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A positive combinatorial formula for symplectic Kostka–Foulkes polynomials I: Rows

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Abstract. We prove a conjecture of Lecouvey, which proposes a closed, positive combinatorial formula for symplectic Kostka–Foulkes polynomials, in the case of rows of arbitrary weight. To show this, we transform the cyclage algorithm in terms of which the conjecture is described into a different, more convenient combinatorial model, free of local constraints. In particular, we show that our model is governed by the situation in type A. We expect our approach to lead to a proof of the conjecture in the general case.

Résumé. Nous prouvons une conjecture de Lecouvey proposant une formule close positive pour les polynômes de Kostka–Foulkes symplectiques, dans le cas des lignes de poids quelconque. Pour cela, nous transformons l'algorithme de cyclage grâce auquel la conjecture est énoncée en un modèle combinatoire plus simple, sans contraintes locales. En particulier, nous démontrons que notre modèle est contrôlé par la situation en type A. Cette approche permet d'entrevoir une preuve de la conjecture dans le cas général.

Keywords: combinatorial representation theory, Kostka–Foulkes polynomials, Lecouvey's conjecture, charge, Type C

1 Introduction

The main motivation for this work is understanding an interplay between combinatorics and representation theory which is highly manifested in the structure of so-called *Kostka–Foulkes polynomials*. Let $\mathfrak{g}$ be a complex, simple Lie algebra of rank n. Kostka–Foulkes polynomials $K_{\lambda,\mu}(q)$ are defined for two dominant integral weights as the transition coefficients between two important bases of the ring of symmetric functions in the variables $x = (x_1, ..., x_n)$ over $\mathbb{Q}(q)$: Hall–Littlewood polynomials $P_\lambda(x; q)$ and Weyl characters

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$\chi_\mu(x)$. They are q-analogues of weight multiplicities [5], affine Kazhdan–Lusztig polynomials [11, 5], and appear in various other situations in geometric and combinatorial representation theory (see [14] and references therein). We refer the reader to [2] for a precise definition of Kostka–Foulkes polynomials and recommend [14] as a thorough reference.

Due to their interpretation as Kazhdan–Lusztig polynomials, we know that Kostka–Foulkes polynomials have nonnegative integer coefficients. This fact leads to one of the most important unsolved problems in combinatorial representation theory:

Problem 1.1. Find a set $\mathcal{T}(\lambda,\mu)$ and a combinatorial statistic $\mathrm{ch} : \mathcal{T}(\lambda,\mu) \to \mathbb{Z}_{\geq 0}$ such that the Kostka–Foulkes polynomial $K_{\lambda,\mu}(q)$ is the generating function of $\mathcal{T}(\lambda,\mu)$ with respect to ch. In other words find $\mathcal{T}(\lambda,\mu)$ and ch such that

$$K_{\lambda,\mu}(q) = \sum_{T\in\mathcal{T}(\lambda,\mu)} q^{\mathrm{ch}(T)}. \tag{1.1}$$

Since $K_{\lambda,\mu}(q)$ are q-deformations of weight multiplicities then $\#\mathcal{T}(\lambda,\mu) = K_{\lambda,\mu}(1)$ is the dimension of the μ-weight space of the irreducible, finite dimensional $\mathfrak{g}$-module of highest weight λ. In particular, in order to tackle Problem 1.1 and find an appropriate set $\mathcal{T}(\lambda,\mu)$, it seems natural to seek for an object which parametrizes the aforementioned μ-weight space of the irreducible, finite dimensional $\mathfrak{g}$-module of highest weight λ. This approach turned out to be very succesful in type A_{n-1}, that is when $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. In this case dominant integral weights are identified with partitions of at most $n-1$ parts, and a natural candidate for $\mathcal{T}(\lambda,\mu)$ is the set $\mathrm{SSYT}(\lambda,\mu)$ of semistandard Young tableaux of shape λ and weight μ. In this context, Foulkes conjectured the existence of such a statistic [3], which was explicitly found by Lascoux and Schützenberger [7]. They called their statistic *charge* (which explains our abbreviation ch used also in the general context of arbitrary type) and established the celebrated formula of Problem 1.1 in type A_{n-1}. A thorough introduction to Kostka–Foulkes polynomials in type A_{n-1} and the charge statistic from a purely combinatorial point of view is carried out in [12]. We refer the reader to [1] for a beautiful exposition and proof of (1.1), which makes use of a recursive formula for computing Kostka–Foulkes polynomials due to Morris [13].

