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On the distribution of the major index on standard Young tableaux

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Abstract. The study of permutation and partition statistics is a classical topic in enumerative combinatorics. The major index statistic on permutations was introduced a century ago by Percy MacMahon in his seminal works. In this extended abstract, we study the well-known generalization of the major index to standard Young tableaux. We present several new results. In one direction, we introduce and study two partial orders on the standard Young tableaux of a given partition shape, in analogy with the strong and weak Bruhat orders on permutations. The existence of such ranked poset structures allows us to classify the realizable major index statistics on standard tableaux of arbitrary straight shape and certain skew shapes, and has representation-theoretic consequences, both for the symmetric group and for Shephard–Todd groups. In a different direction, we consider the distribution of the major index on standard tableaux of arbitrary straight shape and certain skew shapes. We classify all possible limit laws for any sequence of such shapes in terms of a simple auxiliary statistic, aft, generalizing earlier results of Canfield–Janson–Zeilberger, Chen–Wang–Wang, and others. We also study unimodality, log-concavity, and local limit properties.

1 Introduction

For a skew partition $\lambda/\mu \vdash n$, denote by $\mathrm{SYT}(\lambda/\mu)$ the set of all standard Young tableaux of skew shape λ/μ, i.e. the set of all fillings of the cells of the diagram of λ/μ with integers $1, \ldots, n$ that are increasing in rows and columns. We say i is a *descent* in a standard tableau T if $i+1$ appears in a lower row in T than i, where we draw partitions in English notation. Let $\mathrm{maj}(T)$ denote the *major index statistic* on $\mathrm{SYT}(\lambda/\mu)$, which is defined to be the sum of the descents of T.

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This statistic is a generalization of the major index on permutations or words, defined by MacMahon in the early 1900's [20] as the sum of all i for which $\pi_{i+1} > \pi_i$. The distribution of the major index on words is a classic and surprisingly deep topic, implicitly going back before MacMahon to Sylvester's 1878 proof of the unimodality of the q-binomial coefficients [27] and beyond[1]. In this extended abstract, we summarize several recent explorations of the distribution of the major index on ordinary shape tableaux and certain skew tableaux.

The major index generating function for $\mathrm{SYT}(\lambda/\mu)$ for a straight shape λ/μ is given by

$$\mathrm{SYT}(\lambda/\mu)^{\mathrm{maj}}(q) := \sum_{T\in\mathrm{SYT}(\lambda/\mu)} q^{\mathrm{maj}(T)} = \sum_{k\geq 0} b_{\lambda/\mu,k}q^k \tag{1.1}$$

where the coefficients are the *fake degree sequence*

$$b_{\lambda/\mu,k} := \#\{T \in \mathrm{SYT}(\lambda/\mu) : \mathrm{maj}(T) = k\} \text{ for } k = 0, 1, 2, \ldots \tag{1.2}$$

The fake degrees for straight shapes λ and certain skew shapes have appeared in a variety of algebraic and representation-theoretic contexts including Green's work on the degree polynomials of unipotent $\mathrm{GL}_n(\mathbb{F}_q)$-representations [13, Lemma 7.4], the irreducible decomposition of type A coinvariant algebras [22, Proposition 4.11], Lusztig's work on the irreducible representations of classical groups [19], and branching rules between symmetric groups and cyclic subgroups [24, Theorem 3.3]. The term "fake degree" was apparently coined by Lusztig [7], most likely because $\#\mathrm{SYT}(\lambda) = \sum_{k\geq 0} b_{\lambda,k}$ is the degree of the irreducible S_n-representation indexed by λ, so a q-analog of this number is not itself a degree but is related to the degree.

We consider three natural enumerative questions involving the fake degrees:

Q1. Which $b_{\lambda,k}$ are zero?

Q2. Are there efficient asymptotic estimates for $b_{\lambda,k}$?

Q3. Are the fake degree sequences unimodal?

Our results are presented in full in [3, 2]. Given the length of these papers and space limitations for this extended abstract, we present no proofs, and instead provide very rough sketches with references to the full papers. We describe the answer to Q1 in Section 2, a complete answer to one precise version of Q2 in Section 3, and further work and open problems related to Q3 and beyond in Section 4.

