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Symplectic keys and Demazure atoms in type C

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Abstract. We compute, mimicking the Lascoux–Schützenberger type A combinatorial procedure, left and right keys for a Kashiwara–Nakashima tableau in type C. These symplectic keys have a role similar to the keys for semistandard Young tableaux. More precisely, our symplectic keys give a tableau criterion for the Bruhat order on the hyperoctahedral group and cosets, and describe Demazure atoms and characters in type C. The right and the left symplectic keys are related through the Lusztig involution. A type C Schützenberger evacuation is defined to realize that involution.

Résumé. Nous calculons, en imitant la procédure combinatoire du type A de Lascoux–Schützenberger, les clés gauche et droite pour un tableau de Kashiwara–Nakashima du type C. Ces clés symplectiques ont un rôle similaire aux clés des tableaux de Young semistandard. Plus précisément, nos clés symplectiques fournissent un critère de tableau pour l'ordre de Bruhat sur le groupe hyperoctaédrique et ses classes, et décrivent les atomes et les caractères de Demazure du type C. Les clés symplectiques droite et gauche sont liées par l'involution de Lusztig. Une évacuation de Schützenberger du type C est définie pour réaliser cette involution.

Keywords: Keys, Demazure crystal graph, Demazure characters and atoms in type C

1 Introduction

To generate the characters of a given finite dimensional irreducible representation of the symplectic Lie algebra $sp(2n, \mathbb{C})$, two different types of symplectic tableaux have been proposed: the King tableaux [11] and the De Concini tableaux [5]. We work with symplectic Kashiwara and Nakashima tableaux, which are a variation of De Concini tableaux, and with its crystal structure. That crystal structure allows a plactic monoid compatible with insertion and sliding algorithms, and Robinson–Schensted type correspondence, studied by Lecouvey in terms of crystal isomorphisms [13].

Kashiwara [10] and Littelmann [15] have shown that Demazure characters [6], for any Weyl group, can be lifted to certain subsets of Kashiwara–Nakashima tableaux, called Demazure crystals. Demazure characters (key polynomials) are then generated over Demazure crystals. In type C_n, they are non symmetric Laurent polynomials, with respect to the action of the Weyl group, which can be seen as "partial" symplectic characters.

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Given a partition λ, let v be in the orbit of λ under the action of the Weyl group, the Demazure crystal, $\mathfrak{B}_v$, is a union of disjoint sets, *Demazure crystal atoms*, $\widehat{\mathfrak{B}}_u$, over an interval in the Bruhat order, on the cosets modulo the stabilizer of λ. This order, induced on the orbit of λ, gives $\mathfrak{B}_v = \biguplus_{\lambda \le u \le v} \widehat{\mathfrak{B}}_u$.

In type A_{n-1}, Lascoux and Schützenberger identified tableaux with nested columns as key tableaux [12], and defined the right key map that sends tableaux to key tableaux. Their right key map can be used to describe the type A Demazure atoms $\widehat{\mathfrak{B}}_u$, $u \in \mathbb{N}^n$ [12, Theorem 3.8]. Azenhas, in a presentation in *The 69th Séminaire Lotharingien de Combinatoire* [1], identified some type C Kashiwara–Nakashima tableaux as key tableaux, but does not give a construction of the right key map. Motivated by Azenhas [1] and inspired by Lascoux and Schützenberger [12], we give a construction of left and right keys of a type C Kashiwara–Nakashima tableau. Our construction, based on type C frank words, introduced in Section 4, and Sheats *jeu de taquin*, allows us to prove Theorem 4.8, a type C analogue of [12, Theorem 3.8]. We also show, in Section 5, that both keys are related via the Schützenberger evacuation in type C, or Lusztig involution, explicitly realized here using Baker-Lecouvey insertion or Sheats *jeu de taquin*. During the preparation of the paper [17], Jacon and Lecouvey informed us about their paper [8], where, with a different approach, they find the same key map in type C. In the model of alcove paths, Lenart defined an initial key and a final key [14], for any Lie type, related via the Lusztig involution.

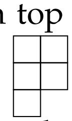
The paper is organized as follows. In Section 2, we discuss the Weyl group of type C, B_n, the Bruhat order on B_n and on its cosets modulo the stabilizer of λ, the Kashiwara–Nakashima tableaux and the symplectic key tableaux. Those key tableaux are used in Proposition 2.5 to explicitly construct the minimal length coset representatives and, recalling some results from Proctor [16], Theorem 2.6 gives a tableau criterion for the Bruhat order on B_n and on those cosets. Section 3 briefly recalls Baker-Lecouvey insertion, the Sheats *jeu de taquin* and Robinson–Schensted type C correspondence, to discuss the plactic and coplactic monoids of type C. These monoids describe connected components and crystal isomorphic connected components of type C Kashiwara crystal, for a $U_q(sp_{2n})$-module. In Section 4, we extend the concept of frank word, in type A, to type C, and our Theorem 4.5 gives right and left key maps. Using the right key map, Theorem 4.8, our main result, describes the tableaux that contribute to a Demazure crystal atom and to a Demazure crystal in type C. In Section 5, we develop a type C evacuation within the plactic monoid, an analogue of the J-operation discussed by Schützenberger for semistandard Young tableaux in [18]. Theorem 5.2 shows that the evacuation of the right key of a Kashiwara–Nakashima tableau is the left key of the evacuation of the same tableau.