In this work, we focus on Problem 1.1 for type C_n, that is, in case of the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$. To the best of our knowledge this is the only case of Problem 1.1 having an explicit conjectural solution, which was formulated by Lecouvey in [9]. In this case, the dominant integral weights λ,μ can again be identified with partitions of at most n parts. The candidate for the set $\mathcal{T}(\lambda,\mu)$ which features in Lecouvey's conjecture is the set of *symplectic tableaux*, which we will denote by $\mathrm{SympTab}_n(\lambda,\mu)$, and which are also known as Kashiwara–Nakashima tableaux [4]. These are defined to be semistandard

Young tableaux with some additional constraints (see [9]) and entries in the ordered alphabet

$$\mathcal{C}_n = \{\overline{n} < \cdots < \overline{1} < 1 < \cdots < n\},$$

such that the shape of a tableau is given by λ and its weight by μ. Here, the weight of a tableau with entries in $\mathcal{C}_n$ is defined slightly differently than the weight of a tableau of type A_{n-1} and is given by the vector $(a_{\overline{n}}, ..., a_{\overline{1}})$, where $a_{\overline{i}}$ is the difference between the number of occurrences of $\overline{i}$'s and i's in T. Lecouvey defined a charge statistic $\mathrm{ch}_n : \mathrm{SympTab}_n(\lambda, \mu) \to \mathbb{Z}_{\geq n}$ by analogy with the situation in type A_{n-1}, which makes use of the column insertion of a letter into a semistandard tableau. See the full version of this work [2] or [9] for more details.

Before we describe Lecouvey's conjectural solution to Problem 1.1 involving cyclage it is worth mentioning that a solution to Problem 1.1 in type C_n in the weight zero case has been given recently in [10, Theorem 6.13], using the combinatorial model for $\mathcal{T}(\lambda, \mu)$ in terms of the so-called King tableaux introduced in [6]. However, this relies on an interpretation of the Kostka–Foulkes polynomials in terms of generalized exponents which only holds in this special case of weight zero, so that there is little hope to tackle the general weight case with this approach.

1.1 Main result and methodology

In order to define the statistic $\mathrm{ch}_n : \mathrm{SympTab}_n(\lambda, \mu) \to \mathbb{N}$ and formulate his conjecture, Lecouvey used a symplectic version of column insertion, which he introduced in [8], to define a symplectic cyclage operation $\mathrm{Cyc}_{\mathrm{C}}$ which transforms a symplectic tableau $T \in \mathrm{SympTab}_n$ into a symplectic tableau $\mathrm{Cyc}_{\mathrm{C}}(T) \in \mathrm{SympTab}_m$ for $m \geq n$. The statistic ch_n is defined as follows. Let $T \in \mathrm{SympTab}_n$ be a symplectic tableau. In [9], Lecouvey showed that there exists a non-negative integer m such that $\mathrm{Cyc}_{\mathrm{C}}^m(T)$ is a column $C(T)$ of weight zero. We denote by $m(T)$ the smallest non-negative integer with this property. For a symplectic column C of weight zero we set $E_C = \{i \geq 1 | i \in C, i+1 \notin C\}$. The charge of C is defined by

$$\mathrm{ch}_n(C) = 2 \sum_{i \in E_C} (n - i),$$

and the charge of an arbitrary symplectic tableau T is defined by

$$\mathrm{ch}_n(T) = m(T) + \mathrm{ch}_n(C(T)).$$

Lecouvey [9] conjectured the following solution of Problem 1.1 in type C_n:

Conjecture 1.2. *Let μ, λ be partitions with at most n parts. Then*

$$K^{\mathrm{C}_n}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SympTab}_n(\lambda,\mu)} q^{\mathrm{ch}_n(T)}. \tag{1.2}$$

Our main result reads as follows.