[1] Indeed, Sylvester's excitement at settling this then-quarter-century-old conjecture is palpable: "I accomplished with scarcely an effort a task which I had believed lay outside the range of human power."

2 Zeros of the fake degree sequence

We completely settle Q1 with the following result. Write $b(\lambda) := \sum(i-1)\lambda_i$ and let λ' denote the conjugate partition of λ.

Theorem 2.1. *For every partition $\lambda \vdash n \geq 1$ and integer k such that $b(\lambda) \leq k \leq \binom{n}{2} - b(\lambda')$, we have $b_{\lambda,k} > 0$ except in the case when λ is a rectangle with at least two rows and columns and k is either $b(\lambda)+1$ or $\binom{n}{2} - b(\lambda') - 1$. Furthermore, $b_{\lambda,k} = 0$ for $k < b(\lambda)$ or $k > \binom{n}{2} - b(\lambda')$.*

The main ingredient of the proof is a map $\varphi : \mathrm{SYT}(\lambda) \setminus \mathcal{E}(\lambda) \to \mathrm{SYT}(\lambda)$ with the property $\mathrm{maj}(\varphi(T)) = \mathrm{maj}(T) + 1$. Here $\mathcal{E}(\lambda)$ is the (small) set of exceptional tableaux where such a map cannot be defined and contains:

i. For all λ, the tableau for λ with the largest possible major index.

ii. If λ is a rectangle, the tableau for λ with the smallest possible major index.

iii. If λ is a rectangle with at least two rows and columns, the unique tableau in $\mathrm{SYT}(\lambda)$ with major index equal to $\binom{n}{2} - b(\lambda') - 2$.

The construction of such a map goes roughly as follows. For most tableaux, we can apply a simple *rotation rule* that increases the major index by 1. More specifically, given $T \in \mathrm{SYT}(\lambda)$, assume that we have an interval $[i,k] \subset [n]$ such that $T' := (i, i+1, \ldots, k-1, k) \cdot T$ is in $\mathrm{SYT}(\lambda)$ and such that there is some j for which

$$\{j\} = \mathrm{Des}(T') - \mathrm{Des}(T) \qquad \text{and} \qquad \{j-1\} = \mathrm{Des}(T) - \mathrm{Des}(T').$$

For example, assume $i < j < k$. Then we require that:

- $i, \ldots, j-1$ form a horizontal strip, $j-1, j$ form a vertical strip, and $j, j+1, \ldots, k$ form a horizontal strip;
- i appears strictly northeast of k and $i-1$ is not in the rectangle bounding i and k;
- k appears strictly northeast of $k-1$ and $k+1$ is not in the rectangle bounding k and $k-1$;

A sketch of the positive rotation in this case is presented in Figure 1. The cases $i = j$ and $j = k$ are slightly different. If i, j, k with the required properties exist, we take the lexicographically smallest such numbers, and define $\varphi(T) := T'$.

We similarly define *negative rotations* where $T' = (k, k-1, \ldots, i) \cdot T$. If no positive rotations are possible on a tableau T, but we have negative rotations, use the lexicographically smallest one to define $\varphi(T)$.

There are very few cases when neither a positive nor a negative rotation can be applied. For example, among the 81,081 tableaux in $\mathrm{SYT}(5442)$, there are only 24 on

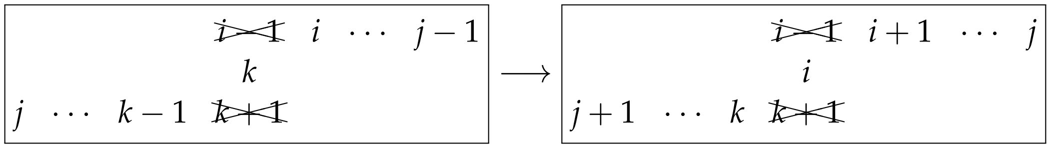


Figure 1: A positive rotation with $i < j < k$.

which we cannot apply any positive or negative rotation rule. In particular, no rotation rules can be applied to the following two tableaux:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & \\ 14 & 15 & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & 2 & 3 & 8 & 12 \\ 4 & 6 & 9 & 13 & \\ 5 & 7 & 10 & 14 & \\ 11 & 15 & & & \end{array} .$$

In that case, we have to apply what we call *block rules* B1–B5. We refer the reader to [2, Definition 4.13] for explicit definitions.

