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Simplicial generation of Chow rings of matroids

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Abstract. We introduce a new presentation of the Chow ring of a matroid whose variables admit a combinatorial interpretation via the theory of matroid quotients and display a geometric behavior analogous to that of nef classes on smooth projective varieties. We apply these properties to produce a bijection between a standard monomial basis of our presentation and a relative generalization of Schubert matroids. As a corollary we obtain the Poincaré duality property for Chow rings of matroids. We then give a formula for the volume polynomial with respect to our presentation and show that it is log-concave in the positive orthant. We recover the portion of the Hodge theory of matroids in [Adiprasito–Huh–Katz, 2018], which implies the Heron–Rota–Welsh conjecture on the log-concavity of the coefficients of the characteristic polynomial. We emphasize that our work eschews the use of flipping, which is the key technical tool employed in [Adiprasito–Huh–Katz, 2018]. Thus our proof does not leave the realm of matroids.

Keywords: matroid, Chow ring, Hodge theory, Lorentzian

1 Chow rings of matroids and the simplicial presentation

First investigated in the context of wonderful compactifications of hyperplane arrangement complements [5], the Chow ring of an arbitrary matroid was introduced in [9].

Definition 1.1 ([9, Definition 3])**.** Let M be a loopless matroid of rank $r = d + 1$ on a ground set E. The **Chow ring** of M is a graded ring $A^\bullet_{FY}(M) = \bigoplus_{i=0}^d A^i_{FY}(M)$ defined as

$$A^\bullet_{FY}(M) := \frac{\mathbb{R}[z_F : F \subseteq E \text{ a nonempty flat of } M]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} z_F \mid a \text{ an atom in the lattice of flats of } M\rangle}$$

The ring $A^\bullet_{FY}(M)$ agrees with the Chow ring of the (non-complete) toric variety of the Bergman fan of M, and when M is realizable as a hyperplane arrangement, is isomorphic to the cohomology ring of its wonderful compactification [9, Corollary 2]. This

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isomorphism is informed by the theory of tropical compactifications [20, Section 6.7], and through toric and tropical intersection theory Adiprasito, Huh, and Katz established Hodge theoretic properties for Chow rings of general matroids in [1]. This development of Hodge theory of matroids has resolved major conjectures in matroid theory and led to various applications in both matroid theory and computer science [17, 15, 16, 2].

We study here a new presentation of the Chow ring of a matroid that further illuminates the interaction between the geometry and combinatorics of matroids.

Definition 1.2. Let M be a loopless matroid on E. The **simplicial presentation** $A^\bullet_\nabla(M)$ of the Chow ring of M is the quotient of a polynomial ring

$$\mathbb{R}[h_F \mid F \subseteq E \text{ nonempty flat of } M]$$

where

$$h_F := -\sum_{F \subseteq G} z_G \in A^\bullet_{FY}(M).$$

We note two distinguished cases:

- When M is the Boolean matroid on E, i.e. E is the basis of M, the associated wonderful compactification is a Losev-Manin space [19], also known as a permutohedral variety. In this case, the generators h_F are divisors on the permutohedral variety corresponding to the (negative) standard simplices $\mathrm{Conv}(-e_i \mid i \in F) \subset \mathbb{R}^E$. These were studied in [23] and inspire our terminology.

- When M is the matroid of a complete graph K_{n-1}, the associated wonderful compactification is the Deligne–Knudsen–Mumford space $\overline{\mathcal{M}}_{0,n}$ of rational curves with marked points [5, Section 4.3]. In this case, our simplicial presentation, after suitable modifications, recovers the Etingof–Henriques–Kamnitzer–Rains–Singh presentation of the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ [7, 26].

In what follows, we exhibit our results on the simplicial presentation. We lay out preliminaries in Sections 2.1 and 2.2, then state the key results connecting the simplicial presentation to both combinatorics and geometry in Section 2.3. In Section 3, we apply our results from Section 2.3 to give direct proofs of Poincaré duality for the Chow ring, as well as the Hodge–Riemann relations and Hard Lefschetz property in degree 1, resulting in a new proof of the Heron–Rota–Welsh conjecture.

2 The basis of relative nested matroids

In this section, we interpret the following monomial basis of the Chow ring.

