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Lattice walks by winding

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Abstract. In 2017, Timothy Budd found exact expressions counting simple walks on the square lattice that wind around a fixed point in the plane a given number of times. Budd found parametric expressions involving Jacobi theta functions for numerous related generating functions. His solution was based on an explicit eigenvalue decomposition of certain matrices. We give an alternative method to derive the same types of results as Budd for four different step sets, including simple walks on the triangular lattice. For each such step set, we can then use the reflection principle to count walks confined to a wedge of opening angle a multiple of $r\pi$, where $r \in \mathbb{Q}$ depends on the step set. One new result in this direction is the enumeration of tandem walks confined to the three quarter plane. For simple walks on the square lattice we rederive many of the results found by Budd.

Abstract. Pendant l'année 2017, Timothy Budd a trouvé les expressions exact pour compter les chemins simple qui enroule un nombre donné de fois autour un point fixé.

Keywords: lattice, path, winding

1 Introduction

We study walks by winding number on four different lattices shown in Figure 1: The triangular lattice, the square lattice, the king lattice and a directed triangular lattice that we call the Kreweras lattice. On the square lattice, the same types of results were discovered by Timothy Budd using a very different method.

Given a point v and a walk w not passing through v, we define the winding angle of w around v as follows: let x be a variable point that moves continuously along the path w, and let $v_x = \frac{x-v}{|x-v|}$ be a variable unit vector pointing from v to x. The *winding angle* of w around v is the total anticlockwise angle that v_x spins around v. If w starts and finishes at the same point, its winding angle θ will necessarily be a multiple of 2π, and in this case the *winding number* is defined as $\frac{\theta}{2\pi}$.

Counting paths by both their endpoint and winding number around v is equivalent to counting paths on the covering space Γ of $\mathbb{C} \setminus \{v\}$, shown in Figure 3.

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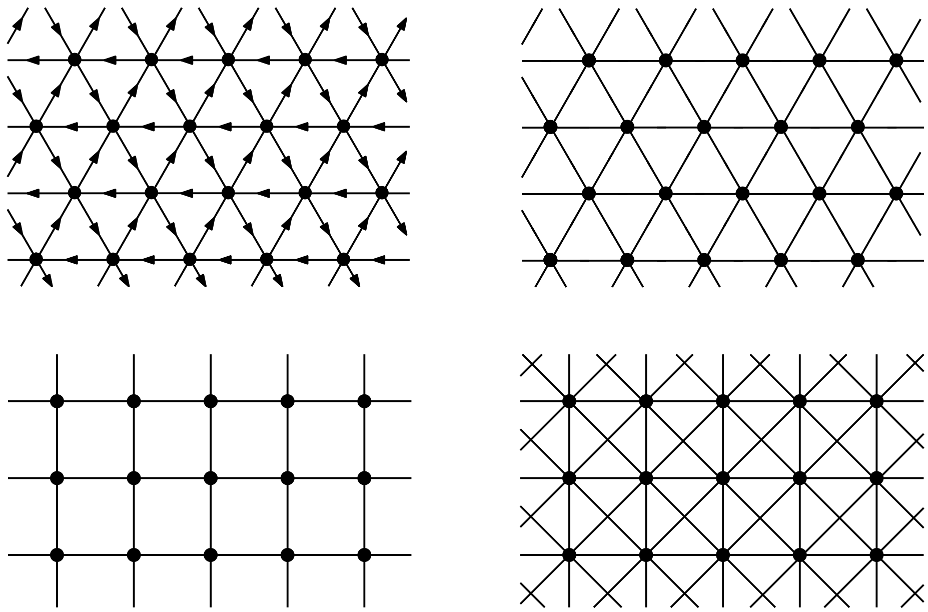

Figure 1: Top left: the Kreweras lattice. Top right: the triangular lattice. Bottom left: the square lattice. Bottom right: the king lattice.

