligned} \tag{2.21}$$

## 2.5 Laguerre histories as permutation encodings

A 2-*Motzkin path* of length $n$ is a word $\mathbf{s} := s_1 \dots s_n$ on the alphabet $\{\mathrm{U}, \mathrm{D}, \mathrm{L}_r, \mathrm{L}_b\}$ such that $|s_1 \dots s_n|_{\mathrm{U}} = [s_1 \dots s_i|_{\mathrm{D}}$ and

$$h_i := |s_1 \dots s_i|_{\mathrm{U}} - [s_1 \dots s_i|_{\mathrm{D}} \geq 0 \quad (i = 1, \dots, n), \tag{2.22}$$

where $|s_1 \dots s_i|_{\mathrm{U}}$ is the number of letters U in the word $s_1 \dots s_i$.

A *Laguerre history* (resp. *restricted Laguerre history*) of length $n$ is a pair $(\mathbf{s}, \mathbf{p})$, where $\mathbf{s}$ is a 2-Motzkin path $s_1 \dots s_n$ and $\mathbf{p} = (p_1, \dots, p_n)$ with $0 \leq p_i \leq h_{i-1}$ (resp. $0 \leq p_i \leq h_{i-1} - 1$ if $s_i = \mathrm{L}_b$ or D) and $h_0 = 0$. Let $\mathcal{LH}_n$ (resp. $\mathcal{LH}_n^*$) be the set of Laguerre histories (resp. restricted Laguerre histories) of length $n$. There are several well-known bijections between $\mathfrak{S}_n$ and $\mathcal{LH}_n^*$ and $\mathcal{LH}_{n-1}$, see [12, 4] and references therein.

## 2.6 Françon–Viennot bijection

We recall a version of Françon and Viennot's bijection $\psi_{FV} : \mathfrak{S}_{n+1} \to \mathcal{LH}_n$. Given $\sigma \in \mathfrak{S}_{n+1}$, the Laguerre history $(\mathbf{s}, \mathbf{p})$ is defined as follows:

$$s_i = U \text{ (resp.} D,\ \mathrm{L}_r,\ \mathrm{L}_b) \text{ if } i \in \mathsf{Val}\sigma \text{ (resp. } i \in \mathsf{Pk}\sigma,\ i \in \mathsf{Da}\sigma,\ i \in \mathsf{Dd}\sigma) \tag{2.23}$$

and $p_i = (\text{2-13})_i\, \sigma$ for $i = 1, \dots, n$.

**Theorem 2.4.** *The mapping $\psi := \psi_{FV} \circ \Psi^{-1}$ is a bijection from $\mathfrak{S}_{n+1}$ to $\mathcal{LH}_n$. If $\psi(\sigma) = (\mathbf{s}, \mathbf{p})$ with $\sigma \in \mathfrak{S}_{n+1}$, then, for $i = 1, \dots, n$,*

$$s_i = U \text{ (resp.} D,\ \mathrm{L}_r,\ \mathrm{L}_b) \text{ if } i \in \mathrm{Cval}^*\sigma \text{ (resp. } i \in \mathrm{Cpk}^*\sigma,\ i \in \mathrm{Cda}^*\sigma \cup \mathrm{Fix}^*\sigma,\ i \in \mathrm{Cdd}^*\sigma) \tag{2.24}$$

*with $p_i = \mathsf{nest}_i\sigma$.*

*Proof.* This follows from Theorem 2.2 by comparing (2.24) with (2.23), see the commutative diagram in Figure 1. □

**Corollary 2.5.** *The two sextuple statistics*

$$(\mathsf{nest}, \mathsf{cros}, \mathsf{exc}, \mathsf{cdd}^*, \mathsf{cda}^* + \mathsf{fix}^*, \mathsf{cpk}^*) \quad \text{and} \quad (\text{2-13}, \text{31-2}, \mathsf{des}, \mathsf{da}, \mathsf{dd}, \mathsf{pk} - 1)$$

*are equidistributed on $\mathfrak{S}_n$.*

We recall two bijections $\phi_{FZ}$ and $\phi_{FV}$ from $\mathfrak{S}_n$ to $\mathcal{LH}_n^*$.

## 2.7 Restricted Françon–Viennot bijection

We recall a restricted version of Françon and Viennot's bijection $\phi_{FV} : \mathfrak{S}_n \to \mathcal{LH}_n^*$. Given $\sigma \in \mathfrak{S}_n$, the Laguerre history $(\mathbf{s}, \mathbf{p})$ is defined as follows:

$$s_i = U \text{ (resp.} D,\ \mathrm{L}_r,\ \mathrm{L}_b) \text{ if } i \in \mathsf{Lval}\sigma \text{ (resp. } i \in \mathsf{Lpk}\sigma,\ i \in \mathsf{Lda}\sigma,\ i \in \mathsf{Ldd}\sigma) \tag{2.25}$$

and $p_i = (2\text{-}31)_i\, \sigma$ for $i = 1, \ldots, n$.