As a consequence of the proof of Theorem 2.1, we identify two new ranked poset structures on $\mathrm{SYT}(\lambda)$ where the rank function is determined by maj. Furthermore, as a corollary of Theorem 2.1 we have a new proof of a complete classification due to the third author [26, Theorem 1.4] generalizing an earlier result of Klyachko [17] for when the counts

$$a_{\lambda,r} := \{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\} \text{ for } \lambda \vdash n$$

are nonzero. We can also classify internal zeros of certain skew shapes, [2, Lemma 6.2].

Remark 2.2. Lascoux–Schützenberger [18] defined an operation called *cyclage* on semistandard tableaux, which decreases *cocharge* by 1. The *cyclage poset* on the set of semistandard tableaux arises from applying cyclage in all possible ways. Cyclage preserves the *content*, i.e. the number of 1's, 2's, etc. See also [21, Section 4.2]. Restricting to standard tableaux, cocharge coincides with maj, so the cyclage poset on $\mathrm{SYT}(n)$ is ranked by maj. However, cyclage does not necessarily preserve the shape, so it does not suffice to prove Theorem 2.1. For example, restricting the cyclage poset to $\mathrm{SYT}(32)$ gives a poset which has two connected components and is not ranked by maj, while both of our poset structures on $\mathrm{SYT}(32)$ are chains. One reviewer posed an interesting question: is there any relation between the cyclage poset covering relations restricted to $\mathrm{SYT}(\lambda)$ and the two ranked poset structures used to prove Theorem 2.1? We have not found one.

Symmetric groups are the finite reflection groups of type A. The classification and invariant theory of both finite irreducible real reflection groups and complex reflection groups developed over the past century builds on our understanding of the type A

case [15]. In particular, these groups are classified by Shephard–Todd into an infinite family $G(m,d,n)$ together with 34 exceptions. Using work of Stembridge on generalized exponents for irreducible representations, the analog of (1.1) can be phrased for all Shephard–Todd groups as

$$g^{\{\underline{\lambda}\}^d}(q) := \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d} \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m) = \sum b_{\{\underline{\lambda}\}^d,k} q^k \tag{2.1}$$

where $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is a sequence of m partitions with n cells total, $\alpha(\underline{\lambda}) = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|) \vDash n$, $d \mid m$, and $\{\underline{\lambda}\}^d$ is the orbit of $\underline{\lambda}$ under the group C_d of (m/d)-fold cyclic rotations; see [2, Corollary 8.2]. The polynomials $\left[\begin{smallmatrix} n \\ \alpha(\underline{\lambda}) \end{smallmatrix}\right]_{q;d}$ are deformations of the usual q-multinomial coefficients which we explore in [2, Section 7]. The coefficients $b_{\{\underline{\lambda}\}^d,k}$ are the fake degrees in this case.

We use (2.1) and Theorem 2.1 to completely classify all nonzero fake degrees for coinvariant algebras for all Shephard–Todd groups $G(m,d,n)$, which includes the finite real reflection groups in types A, B, and D. See [2, Section 8] for details.

3 Asymptotic normality of the major index on SYT

Let us turn our attention to question Q2. It is a well-known fact that the major index statistic on permutations satisfies a central limit theorem. Given a real-valued random variable $\mathcal{X}$, we let $\mathcal{X}^* := \frac{\mathcal{X}-\mu}{\sigma}$ denote the corresponding normalized random variable with mean 0 and variance 1.

Theorem 3.1 ([11])**.** *Let $\mathcal{X}_n[\mathrm{maj}]$ denote the major index random variable on S_n under the uniform distribution. Then, for all $t \in \mathbb{R}$,*