Theorem 1.3. *Let $\lambda = (p)$ and $\mu = (\mu_{\overline{n}}, \dots, \mu_{\overline{1}})$ be an arbitrary partition. Then Conjeture 1.2 holds true:*

$$K^{\mathrm{C}_n}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SympTab}_n(\lambda,\mu)} q^{\mathrm{ch}_n(T)}.$$

A pivotal point in our methodology, and one which we expect will have impact on the study of the general case of Conjeture 1.2, is a reformulation of Lecouvey's construction in the setting of Theorem 1.3 by providing a new algorithm to compute $\mathrm{Cyc}^k_{\mathrm{C}}(T)$ for arbitrary integer k. The big advantage of our approach is that in Algorithm 2, which completes this task, we are able to eliminate local constraints which appear in the original construction in two different contexts:

- we need to compute $\mathrm{Cyc}^{k-1}_{\mathrm{C}}(T)$ in order to compute $\mathrm{Cyc}^k_{\mathrm{C}}(T)$;
- for each column of $\mathrm{Cyc}^{k-1}_{\mathrm{C}}(T)$, we need to insert boxes recursively into consecutive subcolumns of size 2.

In order to eliminate the second constraint we give a formula for inserting an entry into a whole column at once, which is given by [2, Proposition 3.3]. Although more technical in appearance, our new definition allows us to have a full grasp of the symplectic cyclage procedure. We show in Proposition 2.5 that for a partition $\lambda = (p)$ which consists of one row and for an arbitrary partition μ the symplectic tableau $\mathrm{Cyc}^k_{\mathrm{C}}(T)$, where $T \in \mathrm{SympTab}_n(\lambda, \mu)$, is given by Algorithm 2. The main philosophy of Algorithm 2 is that in order to compute $\mathrm{Cyc}^k_{\mathrm{C}}(T)$, it is enough to only apply $\mathrm{Cyc}_{\mathrm{A}}$ to certain standard Young tableaux and then apply a very simple function which changes the entries of the outcome.

As an application, we are able to compute $\mathrm{ch}_n(T)$ directly from T and, using a simple recurrence for Hall–Littlewood polynomials of type C proved by Lecouvey in [9, Theorem 3.2.1.], we deduce Theorem 1.3. We believe that our strategy might lead us to the solution of Conjeture 1.2 in the full generality. Indeed, the restriction $\lambda = (p)$ is due to the fact that symplectic tableaux of row shape coincide with semistandard tableaux with entries in the alphabet $\mathcal{C}_n$ (see Proposition 1.4). In particular, there exists a unique standard tableau of shape (p), and Algorithm 2 consists in applying $\mathrm{Cyc}_{\mathrm{A}}$ multiple times to this unique tableau. It seems likely that in the more general case there exists a "right" labelling of the boxes of any symplectic tableau T of arbitrary shape, such that a very similar procedure could be followed to compute $\mathrm{Cyc}^k_{\mathrm{C}}(T)$ and therefore ch_n. So far, this question remains open and we will be investigating this question in the future. The full version of this work, [2], which contains the full proofs of all the results presented in this extended abstract, will be submitted elsewhere.