## 2.8 Foata–Zeilberger bijection

This bijection $\phi_{FZ}$ encodes permutations using cyclic statistics. Given $\sigma \in \mathfrak{S}_n$, $\phi_{FZ} : \mathfrak{S}_n \to \mathcal{LH}_n^*$ is for $i = 1, \ldots, n$,

$$s_i = U \text{ (resp.} D,\ \mathrm{L}_r,\ \mathrm{L}_b) \text{ if } i \in \mathrm{Cval}\,\sigma \text{ (resp. } i \in \mathrm{Cpk}\,\sigma,\ i \in \mathrm{Cda}\,\sigma \cup \mathrm{Fix}\,\sigma,\ i \in \mathrm{Cdd}\,\sigma) \tag{2.26}$$

with $p_i = \mathsf{nest}_i\sigma$. By (2.7) and (2.8), we can build a comutative diagram, see the right diagram of Figure 1.

By contracting the continued fraction (1.1) we derive the two J-type continued fraction formulae (cf. [6])

$$\sum_{n\geq 0} A_{n+1}(t)z^n = \cfrac{1}{1-(1+t)\cdot z - \cfrac{1\cdot 2\cdot t\cdot z^2}{1-2(1+t)\cdot z-\cfrac{2\cdot 3\cdot t\cdot z^2}{1-\cdots}}} \tag{2.27}$$

and

$$\sum_{n\geq 0} A_{n}(t)z^n = \cfrac{1}{1-(1+0\cdot t)\cdot z - \cfrac{1^2\cdot z^2}{1-(2+1\cdot t)\cdot z-\cfrac{2^2\cdot t\cdot z^2}{1-\cdots}}}. \tag{2.28}$$

In view of Flajolet's combinatorial interpretation in terms of weighted Motzkin paths for generic J-type continued fraction expansions [6], Françon–Viennot's bijection $\psi_{FV}$ (resp. its restricted version $\phi_{FV}$) between permutations and *Laguerre histories* provides a bijective proof of (2.27) (resp. (2.28)), while Foata–Zeilberger's bijection $\psi_{FZ}$ [8] gives a bijective proof of (2.28). More precisely, Françon–Viennot [9] set up a bijection (and its restricted version) from permutations to *Laguarre histories* using *linear statistics of permutation*, while Foata–Zeilberger's bijection [8] uses *cyclic statistics of permutations*. Clarke-Steingrímsson-Zeng [4] gave a direct bijection $\Phi$ on permutations converting statistic des into exc on permutations, and linking the restricted Françon–Viennot's bijection $\phi_{FV}$ to

$$\begin{tikzcd} & \mathcal{LH}_n^* & \\ \mathfrak{S}_n \arrow[ur, "\phi_{FV}"] \arrow[rr, "\Phi"] & & \mathfrak{S}_n \arrow[ul, "\phi_{FZ}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{LH}_n & \\ \mathfrak{S}_{n+1} \arrow[ur, "\psi_{FV}"] \arrow[rr, "\Psi"] & & \mathfrak{S}_{n+1} \arrow[ul, "\psi"'] \end{tikzcd}$$

**Figure 1:** Bijections $\Phi = \phi_{FZ}^{-1} \circ \phi_{FV}$ and $\psi = \psi_{FV} \circ \Psi^{-1}$.

Foata–Zeilberger bijection $\phi_{FZ}$, see Figure 1. As a variation of $\Phi$, Shin and Zeng [14] constructed a bijection $\Psi$ on permutations to derive a cycle version of linear statistics on permutations, which are obtained via Françon–Viennot bijection $\psi_{FV}$. One of our main results (cf. Theorem 2.4) shows that a direct description of the bijection $\psi := \psi_{FV} \circ \Psi^{-1}$ from $\mathfrak{S}_{n+1}$ to $\mathcal{LH}_n$ is straightforward.