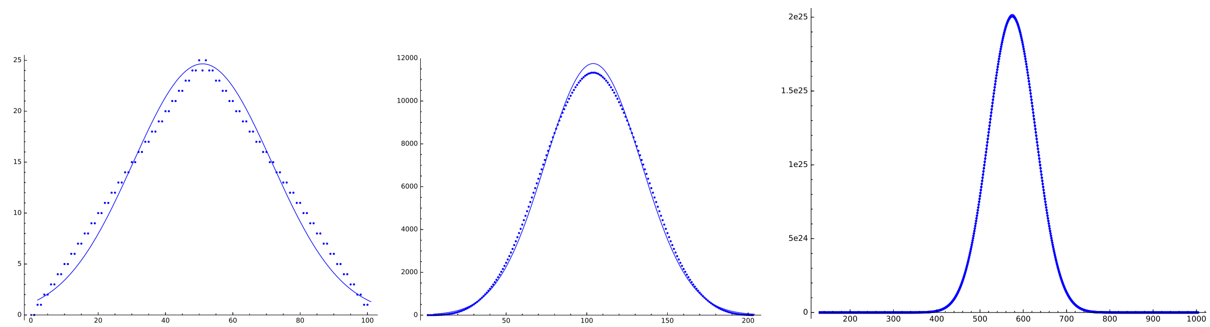
$$\lim_{n\to\infty} \mathbb{P}[\mathcal{X}_n[\mathrm{maj}]^* \le t] = \mathbb{P}[\mathcal{N} \le t]$$

where $\mathcal{N}$ is the standard normal random variable.

Briefly, we say maj on S_n is *asymptotically normal* as $n \to \infty$. See [3, Table 1] for further examples of asymptotic normality. Figure 2 shows some sample distributions for the major index on standard tableaux for three particular partition shapes. Note that Gaussian approximations fit the data well.

In Theorem 3.1, we simply let $n \to \infty$. For partitions, the shape λ may "go to infinity" in many different ways. The following statistic on partitions overcomes this difficulty.

Definition 3.2. Suppose λ is a partition. Let the *aft* of λ be $\mathrm{aft}(\lambda) := |\lambda| - \max\{\lambda_1, \lambda'_1\}$.



(a) $\lambda = (50, 2)$, $\operatorname{aft}(\lambda) = 2$ (b) $\lambda = (50, 3, 1)$, $\operatorname{aft}(\lambda) = 4$ (c) $\lambda = (8^2, 7, 6, 5^3, 2^2)$, aft $= 39$

Figure 2: Plots of $\#\{T \in \mathrm{SYT}(\lambda) : \operatorname{maj}(T) = k\}$ as a function of k for three partitions λ, overlaid with scaled Gaussian approximations using the same mean and variance.

Intuitively, if the first row of λ is at least as long as the first column, then $\operatorname{aft}(\lambda)$ is the number of cells *not* in the first row. This definition is strongly reminiscent of a *representation stability* result of Church and Farb [9, Theorem 7.1], which is proved with an analysis of the major index on standard tableaux.

Our first main result gives the analogue of Theorem 3.1 for maj on $\mathrm{SYT}(\lambda)$. In particular, it completely classifies which sequences of partition shapes give rise to asymptotically normal sequences of maj statistics on standard tableaux.

Theorem 3.3. *Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $\mathcal{X}_N = \mathcal{X}_{\lambda^{(N)}}[\operatorname{maj}]$ be the corresponding random variables for the* maj *statistic on* $\mathrm{SYT}(\lambda^{(N)})$. *Then, the sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ is asymptotically normal if and only if* $\operatorname{aft}(\lambda^{(N)}) \to \infty$ *as $N \to \infty$.*

Remark 3.4. In [3], we more generally consider maj on $\mathrm{SYT}(\underline{\lambda})$ where $\underline{\lambda}$ is a *block diagonal* skew partition. Special cases of this include Canfield–Janson–Zeilberger's main result in [5] classifying asymptotic normality for inv or maj on words (though see [6] for earlier, essentially equivalent results due to Diaconis [10]). The case of words generalizes Theorem 3.1. The $\lambda^{(N)} = (N, N)$ case of Theorem 3.3 also recovers the main result of Chen–Wang–Wang [8], giving asymptotic normality for q-Catalan coefficients.

Our proof of Theorem 3.3 relies on the *method of moments*, which requires useful descriptions of the moments of $\mathcal{X}_\lambda[\operatorname{maj}]$. Adin–Roichman [1] gave exact formulas for the mean and variance of $\mathcal{X}_\lambda[\operatorname{maj}]$ in terms of the hook lengths of λ. These formulas are obtained from Stanley's elegant closed form for the polynomials $\mathrm{SYT}(\lambda)^{\operatorname{maj}}(q)$. Let $h_c = \lambda_i + \lambda'_j - i - j + 1$ denote the hook length of the cell $c = (i, j)$.