1.2 Notation

A composition $\alpha \vDash n$ of size $n \in \mathbb{Z}_{\geq 0}$ is a sequence of non-negative integers $\alpha := (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}_{>0}}$ such that $\sum_i \alpha_i = n$ and such that $\alpha_i = 0$ implies that $\alpha_{i+1} = 0$ for any $i \in \mathbb{Z}_{>0}$ In particular, there are only finitely many non-zero α_i and we denote their number by $\ell(\alpha)$ calling it the *length* of composition α. We will also use the notation $|\alpha| = n$. A *partition* is a composition with non-decreasing entries. We denote the set of compositions of size n by Comp_n, the set of partitions of size n by Part_n and we set $\mathrm{Comp} := \bigcup_n \mathrm{Comp}_n$, respectively $\mathrm{Part} := \bigcup_n \mathrm{Part}_n$. To any $\alpha \in \mathrm{Comp}_n$ we associate its diagram defined by $\mathcal{D}_\alpha = \{(i,j) : 1 \leq i \leq \alpha_{-j}, j \leq -1\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{<0}$. The elements of $\mathcal{D}_\alpha$, referred to as boxes, are linearly ordered by the so-called *reading order*, which is a variant of lexicographic order: $(i_1, j_1) \leq (i_2, j_2) \iff j_1 > j_2$ or $j_1 = j_2, i_1 < i_2$. For an ascending chain $(\mathcal{A}, \prec)$, that is a linearly ordered alphabet with minimal element a and for $b \in \mathcal{A}$ we denote the b-th box of $\mathcal{D}_\alpha$ in reading order by $\square_b$, or by b whenever it does not lead to a confusion. For any composition $\alpha \vDash n$ we define *a tableau* T of shape α and entries in $\mathcal{A}$ to be a filling of the boxes of the diagram of α by elements from alphabet $\mathcal{A}$, that is a map $T : \mathcal{D}_\alpha \to \mathcal{A}$. The *content* of a tableau T of shape α is the multiset of its entries. When $\mathcal{A}$ is a countable ascending chain (with the minimal element a), we say that a tableau has *weight* $\beta = (\beta_1, \beta_2, \dots)$ when its content is given by the multiset $\{a^{\beta_1}, (a+_\succ)^{\beta_2}, \dots, (a+_{\succ^k})^{\beta_{k+1}}, \dots\}$, where $a+_\succ$ denotes the successor of a, and $a+_{\succ^{k+1}} := a +_{\succ^k} +_\succ$. We denote the set of semistandard Young tableaux of shape α and weight β with entries in $\mathcal{A}$ by $\mathrm{SSYT}_{\mathcal{A}}(\alpha, \beta)$.

Let n be a positive integer and λ, μ partitions with at most n parts. The following proposition justifies why we do not need the defining properties of symplectic tableaux in this work. It is a direct consequence of Definition in [9].

Proposition 1.4. *Let $\lambda = (p)$ and μ be a partition. Then*

$$\mathrm{SympTab}_n(\lambda, \mu) = \bigcup_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} \mathrm{SSYT}_{\mathcal{C}_n}(\lambda, (k_n + \mu_{\overline{n}}, k_{n-1} + \mu_{\overline{n-1}}, \dots, k_1 + \mu_{\overline{1}}, k_1, \dots, k_n)).$$

We are interested in the set of symplectic tableaux since these objects give a natural basis of the μ weight space of an irreducible $\mathfrak{g}$–module of highest weight λ in type C. In particular, $K^{C_n}_{\lambda,\mu}(1) = |\mathrm{SympTab}_n(\lambda, \mu)|$.

2 The shift and content algorithms.

In this section we will construct the new algorithm computing $\mathrm{Cyc}^k_C(T)$ for arbitrary $k > 0$ and for $T \in \mathrm{SympTab}((p))$, that is T is a symplectic tableau of row shape. Our algorithm does not rely on the particular form of $\mathrm{Cyc}^{k-1}_C(T)$, which allows us to overcome the problem of controlling many local dependencies present in Lecouvey's original algorithm. This construction is the main ingredient in the proof of Theorem 1.3. Reordering the parts of a composition $\alpha \vDash n$ gives a partition $\lambda \vdash n$. Note that λ can be also seen as the result of lifting all the boxes in each column of α so that after the lift, the boxes in the given column are lying in consecutive rows starting from the first row. For this reason, we denote by grav the map $\mathrm{Comp}_n \to \mathrm{Part}_n, \alpha \mapsto \lambda$ and call it the gravity map. This map is naturally also defined on tableaux.