Caution: Operators, maps and group actions act on the right.

2 Weyl group of type C, Bruhat order and symplectic key tableau

Fix $n \in \mathbb{N}_{>0}$. Define the sets $[n] = \{1 < \cdots < n\}$ and $[\pm n] = \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$ where $\overline{i}$ is just another way of writing $-i$. The hyperoctahedral group is the group, B_n, with generators s_i, $1 \leq i \leq n$, subject to the relations: $s_i^2 = 1$, $1 \leq i \leq n$; $(s_i s_{i+1})^3 = 1, 1 \leq i \leq n-2$; $(s_{n-1}s_n)^4 = 1; (s_i s_j)^2 = 1$, $1 \leq i < j \leq n$, $|i-j| > 1$. This group is a Coxeter group and we consider the (strong) Bruhat order on its elements [3]. Theorem 2.6 gives a symplectic tableau criterion for this order in B_n. The elements of B_n can be seen as odd bijective maps from $[\pm n]$ to itself. The subgroup with the generators $s_1, \ldots, s_{n-1}$ is the symmetric group $\mathfrak{S}_n$. The groups $\mathfrak{S}_n$ and B_n are the Weyl groups for the root systems of types A_{n-1} and C_n, respectively. Given $\sigma \in B_n$, $\sigma = [a_1\, a_2\, \ldots\, a_n]$, where $a_i = (i)\sigma$ for $i \in [n]$, is the window notation of σ. Given a vector $v \in \mathbb{Z}^n$, s_i, with $i \in [n]$, acts on v, vs_i, swapping the i-th and the $(i+1)$-th entries, if $i \in [n-1]$, or changing the sign of the last entry, if $i = n$. The length of $\sigma \in B_n$, $(\sigma)\ell$, is the least number of generators of B_n needed to go from $[1\,2\,\ldots\,n]$, the identity map, to σ. Any expression of σ as a product of $(\sigma)\ell$ generators of B_n is called reduced.

2.1 Kashiwara–Nakashima tableau in type C

We recall the symplectic tableaux used by Kashiwara and Nakashima to label the vertices of the type C crystal graphs [9], which are a variation of De Concini tableaux [5]. A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a partition of $|\lambda| = \sum_{i=1}^{n} \lambda_i$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. A partition λ is identified with its Young diagram of shape λ, an array of boxes, left justified, in which the i-th row, from top to bottom, has λ_i boxes. For example, the Young diagram of shape $\lambda = (2, 2, 1)$ is . Given μ and ν two partitions with $\nu \leq \mu$ entrywise, we write $\nu \subseteq \mu$. The Young diagram of shape μ/ν is obtained after removing the boxes of the Young diagram of ν from the Young diagram of μ. A semistandard Young skew tableau of shape μ/ν on the completely ordered alphabet A is a filling of the diagram μ/ν with letters from A, such that the entries are strictly increasing in each column and weakly increasing in each row. When $|\nu| = 0$ we obtain a semistandard Young tableau (SSYT) of shape μ. Denote by $SSYT(\mu/\nu, A)$ the set of all semistandard skew Young tableaux T of shape μ/ν, with entries in A. When $A = [n]$, we write $SSYT(\mu/\nu, n)$.

From now on we consider tableaux on the alphabet $[\pm n]$. A *column* is a strictly increasing sequence of numbers in $[\pm n]$ and it is usually displayed vertically. A column is said to be *admissible* if the following column condition (1CC) holds for that column:

Definition 2.1 (1CC)**.** Let C be a column. The 1CC holds for C if for all pairs i and $\overline{i}$ in C, where i is in the a-th row counting from the top of the column, and $\overline{i}$ in the b-th row counting from the bottom, we have $a + b \leq i$.

If a column C is admissible then C has at most n letters. If not, we say that C is not admissible at z, where z is the minimal positive integer such that z and $\bar{z}$ exist in C and there are more than z numbers in C with absolute value less or equal than z. For instance, the column $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \bar{1} \\ \hline \end{array}$ is not admissible at 1. We now define splittable columns:

Definition 2.2. Let C be a column and let $I = \{z_1 > \cdots > z_r\}$ be the set of unbarred letters z such that the pair $(z, \bar{z})$ occurs in C. The column C can be split when there exists a set of r unbarred letters $J = \{t_1 > \cdots > t_r\} \subseteq [n]$ such that t_1 is the greatest letter of $[n]$ satisfying $t_1 < z_1$, $t_1 \notin C$, and $\overline{t_1} \notin C$; and for $i = 2, \ldots, r$, t_i is the greatest letter of $[n]$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$, and $\overline{t_i} \notin C$.