Proposition 2.1. *For $c \in \mathbb{Z}_{\geq 0}$, a monomial $\mathbb{R}$-basis for the degree c part $A^c_\nabla(M)$ of the Chow ring $A^\bullet_\nabla(M)$ of a matroid M is*

$$\{h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \mid \sum a_i = c,\ \varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\ 1 \leq a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})\}.$$

*We call this basis of $A^\bullet_\nabla(M)$ the **nested basis** of the Chow ring of M.*

In the following two sections, we review the algebraic and matroidal constructions needed for the interpretation.

2.1 Minkowski weights and the cap product

Let N be a lattice of rank N and let $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be a rational fan. Let $\Sigma(k)$ denote the set of k-dimensional cones of Σ.

Definition 2.2. An ℓ-dimensional **Minkowski weight** $\Delta \in \mathrm{MW}_\ell(\Sigma)$ is a function $\Delta : \Sigma(\ell) \to \mathbb{R}$ such that for each $\tau \in \Sigma(\ell - 1)$, the function Δ satisfies the balancing condition

$$\sum_{\tau \prec \sigma} \Delta(\sigma) u_{\sigma \setminus \tau} \in \mathrm{span}_{\mathbb{R}}(\tau)$$

where $\sigma \setminus \tau$ denotes the unique ray of an ℓ-dimensional cone σ that is not in τ.

When $\Sigma = \Sigma_M$ is the Bergman fan of a matroid M, there is an isomorphism

$$t_{\Sigma_M} : \mathrm{MW}_\ell(\Sigma_M) \xrightarrow{\sim} \mathrm{Hom}(A^\ell(M), \mathbb{R}), \quad \Delta \mapsto \Big(x_\sigma \mapsto \Delta(\sigma)\Big).$$

This isomorphism is analogous to the Kronecker duality map that relates cohomology and homology classes in algebraic topology. The isomorphism t_{Σ_M} was first defined in [10, 11] and generalized to [20, Theorem 6.7.5]; the version above follows the notation of [1, Proposition 5.6]. This isomorphism allows one to define the **cap product** by

$$A^k(M) \times \mathrm{MW}_\ell(\Sigma_M) \to \mathrm{MW}_{\ell-k}(\Sigma_M), \quad (\xi, \Delta) \mapsto \xi \cap \Delta := \Big(\sigma \mapsto (t_{\Sigma_M}\Delta)(\xi \cdot x_\sigma)\Big),$$

which makes $\mathrm{MW}_\bullet(\Sigma_M)$ a graded $A^\bullet(\Sigma_M)$-module. Since $\mathrm{MW}_d(\Sigma_M) \simeq \mathbb{R}$, one can define the **Bergman class** Δ_M to be a generator for $\mathrm{MW}_d(\Sigma_M)$ and define

$$\delta : A^\bullet(M) \to \mathrm{MW}_{d-\bullet}(\Sigma_M) \text{ via } \xi \mapsto \xi \cap \Delta_M. \tag{2.1}$$

In particular, noting that $\mathrm{MW}_0(\Sigma_M) = \mathbb{R}$, we have the **degree map**

$$\int_{\Sigma_M} : A^d(M) \to \mathbb{R}, \quad \xi \mapsto \xi \cap \Delta_M.$$

2.2 Matroid quotients, intersections, and truncations

In this section we review the combinatorial constructions needed to interpret the action of the simplicial generators. Let M and M' be matroids on a common ground set E.

Definition 2.3. A matroid M' is a **(matroid) quotient** of M, written $f : M \twoheadrightarrow M'$, if any of the following equivalent conditions hold ([4, Proposition 7.4.7]):

1. Every flat of M' is also a flat of M,
1. the Bergman fan $\Sigma_{M'}$ is a subfan of Σ_M,
2. $\mathrm{rk}_{M'}(B) - \mathrm{rk}_{M'}(A) \leq \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ for every $A \subset B \subset E$

A matroid quotient $f : M \twoheadrightarrow M'$ defines an inclusion of fans $\iota_f : \Sigma_{M'} \hookrightarrow \Sigma_M$, defining an injective pushforward $\iota_{f_*} : \mathrm{MW}_\bullet(\Sigma_{M'}) \hookrightarrow \mathrm{MW}_\bullet(\Sigma_M)$, which is a $A^\bullet(M)$-module map via the pullback map $\iota_f^* : A^\bullet(M) \to A^\bullet(M')$.