2.1 Shifting

$n \in \mathbb{Z}_{\geq 0}$ and define shift : $\mathrm{Comp}_n \to \mathrm{Comp}_n$ as follows

$$\mathrm{shift}(\alpha) = \begin{cases} \alpha & \text{if } \alpha = (1^l, 0, \dots) \text{ for some } l \in \mathbb{Z}_{\geq 0}; \\ \alpha - e_i + e_{i+1} & \text{otherwise;} \end{cases}$$

where $e_i = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, 0 \dots)$ and $i = \min\{j \mid \alpha_j = \max_k \alpha_k\}$. We define the operator

$$\mathrm{simp} : \mathrm{Comp} \times \mathrm{Part} \to \mathrm{Comp} \times \mathrm{Part}$$

by the following recursive algorithm:

Algorithm 1 Defining $\mathrm{simp}(\alpha, \mu)$.

Input: A partition μ and a composition α.
Output: A pair $(\beta, \nu) \in \mathrm{Comp} \times \mathrm{Part}$.

```
β = α
ν = μ
while max β_k = ν_1 do
    ν = ν \ ν_1
    β = β \ max β_k
end while
```

We extend the domain of the operator shift : Comp × Part → Comp × Part by:

$$\mathrm{shift}(\alpha, \mu) = \begin{cases} (\mathrm{shift}(\alpha), \mu) & \text{if } (\alpha, \mu) \neq \mathrm{simp}(\alpha, \mu); \\ (\mathrm{shift}\left(\mathrm{simp}(\alpha, \mu)_1\right), \mathrm{simp}(\alpha, \mu)_2) & \text{otherwise;} \end{cases}$$

where $\mathrm{simp}(\alpha, \mu)_i$ denotes the i-th coordinate of $\mathrm{simp}(\alpha, \mu)$. For the geometric intuition behind these definitions, see [2, Section 2.4].

Lemma 2.1. *[2, Lemma 2.6] For any pair* $(\alpha, \mu) \in \mathrm{Comp} \times \mathrm{Part}$ *there exists an integer* m *and a partition* ν *such that* $\mathrm{shift}^m(\alpha, \mu) = ((1^l), \nu)$ *and is a fixed point of* shift *(for some* $l \geq 0$*), that is* $\nu_1 \neq 1$*.*

We define

$$m_\mu(\alpha) = \min\{m \mid \mathrm{shift}^{m+1}(\alpha, \mu) = \mathrm{shift}^m(\alpha, \mu)\}. \tag{2.1}$$

Corollary 2.2. *[2, Corollary 2.7]. In the special case* $\alpha = (p), |\mu| \leq p$ *we have*

$$m_\mu(\alpha) = \sum_i (i-1)\mu_i + \frac{(p - |\mu|)(p - |\mu| + 2\ell(\mu) - 1)}{2}.$$

2.2 The content function.

Given a composition α and two boxes b and b' in α such that $b < b'$ in the reading order, their *distance* in α is defined by

$$\delta_\alpha(b, b') = \mathrm{row}_\alpha(b') - \mathrm{row}_\alpha(b) - \chi\big(\mathrm{col}_\alpha(b) \geq \mathrm{col}_\alpha(b')\big),$$

where, for a condition $\mathscr{C}$, we define $\chi(\mathscr{C}) = 1$ if $\mathscr{C}$ is satisfied, and $\chi(\mathscr{C}) = 0$ otherwise, and $\mathrm{row}_k(s)$ and $\mathrm{col}_k(s)$ denote the row and column index of s counted from top to bottom and from left to right, respectively. In Algorithm 2 we define a tableau which we show to be equal to $\mathrm{Cyc}^k(T)$. Note that Algorithm 2 decomposes the set of boxes into two disjoint sets. The first set contains the boxes which had no associated partners; we call such a box b a *single*. All the other boxes are matched into pairs by associating their partners; for such a box b we denote by $\mathrm{partner}(b)$ its partner (note that $\mathrm{partner}(\mathrm{partner}(b)) = b$).