A column C is admissible if and only if C can be split [19, Lemma 3.1]. If C can be split then we define right column of C, Cr, and the left column of C, $C\ell$. The column Cr is obtained by replacing, in C, $\overline{z_i}$ with $\overline{t_i}$ for each letter $z_i \in I$ and reordering, if needed; $C\ell$ is obtained after replacing z_i with t_i for each letter $z_i \in I$ and reordering, if needed. If C is admissible then $C\ell \leq C \leq Cr$ by entrywise comparison. If C does not have symmetric entries, then C is admissible and $C\ell = C = Cr$. Let T be a skew tableau with all of its columns admissible. The split form of a skew tableau T, $(T)spl$, is the skew tableau obtained after replacing each column C of T by the two columns $C\ell\, Cr$. The tableau $(T)spl$ has double the amount of columns of T. A semistandard skew tableau T is a Kashiwara–Nakashima (KN) skew tableau if its split form is a semistandard skew tableau. We define $\mathcal{KN}(\mu/\nu, n)$ to be the set of all KN tableaux of shape μ/ν in the alphabet $[\pm n]$. When $|\nu| = 0$ we obtain $\mathcal{KN}(\mu, n)$. When $T \in SSYT(\mu/\nu, [\pm n])$ with no symmetric entries in any of its columns, T is a KN skew tableau. In particular $SSYT(\mu/\nu, n) \subseteq \mathcal{KN}(\mu/\nu, n)$.

The weight of a word w, $(w)\mathrm{wt}$, on the alphabet $[\pm n]$ is the vector in $\mathbb{Z}^n$ where the entry i is the multiplicity of the letter i minus the multiplicity of the letter $\bar{i}$, for $i \in [n]$. The length of w is its number of letters. The column reading word of a KN tableau T, $(T)cr$, is obtained reading down columns, right to left. The weight of T is the vector $(T)\mathrm{wt} := ((T)cr)\mathrm{wt}$. Let $T = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{3} \\ \cline{1-1} \end{array}$ and $n = 3$. The split form of T is the tableau $(T)spl = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \hline \bar{3} & \bar{1} \\ \cline{1-2} \end{array}$. Hence $T \in \mathcal{KN}((2,2,1), 3)$. Also $(T)cr = 23\,23\bar{3}$ and $(T)\mathrm{wt} = ((T)cr)\mathrm{wt} = (0, 2, 1)$.

Given a partition $\lambda \in \mathbb{Z}^n$, the B_n-orbit of λ is the set $\lambda B_n := \{\lambda\sigma \mid \sigma \in B_n\}$.

Definition 2.3. A key tableau in type C, on the alphabet $[\pm n]$, is a KN tableau in $\mathcal{KN}(\lambda, n)$, for some partition λ, in which the set of elements of each column is contained in the set of elements of the previous column and the letters i and $\bar{i}$ do not appear simultaneously as entries, for any $i \in [n]$. See the example at the end of Section 2.2.

Given $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$, put in the first $|v_i|$ columns the letter i if $v_i > 0$ or $\bar{i}$ if $v_i < 0$. This defines a key tableau of weight v, $(v)K$.

Proposition 2.4. *Let $v \in \lambda B_n$. There is exactly one key tableau $(v)K$ whose weight is v. The shape of $(v)K$ is λ. $(\lambda)K$ is the only KN tableau of weight and shape λ. The map $v \mapsto (v)K$ is a bijection between λB_n and key tableaux in $\mathcal{KN}(\lambda, n)$.*

2.2 The Bruhat order on B_n and cosets of B_n

Given a partition $\lambda \in \mathbb{Z}^n$, let $W_\lambda = \{\rho \in B_n \mid \lambda\rho = \lambda\}$ be the stabilizer of λ, under the action of B_n, a standard parabolic subgroup of B_n generated by a subset of simple generators. Let $W_\lambda \setminus B_n = \{W_\lambda \sigma : \sigma \in B_n\}$ be the set of right cosets of B_n determined by the subgroup W_λ. Given a right coset in $W_\lambda \setminus B_n$, all its elements return the same vector when acting on λ. Hence the vectors v in the B_n-orbit of λ define a labelling for the right cosets. Therefore, the symplectic key tableaux in $\mathcal{KN}(\lambda, n)$ and the cosets of B_n, modulo W_λ, are in bijection: $(v)K \leftrightarrow v \leftrightarrow W_\lambda \sigma_v$, where σ_v is the minimal length coset representative. Key tableaux, $(v)K$, $v \in \lambda B_n$, may be used to explicitly construct the minimal length coset representatives of $W_\lambda \setminus B_n$, a generalization of what Lascoux does for vectors in $\mathbb{N}^n$ (hence $\sigma_v \in \mathfrak{S}_n$).