All of our results are in terms of the power series

$$T_k(u,q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^k q^{n(n+1)/2} (u^{n+1} - (-1)^k u^{-n}).$$

which are related to the Jacobi theta function

$$\begin{aligned}\vartheta(z,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz} \\ &= e^{\frac{\pi\tau i}{4}} \left(e^{iz} - e^{-iz}\right) \prod_{n=1}^{\infty} \left(1 - e^{2\pi\tau n i + 2iz}\right) \left(1 - e^{2\pi\tau n i - 2iz}\right) \left(1 - e^{2\pi\tau n i}\right)\end{aligned}$$

by

$$\vartheta^{(k)}(z,\tau) = e^{\frac{(\pi\tau - 2z)i}{2}} i^k T_k(e^{2iz}, e^{2i\pi\tau}).$$

Our first theorem concerns walks that start and end near the point v. More precisely, let T be a trianglular cell in the Kreweras lattice, let v be a point in the centre of T and let w_0 be one of the vertices of T, as shown in Figure 2. Let $E(t,s)$ be the generating function for walks starting at w_0 and ending on a vertex of T in which each walk w of length n and winding angle θ contributes $t^n s^{\frac{3\theta}{2\pi}}$.

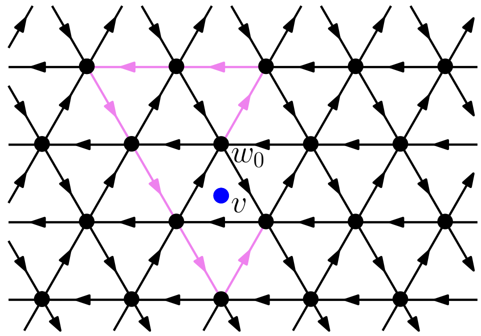

Figure 2: A path with winding angle $\frac{4\pi}{3}$ on the Kreweras lattice. This path contributes s^2t^7 to $E(t,s)$.

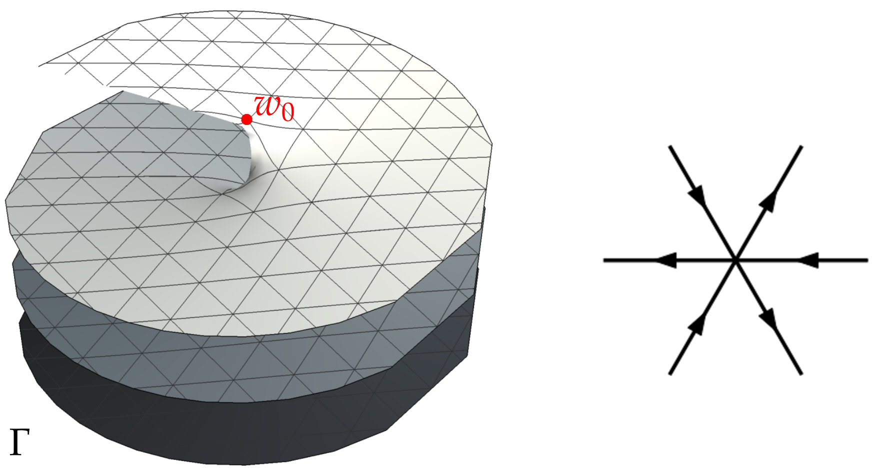

Figure 3: Left: The covering space Γ of $\mathbb{C} \setminus \{v\}$. Right: The allowed steps for Kreweras walks. We enumerate walks on Γ, starting at the red point and following these directions.

Theorem 1. *Let $q(t) \equiv q \in \mathbb{Z}[[t^3]]$ be the unique series with constant term 0 satisfying*

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T(q,q^3)+6T_1(q,q^3)}.$$

Then $E(t,s)$ is given by

$$E(t,s) = \frac{s}{1-s^3}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T(sq^{-2/3},q)}\right)$$

Now, for $c \in \mathbb{Z}$, let $E_c(t) = [s^c]E(t,s)$ denote the generating function of Kreweras walks ending starting at w_0, ending at a vertex adjacent to v and having winding angle

$\frac{2\pi c}{3}$. From Theorem 1 we can extract the following summation formula for $E_c(t)$:

$$E_c(t) =$$

We delay the statement of our most general results to sections 2 and 6, after we have introduced the generating function that we will use to count these walks in general.