Example 2.3 (Weight zero)**.** Let T be a tableau of shape $(2q)$ and weight zero (note that all tableaux of weight zero must have an even number of boxes). We may label its boxes by elements in the interval $[\overline{q}, q] \subset \mathcal{C}$. We have $\alpha = \mathrm{shift}^k((2q), 0)_1 = \mathrm{shift}^k((2q))$, and the content of a given box in T_k is given by

$$T_k(S) = \begin{cases} T(S) + \delta_\alpha(\overline{S}, S) & \text{if } S > 0, \\ \overline{T_k(\overline{S})} & \text{if } S < 0. \end{cases}$$

Boxes $\overline{S}$ and S are always *partners*, and they will hence have opposite contents in T_k for each $k \leq m(T)$.

Algorithm 2 Defining the tableau T_k.

Input: Nonnegative integers $k, k_1, \dots, k_n$ and a partition $\mu = (\mu_{\overline{n}}, \dots, \mu_{\overline{1}})$.
Output: The tableau $T_k : [1, |\alpha|] \to \mathcal{C}$ of shape α.

$p = \sum_{i=1}^{n} (2k_i + \mu_{\bar{i}})$
$\alpha = \text{shift}^k((p), \mu)_1$
$\text{nred} = \ell(\mu) - \ell(\text{shift}^k((p), \mu)_2)$ ▷ nred counts the number of reductions performed so far
$R = n - \text{nred} + 1$
$I_\alpha = \{\overline{n + \text{nred}}^{k_n}, \dots, \overline{r + \text{nred}}^{k_r}, \overline{r - 1 + \text{nred}}^{k_{r-1} + \mu_{\overline{r-1}}}, \dots, \overline{1 + \text{nred}}^{k_1 + \mu_{\bar{1}}}, (1 + \text{nred})^{k_1}, \dots, (n + \text{nred})^{k_n}\}$
$f_\alpha : [1, |\alpha|] \to I_\alpha$ s.t. f_α is the unique non-decreasing bijection ▷ f_α is the natural labeling of elements in the multiset I_α
$D = \min\{S \in [1, |\alpha|] \mid f_\alpha(S) \text{ is unbarred}\}$ ▷ $D = \sum_i^n k_i + \sum_i^{r-1} \mu_{\bar{i}} + 1$
$D' = \max\{S \in [1, |\alpha|] \mid f_\alpha(S) \text{ is barred}\}$ ▷ $D' = \sum_i^n k_i + \sum_i^{r-1} \mu_{\bar{i}}$
$M = 1$
while $D \leq |\alpha|$ **do**
  partners = False
  $X = f_\alpha(D) + \delta_\alpha(D', D)$
  while partners == False **do**
    if $X < M + \text{nred}$ or $M \geq R$ **then**
      partners = True
      $T_k(D') = \overline{X}$, $T_k(D) = X$ (the boxes D' and D are said to be *partners*)
      $D = D + 1$, $D' = D' - 1$
    else
      $T_k(S) = \overline{M + \text{nred}}$ for all $S \in [D' - \mu_{\overline{M}} + 1, D']$
      $D' = D' - \mu_{\overline{M}}$
      $M = M + 1$
    end if
  end while
end while

Example 2.4.

Let $T = $

$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1	1	1	2	2	2	3

.