Proposition 2.5. *Let $v \in \lambda B_n$ and T the tableau obtained after adding the column $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$ to the left of $(v)K$. The aforementioned minimal length coset representative σ_v is given by the reading word T, where entries with the same absolute value are read just once.*

Let $v = (3, \bar{3}, 0, 0, \bar{2})$. Then $(v)K = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \bar{5} & \bar{5} & \bar{2} \\ \hline \bar{2} & \bar{2} \\ \cline{1-2} \end{array}$, $T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & \bar{5} & \bar{5} & \bar{2} \\ \hline 3 & \bar{2} & \bar{2} \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$ and $\sigma_v = [1\bar{2}\bar{5}34]$.

Given v and u in λB_n, we write $v \leq u$ to mean $\sigma_v \leq \sigma_u$ in the Bruhat order. Put $\Lambda_n = (n, n-1, \dots, 1)$. Thanks to Theorem $3BC$ of Proctor's Ph.D. thesis [16], we have a tableau criterion for the Bruhat order on vectors in the same B_n-orbit.

Theorem 2.6. *[16, Theorem 3BC] Let $v, u \in \lambda B_n$. Then $\sigma_v \leq \sigma_u$ if and only if $(v)K \leq (u)K$, by entrywise comparison. In particular, for $\sigma, \rho \in B_n$, $\sigma \leq \rho \Leftrightarrow (\Lambda_n \sigma)K \leq (\Lambda_n \rho)K$.*

For instance, $v = (3, \bar{3}, 0, 0, \bar{2}) \leq u = (\bar{3}, 2, 0, \bar{3}, 0)$, because $(v)K = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \bar{5} & \bar{5} & \bar{2} \\ \hline \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \leq (u)K = \begin{array}{|c|c|c|}\hline 2 & 2 & \bar{4} \\ \hline \bar{4} & \bar{4} & \bar{1} \\ \hline \bar{1} & \bar{1} \\ \cline{1-2} \end{array}$.

3 Type C crystal graphs, plactic and coplactic monoids

Let $[\pm n]^*$ be the free monoid on the alphabet $[\pm n]$. Recall the type C_n simple roots $\{\alpha_i = \mathbf{e_1} - \mathbf{e_2}, i \in [n-1]\} \cup \{\alpha_n = 2\mathbf{e_n}\}$. Here a Kashiwara crystal of type C_n is a nonempty set $\mathfrak{B}$ together with the following maps and statistics [4]: $e_i, f_i : \mathfrak{B} \to \mathfrak{B} \sqcup \{0\}$, $\varepsilon_i, \varphi_i : \mathfrak{B} \to \mathbb{Z}$, $\mathrm{wt} : \mathfrak{B} \to \mathbb{Z}^n$, where $i \in [n]$ and $0 \notin \mathfrak{B}$ is an auxiliary element, such that: if $a, b \in \mathfrak{B}$ then $(a)e_i = b \Leftrightarrow (b)f_i = a$ and in this case $(b)\mathrm{wt} = (a)\mathrm{wt} + \alpha_i$, $(b)\varepsilon_i = (a)\varepsilon_i - 1$ and $(b)\varphi_i = (a)\varphi_i + 1$; for all $a \in \mathfrak{B}$, we have $(a)\varphi_i = \langle (a)\mathrm{wt}, \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \rangle + (a)\varepsilon_i$, where $\langle , \rangle$ is the usual inner product in $\mathbb{R}^n$. For all $a \in \mathfrak{B}$, we have $(a)\varphi_i = \max\{k \in \mathbb{Z}_{\geq 0} \mid (a)f_i^k \neq 0\}$ and $(a)\varepsilon_i = \max\{k \in \mathbb{Z}_{\geq 0} \mid (a)e_i^k \neq 0\}$. An element $u \in \mathfrak{B}$ such that $(u)e_i = 0$ (or $(u)f_i = 0$) for all $i \in [n]$ is called a *highest weight element* (or *lowest weight element*). We associate with $\mathfrak{B}$ a coloured oriented graph with weighted vertices in $\mathfrak{B}$ and edges labelled by $i \in [n]$: $b \xrightarrow{i} b'$ if and only if $b' = (b)f_i$, $i \in [n]$, $b, b' \in \mathfrak{B}$. This is the *crystal graph* of $\mathfrak{B}$. The C_n standard crystal $\mathbb{B}$ is $1 \xrightarrow{1} 2 \xrightarrow{2} \dots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \dots \xrightarrow{1} \overline{1}$, with set $\mathbb{B} = [\pm n]$, where $(i)\mathrm{wt} = \mathbf{e_i}$, $(\overline{i})\mathrm{wt} = -\mathbf{e_i}$. The highest weight word is the word 1, and the lowest weight word is $\overline{1}$.