Theorem 3.5 ([23, p. 7.21.5])**.** *Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then*

$$\mathrm{SYT}(\lambda)^{\operatorname{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c \in \lambda}[h_c]_q}. \tag{3.1}$$

More generally, formulas for the dth moment μ_d^λ, dth central moment α_d^λ, and dth *cumulant* κ_d^λ of maj on SYT(λ) may be derived from Theorem 3.5. Here the cumulants $\kappa_1, \kappa_2, \ldots$ of $\mathcal{X}$ are defined to be the coefficients of the exponential generating function

$$K_{\mathcal{X}}(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_{\mathcal{X}}(t) = \log \mathbb{E}[e^{t\mathcal{X}}].$$

The most elegant of these formulas is for the cumulants, from which the moments and central moments are all easy to compute.

Theorem 3.6. *Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. We have*

$$\kappa_d^\lambda = \frac{B_d}{d} \left[\sum_{j=1}^{n} j^d - \sum_{c \in \lambda} h_c^d \right] \tag{3.2}$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers.

See [3, Theorem 2.9] for a generalization of (3.2) along with exact formulas for the moments and central moments. See [3, Remark 2.10] for some of the history of this formula.

For "most" partition shapes, one expects the term $\sum_{j=1}^n j^d$ in (3.2) to dominate $\sum_{c\in\lambda} h_c^d$, in which case asymptotic normality is quite straightforward. However, for some shapes there is a very large amount of cancellation in (3.2) and determining the limit law can be quite subtle.

Remark 3.7. $\mathcal{X}_\lambda[\mathrm{maj}]$ can be written as the sum of scaled indicator random variables $D_1, 2D_2, 3D_3, \ldots, (n-1)D_{n-1}$ where D_i determines if there is a descent at position i. However, the D_i are not at all independent, so one may not simply apply standard central limit theorems. Interestingly, the D_i are identically distributed [23, Proposition 7.19.9]. The lack of independence of the D_i's likewise complicates related work by Fulman [12] and Kim–Lee [16] considering the limiting distribution of descents.

The non-normal continuous limit laws for maj on SYT(λ) turn out to be the *Irwin–Hall distributions* $\mathcal{IH}_M := \sum_{k=1}^M \mathcal{U}[0,1]$, which are the sum of M i.i.d. continuous $[0,1]$ random variables. The following result completely classifies all possible limit laws for maj on SYT(λ) for any sequence of partition shapes.

Theorem 3.8. *Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Then $(\mathcal{X}^*_{\lambda^{(N)}}[\mathrm{maj}])$ converges in distribution if and only if*

1. $\mathrm{aft}(\lambda^{(N)}) \to \infty$*; or*
2. $|\lambda^{(N)}| \to \infty$ *and* $\mathrm{aft}(\lambda^{(N)}) \to M < \infty$*; or*
3. *the distribution of* $\mathcal{X}^*_{\lambda^{(N)}}[\mathrm{maj}]$ *is eventually constant.*

The limit law is $\mathcal{N}$ in case (i), $\mathcal{IH}^*_M$ in case (ii), and discrete in case (iii).

Case (iii) naturally leads to the question: when does $\mathcal{X}^*_\lambda[\text{maj}] = \mathcal{X}^*_\mu[\text{maj}]$? Such a description in terms of hook lengths is given in [3, Theorem 7.1].

Example 3.9. We illustrate each possible limit. For (i), let $\lambda^{(N)} := (N, \lfloor \ln N \rfloor)$, so that $\text{aft}(\lambda^{(N)}) = \lfloor \ln N \rfloor \to \infty$ and the distributions are asymptotically normal. For (ii), fix $M \in \mathbb{Z}_{\geq 0}$ and let $\lambda^{(N)} := (N + M, M)$, so that $\text{aft}(\lambda^{(N)}) = M$ is constant and the distributions converge to $\mathcal{IH}^*_M$. For (iii), let $\lambda^{(2N)} := (12, 12, 3, 3, 3, 2, 2, 1, 1)$ and $\lambda^{(2N+1)} := (15, 6, 6, 6, 4, 2)$, which have the same multisets of hook lengths despite not being transposes of each other, and consequently the same normalized distributions.