2 Walks by winding number on the Kreweras lattice

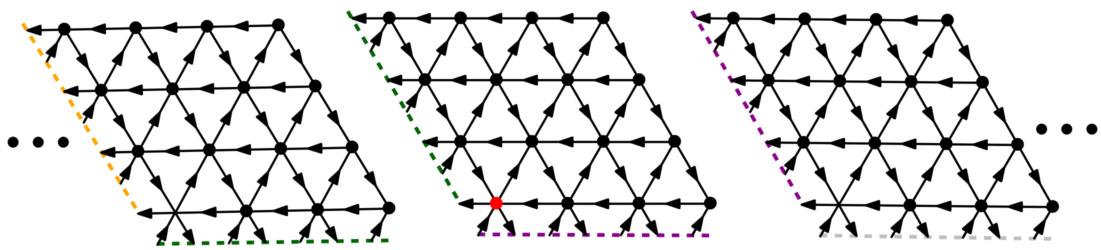

Figure 4: Decomposition of the covering space Γ into an infinite sequence of wedges. Each wedge is attached to its neighbours along the dotted lines.

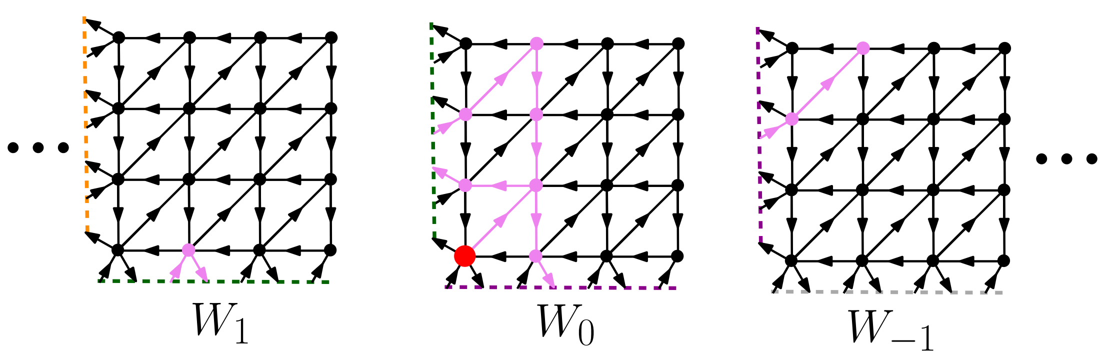

Figure 5: An example of a walk on Γ using Kreweras steps. this walk contributes t^9xy^3s to $\mathsf{K}(t,s,x,y)$.

2.1 Decomposition and functional equations

Counting walks in the plane by end point and winding number around the point v is equivalent to counting walks in the covering space Γ of $\mathbb{C}\setminus\{v\}$ by endpoint (see Figure 3). For each point $w\in\Gamma$ and each $n\in\mathbb{N}_0$, let $p_{n,w}$ be the number of paths of length n in Γ

starting at u and ending at w. In order to characterise the different points $w \in \Gamma$, we partition this covering space into an infinite sequence of wedges, as shown in Figure 4, then we transform these wedges into quarter-planes labelled $\ldots, W_2, W_1, W_0, W_{-1}, W_{-2}, \ldots$ as in Figure 5. Each possible endpoint w then corresponds to a triple (a, b, k), where $w \in W_k$ and (a, b) are the coordinates of w in W_k. This way we associate each point $w \in \Gamma$ with the unique monomial $f_w(s, x, y) = x^a y^b s^k$. Finally, the most general series $\mathsf{K}(t, s, x, y) \in \mathbb{Z}[s, x, y][[t]]$ that we determine in this section is

$$\mathsf{K}(t, s, x, y) = \sum_{n=0}^{\infty} t^n \sum_{w \in \Gamma} p_{n,w} f_w(s, x, y).$$