## 2.9 Pattern avoidances and 2-Motzkin paths

A permutation $\sigma$ is called 231-*avoiding permutation* if there is no triple of indices $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$. The Narayana polynomials are defined by

$$N_n(t) = \sum_{\sigma \in \mathfrak{S}_n(231)} t^{\mathsf{des}\,\sigma},$$

where $\mathfrak{S}_n(231)$ is the set of 231-avoiding permutations in $\mathfrak{S}_n$. It is well known that Narayana polynomial is $\gamma$-positive and have the expansion [13, Chapter 4]:

$$N_n(t) = \sum_{k=0}^{n/2} \widetilde{\gamma}_{n,j} t^j (1+t)^{n-1-2j}, \tag{2.29}$$

where $\widetilde{\gamma}_{n,j} = |\{\sigma \in \mathfrak{S}_n(231) : \ \mathsf{des}(\sigma) = \mathsf{pk}(\sigma) = j\}|$. As for Eulerian polynomials, by contraction, from

$$\sum_{n\geq 0} N_n(t) z^n = 1/1 - z/1 - t \cdot z/1 - z/1 - t \cdot z/1 - \ldots \tag{2.30}$$

we derive immediately the following continued fractions

$$\sum_{n\geq 0} N_{n+1}(t) z^n = \cfrac{1}{1-(1+t)\cdot z - \cfrac{t \cdot z^2}{1-(1+t)\cdot z - \cfrac{t\cdot z^2}{1-\cdots}}} \tag{2.31}$$

and

$$\sum_{n\geq 0} N_n(t)z^n = \cfrac{1}{1-z-\cfrac{t\cdot z^2}{1-(1+t)\cdot z-\cfrac{t\cdot z^2}{1-(1+t)\cdot z-\cdots}}}. \tag{2.32}$$

Note that $N_n(1)$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, there are several well-known $q$-Narayana polynomials in the literature, see [3] and [10].

Given two permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, we say that $\sigma$ *contains the pattern* $\tau$ if there exists a set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the subsequence $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ of $\sigma$ is order-isomorphic to $\tau$. Otherwise, $\sigma$ is said to *avoid* $\tau$. For example, the permutation 15324 contains the pattern 321 and avoids the pattern 231. The set of permutations of length $n$ that avoid patterns $\tau_1, \tau_2, \cdots, \tau_m$ is denoted as $\mathfrak{S}_n(\tau_1, \tau_2, \cdots, \tau_m)$.

Moreover we shall consider the so-called *vincular patterns*. The number of occurrences of vincular patterns 31-2, 2-31, 2-13 and 13-2 in $\pi \in \mathfrak{S}_n$ are defined (cf. (2.1)) by

$$\begin{aligned}
(\text{31-2})\,\pi &= \#\{(i,j) : i+1 < j \leq n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\},\\
(\text{2-31})\,\pi &= \#\{(i,j) : j < i < n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\},\\
(\text{2-13})\,\pi &= \#\{(i,j) : j < i < n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\},\\
(\text{13-2})\,\pi &= \#\{(i,j) : i+1 < j \leq n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\}.
\end{aligned} \tag{2.33}$$

Similarly, we use $\mathfrak{S}_n(\text{31-2})$ to denote the set of permutations of length $n$ that avoid the vincular pattern 31-2, etc. In order to apply Laguerre history to count pattern-avoiding permutations, we will need the following results in [10, Lemmas 2.8 and 2.9].

**Lemma 2.6** ([10, Lemma 2.8]). *For any $n \geq 1$, we have*

$$\mathfrak{S}_n(\text{2-13}) = \mathfrak{S}_n(213), \quad \mathfrak{S}_n(\text{31-2}) = \mathfrak{S}_n(312), \tag{2.34}$$

$$\mathfrak{S}_n(\text{13-2}) = \mathfrak{S}_n(132), \quad \mathfrak{S}_n(\text{2-31}) = \mathfrak{S}_n(231). \tag{2.35}$$

**Lemma 2.7** ([10, Lemma 2.9]). *(i) A permutation $\pi \in \mathfrak{S}_n$ belongs to $\mathfrak{S}_n(321)$ if and only if* $\mathsf{nest}\,\pi = 0$

*(ii) The mapping $\Phi$ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$.*

We use $\mathcal{CM}_n$ to denote the set of 2-Motzkin paths of length $n$ and $\mathcal{CM}_n^*$ to denote its subset that is composed of 2-Motzkin paths without $\mathrm{L}_b$-step at level zero, i.e., if $h_{i-1} = 0$, then $s_i \neq \mathrm{L}_b$. Noticing that the generating function $\sum_{n\geq 0} |\mathcal{CM}_n^*| z^n$ has the continued fraction expansion (2.32) with $t = 1$, we derive that $|\mathcal{CM}_n^*| = C_n$. Similarly, by (2.31) we see that $|\mathcal{CM}_n| = C_{n+1}$.