In order to be able to use the method of moments, we need the following result.

Theorem 3.10 (Frechét–Shohat Theorem, [4, Theorem 30.2])**.** *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots$ be a sequence of real-valued random variables, and let $\mathcal{X}$ be a real-valued random variable. Suppose the moments of $\mathcal{X}_n$ and $\mathcal{X}$ all exist and the moment generating functions all have a positive radius of convergence. If*

$$\lim_{n\to\infty} \mu_d^{\mathcal{X}_n} = \mu_d^{\mathcal{X}} \quad \forall d \in \mathbb{Z}_{\geq 1}, \tag{3.3}$$

then $\mathcal{X}_1, \mathcal{X}_2, \ldots$ converges in distribution to $\mathcal{X}$.

By Theorem 3.10 we may test for asymptotic normality on level of individual normalized moments, which is often referred to as the *method of moments*. By the formula

$$\mu_d = \kappa_d + \sum_{m=1}^{d-1} \binom{d-1}{m-1} \kappa_m \mu_{d-m}, \tag{3.4}$$

which is not hard to derive, we may further replace the moment condition (3.3) with the corresponding cumulant condition. For instance, we have the following explicit criterion.

Corollary 3.11. *A sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ of real-valued random variables on finite sets is asymptotically normal if for all $d \geq 3$ we have*

$$\lim_{n\to\infty} \frac{\kappa_d^{\mathcal{X}_n}}{(\sigma^{\mathcal{X}_n})^d} = 0. \tag{3.5}$$

The hardest part of the proof of Theorem 3.8 is asymptotic normality. We can prove that part by using the following three lemmas. We refer to [3, Section 5] for proofs and further background.

Definition 3.12. A *reverse standard Young tableau* of shape λ/ν is a bijective filling of λ/ν which strictly decreases along rows and columns. The set of reverse standard Young tableaux of shape λ/ν is denoted $\text{RSYT}(\lambda/\nu)$.

Lemma 3.13. *Let $\lambda/\nu \vdash n$ and $T \in \mathrm{RSYT}(\lambda/\nu)$. Then for all $c \in \lambda/\nu$,*

$$T_c \geq h_c. \tag{3.6}$$

Furthermore, for any positive integer d,

$$\sum_{j=1}^{n} j^d - \sum_{c\in\lambda/\nu} h_c^d = \sum_{c\in\lambda/\nu} (T_c^d - h_c^d) = \sum_{c\in\lambda/\nu} (T_c - h_c)\mathbf{h}_{d-1}(T_c, h_c), \tag{3.7}$$

where $\mathbf{h}_{d-1}$ denotes the complete homogeneous symmetric function.

Lemma 3.14. *Let $\lambda/\nu \vdash n$ such that $\max_{c\in\lambda/\nu} h_c < 0.8n$. Let d be any positive integer. Then*

$$\frac{n^{d+1}}{26(d+1)} - 2(0.8)^d n^d < \sum_{j=1}^{n} j^d - \sum_{c\in\lambda/\nu} h_c^d < \frac{n^{d+1}}{d+1} + n^d.$$

Lemma 3.15. *Let $\lambda/\nu \vdash n$ such that $\max_{c\in\lambda/\nu} h_c \geq 0.8n$, and let d be any positive integer. Furthermore, suppose $n \geq 10$. Then,*

$$\mathrm{aft}(\lambda/\nu)\frac{\lfloor 0.1n \rfloor^d}{d} \;\leq\; \sum_{j=1}^{n} j^d - \sum_{c\in\lambda/\nu} h_c^d \;\leq\; 2\,\mathrm{aft}(\lambda/\nu)\left(n^d + dn^{d-1}\right). \tag{3.8}$$

Corollary 3.16. *For fixed $d \in \mathbb{Z}_{\geq 1}$, uniformly for all skew shapes λ/ν,*

$$\sum_{k=1}^{|\lambda/\nu|} k^d - \sum_{c\in\lambda/\nu} h_c^d = \Theta(\mathrm{aft}(\lambda/\nu)\cdot|\lambda/\nu|^d). \tag{3.9}$$

Proof. Let $n = |\lambda/\nu|$. When $\max_{c\in\lambda/\nu} h_c \geq 0.8n$, the result follows from Lemma 3.15. On the other hand, when $\max_{c\in\lambda/\nu} h_c < 0.8n$, then $n \geq \mathrm{aft}(\lambda/\nu) \geq 0.2n$, and the result follows from Lemma 3.14. □