Let $\mathfrak{B}$ and $\mathfrak{C}$ be two crystals associated to the same root system. The *tensor product* $\mathfrak{B} \otimes \mathfrak{C}$ is a crystal whose set is the Cartesian product $\mathfrak{B} \times \mathfrak{C}$, where its elements are $b \otimes c$, $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$, with $(b \otimes c)\mathrm{wt} = (b)\mathrm{wt} + (c)\mathrm{wt}$. The crystal operator f_i is defined by $(b \otimes c)f_i = \begin{cases} (x)f_i \otimes y \text{ if } (c)\varphi_i \leq (b)\varepsilon_i \\ x \otimes (y)f_i \text{ if } (c)\varphi_i > (b)\varepsilon_i \end{cases}$, and e_i is its inverse. Using the tensor product we can define the crystal $\mathbb{B}^{\otimes k}$ of words of length k. Thus, we define how the crystal operators f_i and e_i act on any finite word. This operators can be described via the signature rule, see [4]. Let $G_n = \bigoplus_{k \geq 0} \mathbb{B}^{\otimes k}$ be the type C_n crystal of all words in $[\pm n]^*$. The crystal G_n is the union of connected components where each component has a unique highest (lowest) weight word. Two connected components are isomorphic if and only if they have the same highest weight [13].

The Robinson–Schensted (RS) type C correspondence is a bijection between words $w \in G_n$ of length k, and tuples consisting of a KN tableau $(w)P$ and an *oscillating tableau* Q, of length k, with the same final shape as $(w)P$, see [13]. We denote this map by $w \mapsto ((w)P, Q)$, where $(w)P$ can be computed via Sheats *jeu de taquin* or *Baker-Lecouvey insertion*. The symplectic *jeu de taquin* [13, 19] is a weight-preserving procedure that allows us to change the shape of a KN skew tableau and eventually rectify it, i.e., make it to have partition shape. It is a variation of the ordinary *jeu de taquin* for skew SSYTs. The rectification is independent of the order in which the inner corners of ν are filled [13, Corollary 6.3.9].

The Baker-Lecouvey insertion [2, 13] is a bumping algorithm that, given a word w in the alphabet $[\pm n]$, returns the KN tableau $(w)P$. It depends on the symplectic jeu de taquin. This insertion is similar to the usual column insertion for SSYTs. However, when inserting a letter it may happen that we remove a cell from the inserted tableau, instead of adding. The length of $((w)P)cr$ might be less than the length of w, but the weight is preserved, $(w)\mathrm{wt} = ((w)P)\mathrm{wt}$. If the word w does not have symmetric letters, then the insertion works just like the column insertion for SSYTs. If l is the length of w, $(w)P$ is the rectification of the skew tableau of shape Λ_l/Λ_{l-1} and reading word w [13, Corollary 6.3.9]. More generally, if $T \in \mathcal{KN}(\mu/\nu, n)$, the rectification of T coincides with $((T)cr)P$.

Given $w_1, w_2 \in [\pm n]^*$, the relation $w_1 \sim w_2 \Leftrightarrow (w_1)P = (w_2)P$ defines an equivalence relation on $[\pm n]^*$ known as Knuth equivalence. The type C plactic monoid is the quotient $[\pm n]^*/\sim$ where each Knuth (plactic) class is uniquely identified with a KN tableau [13]. Hence two Knuth-related words have the same weight. It is also described as the quotient of $[\pm n]^*$ by the elementary Knuth relations; see [13] for details. If $w_1 \sim w_2$ then they occur in the same place in two isomorphic connected components of G_n [13], *i.e.*, $(w_1)e_i \sim (w_2)e_i$ and $(w_1)f_i \sim (w_2)f_i$, $i \in [n]$.

Two words $w_1, w_2 \in [\pm n]^*$ arecoplactic equivalent if and only if they belong to the same connected component of G_n. The connected components of G_n are the coplactic classes in the RS correspondence that identify words with the same oscillating tableau [13, Proposition 5.2.1].

Choose a word $w \in [\pm n]^*$ where the shape of $(w)P$ is λ. If we replace every word of its coplactic class with its insertion tableau we obtain the crystal $\mathfrak{B}^\lambda$ of tableaux $\mathcal{KN}(\lambda, n)$. The crystal $\mathfrak{B}^\lambda$ does not depend on the initial choice of word w in the plactic class of w [13, Theorem 6.3.8]. A word w of G_n is a highest weight word if and only if the weight of all its prefixes (including itself) is a partition. In this case,

4 Right and left keys and Demazure atoms in type C

We generalize Lascoux–Schützenberger frank words, in type A [12], to type C to create right and left key maps in type C. Our Theorem 4.8 detects the type C KN tableaux for Demazure atoms. It is the type C version of Lascoux and Schützenberger [12, Theorem 3.8].