This is related to the series $\mathsf{E}(t, s)$ from Theorem 1 by

$$\mathsf{E}(t, s) = \mathsf{K}(t, s, 0, 0).$$

By considering all possible final steps of a path in Γ, we derive the following functional equation, which characterises $\mathsf{K}(x, y) \equiv \mathsf{K}(t, s, x, y) \in \mathbb{Z}[s, x, y][[t]]$:

$$\mathsf{K}(x, y) = 1 + txy\mathsf{K}(x, y) + \frac{t}{x}(\mathsf{K}(x, y) - \mathsf{K}(0, y)) + \frac{t}{y}(\mathsf{K}(x, y) - \mathsf{K}(x, 0)) \tag{2.1}$$

$$+ ts\mathsf{K}(0, x) + ts^{-1}y\mathsf{K}(y, 0). \tag{2.2}$$

The last two terms in this equation come from the steps from one wedge to an adjacent wedge, so removing these terms yields the well known functional equation for the generating function $\hat{K}(t, x, y)$ of Kreweras paths confined to a quarter plane.

2.2 Solution in terms of Theta functions

In order to solve the functional equation (2.1), we fix $t, s \in \mathbb{C}$ and think of $K(x, y)$ as an analytic function of x and y. More precisely, we assume $|t| < 1/3$ and $|s| = 1$, as this ensures that the series defining $\mathsf{K}(x, y) \equiv \mathsf{K}(t, s, x, y)$ converges for $|x|, |y| < 1$, using the fact that the total number of paths of length n in Γ is 3^n.

the next step is to parametrise the kernel equation

$$K(x, y) = 1 - txy - \frac{t}{x} - \frac{t}{y} = 0.$$

The kernel here is exactly the same as the case for Kreweras paths confined to a quarter plane. Moreover, (some people) showed that the Kernel equation $K(x, y) = 0$ can be characterised in terms of the Weierstrass elliptic function $\wp$ for any step set. In this case, as well as the other step sets that we consider, the parametrisation can be written more simply in terms of the jacobi theta function

$$\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

as follows:

Lemma 1. *Let $\tau \in i\mathbb{R}_{>0}$ satisfy*

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}$$

and let $X(z)$ and $Y(z)$ be defined by

$$X(z) = e^{-\frac{4\pi\tau i}{3}} \frac{\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)} \quad \text{and} \quad Y(z) = X(z+\pi\tau).$$

Then $K(X(z),Y(z)) = 0$ for all $z \in \mathbb{C}$.

since $X(0) = Y(0) = 0$, we can substitute $x = X(z)$ and $y = Y(z)$ into (2.1) for z in a neighbourhood of 0. Writing

$$L(z) = st\mathsf{K}(0,X(z)) - \frac{t}{Y(z)}\mathsf{K}(X(z),0), \tag{2.3}$$

the resulting equation simplifies to

$$1 = -L(z) + \frac{L(z+\pi\tau)}{sX(z)}.$$

Moreover, we can prove that the only poles of $L(z)$ in $0 \le \mathrm{Im}(z) \le \mathrm{Im}(\pi\tau)$ are simple poles at the points in $\pi\mathbb{Z}$. This uniquely defines $L(z)$:

$$\begin{aligned} L(z) = &\frac{1}{1-e^{3i\alpha}}\left(e^{3i\alpha} + \frac{e^{2i\alpha}}{X(z)} + e^{i\alpha}X(z-\pi\tau)\right) \\ &- \frac{e^{i\alpha+\frac{5i\pi\tau}{3}}\vartheta(\pi\tau,3\tau)\vartheta'(0,\tau)}{(1-e^{3i\alpha})\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)\vartheta'(0,3\tau)} \frac{\vartheta(z-2\pi\tau,3\tau)\vartheta(z+\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta(z,\tau)\vartheta(z,3\tau)} \end{aligned}$$