Then $n = 3$, $(k_3, k_2, k_1) = (1, 3, 4)$ and $\mu = (\mu_{\bar{3}}, \mu_{\bar{2}}, \mu_{\bar{1}}) = (3, 2, 1)$. We would like to compute T_{78}. Since $p = 22$, we know that $\alpha = \text{shift}^{78}((22), \mu)_1$ which is equal to $(2, 2, 3, 4, 3, 3, 2)$. Moreover $\text{shift}^{78}((22), \mu)_2 = (2, 1)$, thus $\text{nred} = 1$, $R = 3$ and $I_\alpha = \{\bar{4}, \bar{3}^5, \bar{2}^5, 2^4, 3^3, 4\}$.

Let us first assign labels to all the boxes in α according to the reading order:

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 2 \\
\cline{1-2}
3 & 4 \\
\cline{1-3}
5 & 6 & 7 \\
\hline
8 & 9 & 10 & 11 \\
\hline
12 & 13 & 14 \\
\cline{1-3}
15 & 16 & 17 \\
\cline{1-3}
18 & 19 \\
\cline{1-2}
\end{array}\,.
$$

Note that at the beginning of our algorithm $D = 12, D' = 11$ and $M = 1$. We perform the algorithm described above to find a partner box for 12 in α and to calculate their contents. Since

$$\delta_\alpha(11, 12) = 0,$$

we have that $X = f_\alpha(12) + \delta_\alpha(11, 12) = 2 = M + \text{nred}$. Therefore 12 and 11 are not partners, and $T_{78}(11) = \overline{2}, D' = 10, M = 2$. Thus 11 is a single and we are still looking for a partner for 12. $X = f_\alpha(12) + \delta_\alpha(10, 12) = 2 < 3 = M + \text{nred}$ now, which means that partner$(12) = 10$ so $\overline{T_k(12)} = T_k(10) = 2$ and $D = 13, D' = 9$. Similarly as before partner$(13) = 9$ so $\overline{T_k(13)} = T_k(9) = 2$ and $D = 14, D' = 8$. At this step our tableau has the following form:

$$
\begin{array}{|c|c|c|c|}
\hline
 & \\
\cline{1-2}
 & \\
\cline{1-3}
 & & \\
\hline
 & \overline{2} & \overline{2} & \overline{2} \\
\hline
2 & 2 & \\
\cline{1-3}
 & & \\
\cline{1-3}
 & \\
\cline{1-2}
\end{array}\,.
$$

Note that now

$$\delta_\alpha(8, 14) = 1,$$

so 8 and 14 are not partners since $X = f_\alpha(14) + \delta_\alpha(8, 14) = 2 + 1 = M + \text{nred}$. Therefore 8 and 9 are singles and $T_{78}(7) = T_{78}(8) = \overline{3}, D' = 6, M = 3$. But now $M \geq 3 = R$ so our algorithm will assign partners at every step: $(14, 6), (15, 5), (16, 4), (17, 3), (18, 2), (19, 1)$. Moreover, the distances between partners are as follows:

$$
\begin{aligned}
\delta_\alpha(6, 14) = \delta_\alpha(5, 15) &= 2, \\
\delta_\alpha(4, 16) &= 3, \\
\delta_\alpha(3, 17) &= 4, \\
\delta_\alpha(2, 18) &= 5, \\
\delta_\alpha(1, 19) &= 6.
\end{aligned}
$$

Since

$$\begin{aligned} f_\alpha(6) = f_\alpha(5) &= 2, \\ f_\alpha(4) = f_\alpha(3) = f_\alpha(2) &= 3, \\ f_\alpha(1) &= 4, \end{aligned}$$

we obtain the following tableau T_{78}:

$$\begin{array}{|c|c|c|c|} \cline{1-2} \overline{10} & \overline{8} \\ \cline{1-2} \overline{7} & \overline{6} \\ \cline{1-3} \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} & \overline{2} \\ \hline 2 & 2 & 4 \\ \cline{1-3} 4 & 6 & 7 \\ \cline{1-3} 8 & 10 \\ \cline{1-2} \end{array}.$$

In [2], we show the following proposition, which is the key ingredient in the proof of our main result.