Definition 4.1. The word $w \in [\pm n]^*$ is a type C frank word if the lengths of its maximal column factors form a multiset equal to the multiset formed by the lengths of the columns of the tableau $(w)P$.

For instance, $(23\bar{2}\bar{3}1)P = (\bar{1}113\bar{3})P = \begin{array}{|c|c|c|}\hline 1 & 1 & \bar{1} \\ \hline 3 \\ \cline{1-1} \bar{3} \\ \cline{1-1}\end{array}$. Since $23\bar{2}\bar{3}1$ and $\bar{1}113\bar{3}$ have one column of length 3 and two columns of length 1, they are frank words.

Given a frank word w, the number of letters of w is the same as the number of cells of $(w)P$. This implies that all columns of w are admissible. The following proposition is an extension of [7, Proposition 7] on SSYTs to KN tableaux.

Proposition 4.2. *Let $T \in \mathcal{KN}(\lambda, n)$. Let μ/ν be a skew diagram with same number of columns of each length as T. Then there is a unique KN skew tableau S with shape μ/ν that rectifies to T and $(S)cr$ is a frank word.*

Corollary 4.3. *Let S be as in the previous proposition. The last column of S depends only on the length of that column.*

Fixed a KN tableau T, consider the set of all possible last columns taken from skew tableaux with same number of columns of each length as T. Corollary 4.3 implies that this set has one element for each distinct column length of T. For each column C in this set, consider the column Cr, its right column. The next proposition implies that this set of right columns is nested, if we see each column as the set formed by its letters.

Proposition 4.4. *Consider T a two-column KN skew tableau C_1C_2 with empty cells in the first column. Slide via symplectic jeu de taquin the bottommost of those empty cell, obtaining a two-column KN skew tableau $C_1'C_2'$. Then $C_2'r \subseteq C_2r$.*

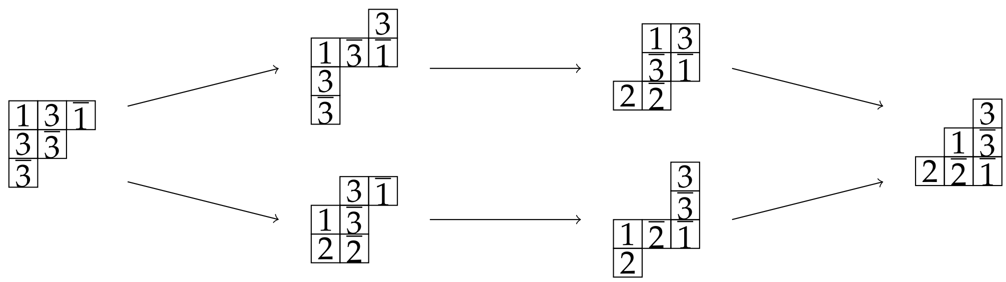
Next, one gives the type C right key map. It extends the one defined for type A in [12].

Theorem 4.5 (Right key map)**.** *Given a KN tableau T, if we replace each column with a column of the same size taken from the right columns of the last columns of all skew tableaux associated to T, then we obtain a key tableau. This tableau is the right key tableau of T and we denote it by $(T)K_+$. (See Example 4.7.)*

Remark 4.6. Recall the set up of Proposition 4.2. If the shape of S, μ/ν, is such that every two consecutive columns have at least one cell in the same row, then each column of S is a maximal column factor of the word $(S)cr$, hence $(S)cr$ is a frank word. Moreover, the columns of S appear in reverse order in $(S)cr$. Therefore, given a KN tableau T, the columns of $(T)K_+$ consist of right columns of the first columns of the frank words associated to T.

In the set up of Proposition 4.4, we also can prove that $C_1\ell \subseteq C_1'\ell$, hence the set of left columns of the first columns of all skew tableaux with the same number of columns of each length as T will be nested. The left key $(T)K_-$ is obtained after replacing each column of T with a column of the same size taken from this set.

Example 4.7. The tableau $T = \begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline 3 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$ has the following six KN skew tableaux with same number of columns of each length as T, each one corresponding to a permutation of its column lengths, and each one is associated to the frank word given by its column reading.



The right key of T has as columns $\begin{array}{|c|}\hline 3\\\hline \bar{3}\\\hline \bar{1}\\\hline\end{array}\, r$, $\begin{array}{|c|}\hline 3\\\hline \bar{1}\\\hline\end{array}\, r$ and $\begin{array}{|c|}\hline \bar{1}\\\hline\end{array}\, r$, hence $(T)K_+ = \begin{array}{|c|c|c|}\hline 3 & 3 & \bar{1}\\\hline \bar{2} & \bar{1} \\\cline{1-2} \bar{1} \\\cline{1-1}\end{array}$.