We can then extract $\mathsf{K}(X(z),0)$ from 2.3 using the fact that $X(z) = X(\pi\tau - z)$:

$$\mathsf{K}(X(z),0) = \frac{L(z) - L(\pi\tau - z)}{tX(z)(X(z+\pi\tau) - X(z-\pi\tau))}.$$

Hence we have an exact, parametric expressions for $\mathsf{K}(x,0)$. We can simarly derive such an expression for $\mathsf{K}(0,y)$, which yields the exact solution for $\mathsf{K}(x,y)$, using (2.1). Substituting $z \to 0$ yields the exact solution for $\mathsf{E}(t,s) = \mathsf{K}(0,0)$, which is:

$$E(t,e^{i\alpha}) = \frac{e^{i\alpha}}{1-e^{3i\alpha}}\left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}}\frac{\vartheta'(2\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}\right)$$

I'm a bit confused about whether the second $-$ above should be $+$. Expanding both t and $E(t,e^{i\alpha})$ as series in $s = e^{i\alpha}$ and $q = e^{\pi\tau i}$ yields Theorem 1.

3 Distribution of winding angles

4 Walks with confined winding angles

In this section we will show how to use our results to count walks confined to wedges. One new result of this type is the enumeration of tandem walks confined to the three quarter-plane.

Let v be a vertex of the triangular lattice. For integers $k_1 < 0 < k_2$ and $k \in (k_1, k_2)$, let $\tilde{E}_{k,k_1,k_2}(t)$ be the generating function for Kreweras walks w in $\mathbb{C} \setminus \{v\}$ with the following properties:

- w starts at a fisxed vertex u_0, adjacent to v
- w ends at a vertex adjacent to v
- w has winding angle $\frac{k\pi}{3}$
- The winding angle is confined to stay within the interval $(\frac{k_1\pi}{3}, \frac{k_2\pi}{3})$ for all intermediate points along w

Let $\hat{E}_k(t)$ denote the generating function of Kreweras walks satisfying only the first three properties, and let

$$\hat{E}(t,u) = \sum_{k=-\infty}^{\infty} u^k \hat{E}_k(t).$$

Using the reflection principle we can show that

$$\tilde{E}_{k,k_1,k_2}(t) = \sum_{j=-\infty}^{\infty} \hat{E}_{k-2jk_1+2jk_2}(t) - \hat{E}_{-k-2jk_1+(2j+2)k_2}(t).$$

This can be written as a sum of values of $\hat{E}(t,u)$ as follows:

$$\tilde{E}_{k,k_1,k_2}(t) = \frac{1}{2(k_2-k_1)} \sum_{j=0}^{2k_2-2k_1-1} \left(e^{-\frac{\pi i jk}{k_2-k_1}} - e^{\frac{\pi i j(k-2k_1)}{k_2-k_1}} \right) \hat{E}\left(t, e^{\frac{\pi i j}{k_2-k_1}}\right).$$

Example 1: taking $v = (-1,0)$, the generating function $E_{0,-1,1}(t)$ counts walks starting and ending at $(0,0)$, which are confined to stay within the wedge with opening angle $120°$, shown in figure **??**. Equivalently, $E_{0,-1,1}(t)$ counts quarter-plane excursions with the reverse Kreweras step set. Moreover, it follows from the equation above that

$$E_{0,-1,1}(t) = \frac{E(t,i) + E(t,-i)}{2}.$$

Example 2: The generating function $E_{0,-2,3}(t)$ counts walks in the 5/6-plane. Equivalently, this counts walks in the three quarter plane, starting and ending at $(0,-1)$, using the tandem step-set (See Figure **??**).

5 Algebraicity results

6 Conclusion and other lattices

In the previous sections we described how to count Kreweras walks on the triangular lattice by length and winding number. Using our results we extracted the asymptotic distribution of the final winding angle. We were also able to enumerate walks on a wedge using the reflection principle. We can use the same method to extract analogous results for walks on the triangular lattice, the square lattice and the king lattice, shown in Figure 1. On the square lattice, many of the same results were found by Timothy Budd in 2017. results were found

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