Let $\widetilde{\phi}_{FV}$ (resp. $\widetilde{\phi}_{FZ}$, $\widetilde{\psi}_{FV}$, $\widetilde{\psi}$, $\widetilde{\Psi}$, $\widetilde{\Phi}$) be the restriction of $\phi_{FV}$ (resp. $\phi_{FZ}$, $\psi_{FV}$, $\psi$, $\Psi$, $\Phi$) on the sets $\mathfrak{S}_n(231)$ (resp. $\mathfrak{S}_n(321)$, $\mathfrak{S}_{n+1}(213)$, $\mathfrak{S}_{n+1}(321)$, $\mathfrak{S}_{n+1}(213)$, $\mathfrak{S}_n(231)$). By Lemmas 2.6 and 2.7 and Figure 1, we obtain the diagrams in Figure 2.

$$
\begin{tikzcd}
 & \mathcal{CM}_n & \\
\mathfrak{S}_{n+1}(213) \arrow[ur, "\widetilde{\psi}_{FV}"] \arrow[rr, "\widetilde{\Psi}"] & & \mathfrak{S}_{n+1}(321) \arrow[ul, "\widetilde{\psi}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{CM}_n^{*} & \\
\mathfrak{S}_{n}(231) \arrow[ur, "\widetilde{\phi}_{FV}"] \arrow[rr, "\widetilde{\Phi}"] & & \mathfrak{S}_{n}(321) \arrow[ul, "\widetilde{\phi}_{FZ}"']
\end{tikzcd}
$$

**Figure 2:** Bijections $\widetilde{\Psi} = \widetilde{\psi}^{-1} \circ \widetilde{\psi}_{FV}$ and $\widetilde{\Phi} = \widetilde{\phi}_{FZ}^{-1} \circ \widetilde{\phi}_{FV}$.

# 3 Main results

For a finite set of permutations $\Omega$ and $m$ statistics $\mathsf{stat}_1, \ldots, \mathsf{stat}_m$ on $\Omega$, we define the generating polynomial

$$P^{(\mathsf{stat}_1,\ldots,\mathsf{stat}_m)}(\Omega; t_1, \ldots, t_m) := \sum_{\sigma \in \Omega} t_1^{\mathsf{stat}_1 \sigma} \ldots t_m^{\mathsf{stat}_m \sigma}. \tag{3.1}$$

**Theorem 3.1.** *For $n \geq 1$,*

$$\begin{aligned} &P^{(\mathsf{nest},\mathsf{cros},\mathsf{exc},\mathsf{fix})}(\mathfrak{S}_n; p, q, tq, r) \\ &= \left(\frac{1+xt}{1+x}\right)^n P^{(\mathsf{nest},\mathsf{cros},\mathsf{cpk},\mathsf{exc},\mathsf{fix})}\left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{q(x+t)}{1+xt}, \frac{(1+x)r}{1+xt}\right), \end{aligned} \tag{3.2}$$

*equivalently,*

$$\begin{aligned} &P^{(\mathsf{nest},\mathsf{cros},\mathsf{cpk},\mathsf{exc},\mathsf{fix})}(\mathfrak{S}_n; p, q, x, qt, r) \\ &= \left(\frac{1+u}{1+uv}\right)^n P^{(\mathsf{nest},\mathsf{cros},\mathsf{exc},\mathsf{fix})}\left(\mathfrak{S}_n; p, q, qv, \frac{(1+uv)r}{1+u}\right), \end{aligned} \tag{3.3}$$

*where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.*

**Remark 3.2.** Cooper et al. [5, Theorem 11] have recently proved the $p = q = 1$ case of (3.2) by applying Sun and Wang's CMFS action [17].

We define the polynomial

$$A_n(p, q, t) := \sum_{\sigma \in \mathfrak{S}_n} p^{\mathsf{nest}\, \sigma} q^{\mathsf{cros}\, \sigma} t^{\mathsf{exc}\, \sigma}. \tag{3.4}$$

The following is a generalization of Stembridge's identity (1.4).

**Theorem 3.3.** *For $n \geq 1$, we have*

$$A_n(p,q,t) = \left(\frac{1+xt}{1+x}\right)^{n-1} P^{(\mathsf{nest},\mathsf{cros},\mathsf{cpk}^*,\mathsf{exc})}\left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}\right), \tag{3.5}$$

*equivalently,*

$$P^{(\mathsf{nest},\mathsf{cros},\mathsf{cpk}^*,\mathsf{exc})}(\mathfrak{S}_n; p, q, x, t) = \left(\frac{1+u}{1+uv}\right)^{n-1} A_n(p,q,v), \tag{3.6}$$

*where* $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ *and* $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

**Remark 3.4.** By Corollary 2.5 and (3.5), when $x = 1$ or $p = q = 1$ we recover two special cases of (3.5) due to Brändén's result [2, (5.1)] and Zhuang [18, Theorem 4.2], respectively.


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