The left key of T has as columns $\begin{array}{|c|}\hline 1\\\hline 3\\\hline \bar{3}\\\hline\end{array}\, \ell$, $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\, \ell$ and $\begin{array}{|c|}\hline 2\\\hline\end{array}\, \ell$, hence $(T)K_- = \begin{array}{|c|c|c|}\hline 1 & 1 & 2\\\hline \bar{2} & 2 \\\cline{1-2} \bar{3} \\\cline{1-1}\end{array}$.

4.1 Demazure crystals and right key tableaux

Let $\lambda \in \mathbb{Z}^n$ be a partition and $v \in \lambda B_n$. We define $(v)\mathfrak{U} = \{T \in \mathcal{KN}(\lambda, n) \mid (T)K_+ = (v)K\}$ the set of KN tableaux of B^λ with right key $(v)K$. Given a subset X of $\mathfrak{B}^\lambda$, consider the operator $\mathfrak{D}_i$ on X, $i \in [n]$, defined by $X\mathfrak{D}_i = \{x \in \mathfrak{B}^\lambda \mid (x)e_i^k \in X$ for some $k \geq 0\}$ [4]. If $v = \lambda\sigma$ where $\sigma = s_{i_1} \ldots s_{i_{(\sigma)\ell}} \in B_n$ is a reduced word, we define the Demazure crystal to be $\mathfrak{B}_v = \{(\lambda)K\}\mathfrak{D}_{i_1} \ldots \mathfrak{D}_{i_{(\sigma)\ell}}$. This definition is independent of the reduced word for σ [4, Theorem 13.5], as well as of the coset representative of $W_\lambda\sigma$, that is, $\mathfrak{B}_{\lambda\sigma} = \mathfrak{B}_{\lambda\sigma_v}$.

From [3, Proposition 2.5.1], if $\rho \leq \sigma$ in B_n then $\rho_u = \sigma_u \leq \sigma_v$ where $u = \lambda\rho$. Since $(x)e_i^0 = x$, if $\rho \leq \sigma$ then $\mathfrak{B}_{\lambda\rho} = \mathfrak{B}_{\lambda\rho_u} \subseteq \mathfrak{B}_{\lambda\sigma_v} = \mathfrak{B}_v$. Thus we define the Demazure crystal atom $\hat{\mathfrak{B}}_v$ to be $\hat{\mathfrak{B}}_v = \hat{\mathfrak{B}}_{\lambda\sigma} := \mathfrak{B}_{\lambda\sigma_v} \setminus \bigcup_{\rho_u < \sigma_v} \mathfrak{B}_{\lambda\rho_u} = \mathfrak{B}_v \setminus \bigcup_{u<v} \mathfrak{B}_u \overset{\text{Theorem 2.6}}{=} \mathfrak{B}_v \setminus \bigcup_{(u)K<(v)K} \mathfrak{B}_u$.

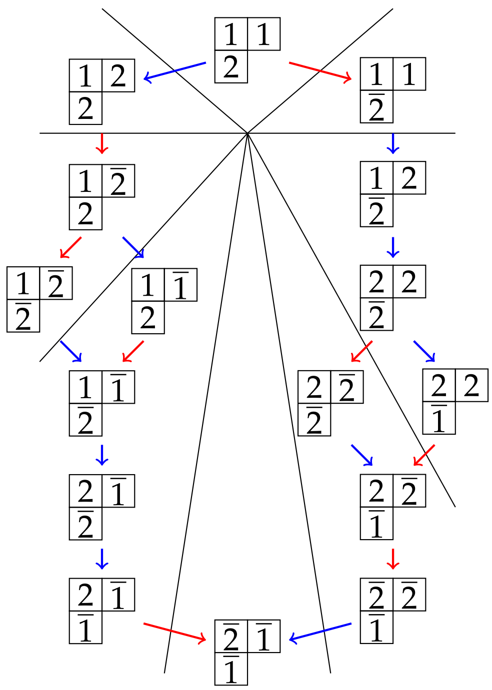
Thanks to the right key map, Theorem 4.5, we now describe the Demazure crystal atom in type C. Lascoux and Schützenberger, [12, Theorem 3.8], have given the type A version. Full details can be seen in the full version of this extended abstract [17].