Corollary 3.17. *Fix d to be an even positive integer. Uniformly for all block diagonal skew shapes $\underline{\lambda}$, the absolute value of the normalized cumulant $|\kappa_d^{\underline{\lambda}*}|$ of $\mathcal{X}_{\underline{\lambda}}[\mathrm{maj}]$ is $\Theta(\mathrm{aft}(\underline{\lambda})^{1-d/2})$.*

Proof. For d even, by a generalization of Theorem 3.6 to block diagonal shapes and Corollary 3.16, we have $|\kappa_d^{\underline{\lambda}}| = \Theta(\mathrm{aft}(\underline{\lambda})n^d)$, where $n = |\underline{\lambda}|$. Consequently,

$$|\kappa_d^{\underline{\lambda}*}| = \left|\frac{\kappa_d^{\underline{\lambda}}}{(\kappa_2^{\underline{\lambda}})^{d/2}}\right| = \Theta\left(\frac{\mathrm{aft}(\underline{\lambda})n^d}{\mathrm{aft}(\underline{\lambda})^{d/2}n^d}\right) = \Theta(\mathrm{aft}(\underline{\lambda})^{1-d/2})$$

by the homogeneity of cumulants. □

A natural generalization of Theorem 3.3 to block diagonal skew shapes $\underline{\lambda}$ follows by combining Corollary 3.11, Corollary 3.17, and similar estimates in the "degenerate" case when $\mathrm{aft}(\underline{\lambda})$ is bounded. See [3, Theorem 6.3].

4 Unimodality and beyond

The easy answer to question Q3 is "no": the fake degree sequences are not always unimodal. For example, $\mathrm{SYT}(42)^{\mathrm{maj}}(q)$ is not unimodal. Nonetheless, certain inversion number generating functions $p_\alpha^{(k)}(q)$ which appear in a generalization of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are in fact unimodal; see [2, Definition 7.7, Corollary 7.10]. Furthermore, computational evidence suggests $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ is typically not far from unimodal. See also [3, Section 8] concerning unimodality, log-concavity, and asymptotic normality for skew shapes.

Conjecture 4.1. *The polynomial* $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *is unimodal if* λ *has at least* 4 *corners.*

In another direction, one may ask for a precise estimate of the deviation of $b_{\lambda,k}$ from normal with an explicit error bound. Such a result is called a *local limit theorem*.

Conjecture 4.2. *Let* $\lambda \vdash n$ *be any partition. Uniformly for all* n*, for all integers* k*, we have*

$$\left|\mathbb{P}[X_\lambda[\mathrm{maj}] = k] - f(k; \kappa_1^\lambda, \kappa_2^\lambda)\right| = O\left(\frac{1}{\sigma_\lambda \, \mathsf{aft}(\lambda)}\right).$$

A sequence $a_0, a_1, a_2, \ldots$ is *parity-unimodal* if $a_0, a_2, a_4, \ldots$ and $a_1, a_3, a_5, \ldots$ are each unimodal. Stucky [25, Theorem 1.3] recently showed that the q-Catalan polynomials, namely $\mathrm{SYT}((n,n))^{maj}(q)$ up to a q-shift, are parity-unimodal. The argument involves constructing an $\mathfrak{sl}_2$-action on rational Cherednik algebras. See [14, Section 3.1] for a prototype of the argument in a highly related context. Based on Stucky's result, our internal zeros classification, and a brute-force check for $n \leq 50$, we conjecture the following.

Conjecture 4.3. *The fake-degree polynomials* $f^\lambda(q)$ *are parity-unimodal for all* λ*.*

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References

[1] R. M. Adin and Y. Roichman. "Descent functions and random Young tableaux". *Combin. Probab. Comput.* **10**.3 (2001), pp. 187–201. Link.

[2] S. C. Billey, M. Konvalinka, and J. P. Swanson. "Tableau posets and the fake degrees of coinvariant algebras". *Advances in Mathematics* **371** (2020). Link.

[3] S. C. Billey, M. Konvalinka, and J. P. Swanson. "Asymptotic normality of the major index on standard tableaux". *Adv. in Appl. Math.* **113** (2020), pp. 101972, 36. Link.