Proposition 2.5. *Let $\mu = (\mu_{\overline{n}}, \mu_{\overline{n-1}}, \ldots, \mu_{\overline{1}})$ be a partition, $\ell(\mu) = \#\{i\colon \mu_i > 0\}$, and let $T \in \mathrm{SympTab}_n((p), \mu)$. Then*

$$\mathrm{Cyc}_C^k(T) = \mathrm{red}(\mathrm{grav}(T_k)), \tag{2.2}$$

where the reduction operation is as defined in [9, Section 4.3], see also [2, Section 3.4].

Idea of proof. The idea of the proof is to carry out a careful, recursive study of the cyclage, and in particular insertion algorithms of Lecouvey. In particular, we need a slightly different definition of Lecouvey's insertion algorithm, which can be found in [2, Proposition 3.3]. Although the idea of the proof is simple, it is rather involved due to the technical nature of the symplectic insertion algorithm. The reader is therefore referred to the full version of this work, see [2, Section 4]. □

3 A proof of Lecouvey's conjecture for one row and general weight

In this section we are going to apply (2.2) to prove Theorem 1.3. We need the following proposition due to Lecouvey, which is an easy consequence of the following Morris recurrence formula described in [9]:

Proposition 3.1. *[9, Proposition 3.2.3.] Let $\mu = (\mu_{\overline{n}}, \mu_{\overline{n-1}}, \ldots, \mu_{\overline{1}})$ be a partition and $p \geq |\mu|$ be a positive integer. Then*

$$K^{\mathrm{C}_n}_{(p),\mu}(q) = q^{f_n(\mu)} \cdot \sum_{T \in \mathrm{SympTab}_n((p),\mu)} q^{\theta_n(T)}$$

where $f_n(\mu) = \sum_{i=1}^{n}(n-i)\mu_{\bar{i}}$ and

$$\theta_n(T) = \sum_{i=1}^{n}(2(n-i)+1)k_i,$$

where $T \in \mathrm{SSYTab}_{\mathcal{C}_n}((p),(k_n+\mu_{\overline{n}}, k_{n-1}+\mu_{\overline{n-1}}, \ldots, k_1+\mu_{\overline{1}}, k_1, \ldots, k_n))$.

Proof of Theorem 1.3. Let T be as in the statement of Proposition 3.1. Proposition 2.5 (see [2, Section 3.4]) implies that $m(T) = \min\{k : T_k = T_{k+1}\}$, which is simply equal to $m_\mu((p))$ defined by (2.1). Let us compute $E_{C(T)}$. Notice that $C(T)$ is a column of weight 0 and length $\sum_i k_i$. Therefore, for any $\square, \square+1 \in I_{>0}$ we have

$$\begin{aligned} C(T)(\square+1) - C(T)(\square) = \delta_{\mathrm{shape}(C(T))}\big(\mathrm{partner}(\square+1), \square+1\big) - \\ - \delta_{\mathrm{shape}(C(T))}\big(\mathrm{partner}(\square), \square\big) = 2. \end{aligned}$$

Therefore $E_{C(T)}$ consists of all positive entries of $C(T)$ and due to the construction given by Algorithm 2 we know that $\mathrm{nred} = \ell(\mu)$, thus

$$E_{C(T)} = \{i + \ell(\mu) + 2j \colon 1 \leq i \leq n, \sum_{l \leq i-1} k_l \leq j < \sum_{l \leq i} k_l\}.$$

Finally, using Corollary 2.2 we may calculate (see [2]):

$$\mathrm{ch}_n(T) = f_n(\mu) + \theta_n(T)$$

and comparing this with Proposition 3.1 finishes the proof. □

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