Theorem 4.8 (Main Theorem)**.** *Let $v \in \lambda B_n$. Then $(v)\mathfrak{U} = \hat{\mathfrak{B}}_v$.*

Given $v \in \lambda B_n$ define the Demazure character (or key polynomial), κ_v, and the Demazure atom in type C, $\widehat{\kappa}_v$, as the generating functions of the KN tableau weights in $\mathfrak{B}_v$ and $\widehat{\mathfrak{B}}_v$, respectively: $\kappa_v = \sum_{T \in \mathfrak{B}_v} x^{(T)\mathrm{wt}}$, $\hat{\kappa}_v = \sum_{T \in \hat{\mathfrak{B}}_v} x^{(T)\mathrm{wt}}$. Theorem 4.8 detects the KN tableaux in $\mathfrak{B}^\lambda$ contributing to the Demazure atom $\hat{\kappa}_v = \sum_{\substack{(T)K_+=(v)K \\ T \in \mathfrak{B}^\lambda}} x^{(T)\mathrm{wt}}$. Moreover, one has $\kappa_v = \sum\limits_{u \leq v} \hat{\kappa}_u = \sum\limits_{\substack{u \leq v \\ T \in (u)\mathfrak{U}}} x^{(T)\mathrm{wt}} = \sum\limits_{\substack{(u)K \leq (v)K \\ T \in (u)\mathfrak{U}}} x^{(T)\mathrm{wt}} = \sum\limits_{(T)K_+ \leq (v)K} x^{(T)\mathrm{wt}}$. In Example 4.9,

$$\mathfrak{B}_{(1,\bar{2})} = \{T \in \mathfrak{B}^{\lambda} \mid (T)K_+ \leq ((1,\bar{2}))K\} = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{2} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array} \right\}.$$

Example 4.9. Consider the crystal graph of $\mathfrak{B}^{(2,1)}$:



The crystal is split into pieces. Each piece is a Demazure atom and contains exactly one symplectic key tableau, so we can identify each part with the weight of that key tableau, a vector in the B_2-orbit of $(2,1)$. From the previous theorem we have that all tableaux in the same piece have the same right key. One can check that $((1,\bar{2}))\mathfrak{U} = \left\{ \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array} \right\} = \hat{\mathfrak{B}}_{\lambda s_1 s_2}$, for example.

5 Lusztig involution, right and left keys

Let $\mathfrak{B}^{\lambda}$ be the crystal of tableaux in $\mathcal{KN}(\lambda, n)$. The type C_n Lusztig involution $L : \mathfrak{B}^{\lambda} \to \mathfrak{B}^{\lambda}$ is the only involution such that, for all $i \in [n]$, $x \in \mathcal{KN}(\lambda, n)$: (1) $((x)L)\mathrm{wt} = ((x)\mathrm{wt})\omega_0 = -(T)\mathrm{wt}$, where ω_0 is the longest element of B_n; (2) $((x)L)e_i = ((x)f_i)L$ and $((x)L)f_i = ((x)e_i)L$; and (3) $((x)L)\varepsilon_i = (x)\varphi_i$ and $((x)L)\varphi_i = (x)\varepsilon_i$.

The involution L flips the crystal upside down. The Schützenberger evacuation is a realization of Lusztig involution in type A. The algorithm below adapts it to KN tableaux. It sends a tableau $T \in \mathcal{KN}(\lambda, n)$ to $T^{\mathrm{Ev}} \in \mathcal{KN}(\lambda, n)$, where $(T)\mathrm{wt} = -((T^{\mathrm{Ev}})\mathrm{wt})$.

Algorithm 5.1. 1. Let $(T)cr^{\star}$ be the word obtained by applying ω_0 to the letters of $(T)cr$ and writing it backwards (or define $T^{\#}$ by π-rotating T and applying ω_0 to its entries).

2. Insert $(T)cr^{\star}$ (or rectify $T^{\#}$). Define $T^{Ev} := ((T)cr^{\star})P =$rectification of $T^{\#}$.

Consider the KN tableau $T = \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{3} & \bar{1} \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$. Then, $w = (T)cr = \bar{2}\bar{1}1\bar{3}\bar{2}$ and $w^\star = 23\bar{1}12$. Then $(w^\star)P = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ which is the rectification of $T^\# = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\bar{1}} \\ \hline \end{array}$.

Let $w \in [\pm n]$. The connected component of the crystal G_n that contains the word $w^\star$ is obtained applying * to each vertex of the one containing w and reverting arrows. They are isomorphic because they have the same highest weight, say λ. Therefore $(w)P$, $(w^\star)P \in \mathfrak{B}^\lambda$ and have symmetric weights. T^{Ev} is the only KN tableau with the same shape as T, and Knuth equivalent to $(T)cr^\star$. The algorithm is a realization of the type C Lusztig involution. It follows that evacuation of the right key of a tableau is the left key of the evacuation of the same tableau. See [17] for the proof.

Theorem 5.2. *Let T be a KN tableau and ^{Ev} the type C Lusztig (Schützenberger) involution. Then*

$$(T)K_+{}^{Ev} = (T^{Ev})K_-.$$

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