[4] P. Billingsley. *Probability and measure*. Third. Wiley Series in Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995, pp. xiv+593.

[5] E. R. Canfield, S. Janson, and D. Zeilberger. "The Mahonian probability distribution on words is asymptotically normal". *Adv. in Appl. Math.* **46**.1-4 (2011), pp. 109–124. Link.

[6] E. R. Canfield, S. Janson, and D. Zeilberger. "Corrigendum to "The Mahonian probability distribution on words is asymptotically normal" [Adv. in Appl. Math. 46 (1–4) (2011) 109–124] [MR2794017]". *Adv. in Appl. Math.* **49**.1 (2012), p. 77. Link.

[7] R. W. Carter. *Simple Groups of Lie Type*. Wiley Classics Library. Reprint of the 1972 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989, pp. x+335.

[8] W. Y. C. Chen, C. J. Wang, and L. X. W. Wang. "The limiting distribution of the coefficients of the q-Catalan numbers". *Proc. Amer. Math. Soc.* **136**.11 (2008), pp. 3759–3767. Link.

[9] T. Church and B. Farb. "Representation theory and homological stability". *Adv. Math.* **245** (2013), pp. 250–314. Link.

[10] P. Diaconis. *Group representations in probability and statistics*. Vol. 11. Institute of Mathematical Statistics Lecture Notes—Monograph Series. Institute of Mathematical Statistics, Hayward, CA, 1988, pp. vi+198.

[11] W. Feller. "The fundamental limit theorems in probability". *Bull. Amer. Math. Soc.* **51** (1945), pp. 800–832. Link.

[12] J. Fulman. "The distribution of descents in fixed conjugacy classes of the symmetric groups". *J. Combin. Theory Ser. A* **84**.2 (1998), pp. 171–180. Link.

[13] J. A. Green. "The characters of the finite general linear groups". *Trans. Amer. Math. Soc.* **80** (1955), pp. 402–447. Link.

[14] M. Haiman. "Conjectures on the quotient ring by diagonal invariants". *J. Algebraic Combin.* **3**.1 (1994), pp. 17–76. Link.

[15] J. E. Humphreys. *Reflection groups and Coxeter groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1990.

[16] G. B. Kim and S. Lee. "Central limit theorem for descents in conjugacy classes of S_n". *J. Combin. Theory Ser. A* **169** (2020), pp. 105123, 13. Link.

[17] A. A. Klyachko. "Lie elements in the tensor algebra". *Siberian Mathematical Journal* **15**.6 (1974), pp. 914–920. Link.

[18] A. Lascoux and M.-P. Schützenberger. "Le monoïde plaxique". *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*. Vol. 109. Quad. "Ricerca Sci." Rome: CNR, 1981, pp. 129–156.

[19] G. Lusztig. "Irreducible representations of finite classical groups". *Invent. Math.* **43**.2 (1977), pp. 125–175. Link.

[20] P. A. MacMahon. "The Indices of Permutations and the Derivation Therefrom of Functions of a Single Variable Associated with the Permutations of any Assemblage of Objects". *Amer. J. Math.* **35**.3 (1913), pp. 281–322. Link.

[21] M. Shimozono and J. Weyman. "Graded characters of modules supported in the closure of a nilpotent conjugacy class". *European J. Combin.* **21**.2 (2000), pp. 257–288.

[22] R. P. Stanley. "Invariants of finite groups and their applications to combinatorics". *Bull. Amer. Math. Soc. (N.S.)* **1**.3 (1979), pp. 475–511. Link.

[23] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xii+581. Link.

[24] J. R. Stembridge. "On the eigenvalues of representations of reflection groups and wreath products". *Pacific J. Math.* **140**.2 (1989), pp. 353–396. Link.

[25] E. Stucky. "Cyclic Sieving, Necklaces, and Bracelets". 2018. arXiv:1812.04578.

[26] J. P. Swanson. "On the existence of tableaux with given modular major index". *Algebraic Combinatorics* **1**.1 (2018), pp. 3–21. Link.

[27] J. Sylvester. "XXV. Proof of the hitherto undemonstrated fundamental theorem of invariants". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **5**.30 (1878), pp. 178–188. Link.