For a matroid quotient $f : M \twoheadrightarrow M'$, the f**-nullity** of $A \subset E$ is defined to be

$$n_f(A) := \mathrm{rk}_M(A) - \mathrm{rk}_{M'}(A).$$

We say that a flat F of M' is f**-cyclic** if it is minimal among flats of M' of the same nullity, and we say that M' is a **relative nested quotient** of M if the f-cyclic flats of M' form a chain. An example of matroid quotients that will be important to us is matroid truncation with respect to a flat.

Definition 2.4. [22, Exercise 7.2.4.] The **principal truncation** $T_F(M)$ of a matroid M with respect to $F \in \mathscr{L}_M$ has bases

$$\mathcal{B}(T_F(M)) = \{B \setminus f : B \in \mathcal{B}(M), f \in B \cap F \neq \emptyset\}.$$

The **Higgs factorization** (see [14, 4, Exercise 7.20]) implies that a quotient matroid $f : M \twoheadrightarrow M'$ can be recovered from its f-cyclic flats and their nullities, and that the relative nested quotients of M are precisely the loopless quotients M' of M that can be obtained by taking iterated principal truncations with respect to the elements of a multichain of flats of M.

Principal truncations can in turn be realized as special cases of matroid intersection. For two matroids M, N on a common ground set E, we define their **matroid intersection** to be a new matroid $M \wedge N$ on E whose spanning sets $\mathcal{S}$ are

$$\mathcal{S}(M \wedge N) = \{S \cap S' \mid S \in \mathcal{S}(M), S' \in \mathcal{S}(N)\}.$$

The matroid $M \wedge N$ is a quotient of both M and N.

2.3 Interpreting the basis

We now show how the notions in the previous two subsections can be used to interpret the basis of Proposition 2.1. Matroid intersection behaves well in relation to Minkowski weights. By [11, Theorem 3.1, Proposition 4.1.(b), Theorem 4.2], when $\Sigma = \Sigma_{U_{E,|E|}}$ is the Bergman fan of a full rank uniform matroid, the map of equation (2.1) extends to an isomoprhism of graded rings $\delta : A^\bullet(U_{E,|E|}) \to \mathrm{MW}^\bullet(\Sigma) := \mathrm{MW}_{d-\bullet}(\Sigma)$ where the ring structure on $\mathrm{MW}^\bullet$ is given by stable intersection of tropical cycles.

Any matroid on E is a quotient of $U_{E,|E|}$, so the pushforwards of the Bergman classes of loopless matroids M, N on E define Minkowski weights $\Delta_M, \Delta_N \in \mathrm{MW}^\bullet(\Sigma_{U_{E,|E|}})$. By [27, Theorem 4.11] or [12, Remark 2.31], their product is

$$\Delta_M \cdot \Delta_N = \begin{cases} \Delta_{M\wedge N} & \text{if } M \wedge N \text{ is loopless} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $S \subset E$, and H_S is the matroid with bases $\mathcal{B}(H_S) := \{E \setminus i : i \in S\}$, and M is a loopless matroid on E, and $F = \mathrm{cl}_M(S)$ is the closure of S in M then

$$T_F(M) = M \wedge H_S, \text{ so } \Delta_M \cdot \Delta_{H_S} = \Delta_{T_F(M)}.$$

The importance of this special case stems from the following lemma, which reflects the behavior of linear series of divisors on blow-ups. In particular, when M is realizable by a hyperplane arrangement $\mathcal{A}$, the variable h_F is the class of a divisor on the wonderful compactification of $\mathcal{A}$ whose general section corresponds to a hyperplane containing the flat F.

Lemma 2.5. *Let $h_S \in A_H(U_{|E|,E})$ for $\emptyset \subsetneq S \subset E$, and let M be a loopless matroid on E. Let $F = \mathrm{cl}_M(S)$ be the closure of S in M. We have*

$$h_S \cap \Delta_M = \Delta_{T_F(M)}, \quad \text{and} \quad h_F(M) \cap \Delta_M(M) = \Delta_{T_F(M)}(M).$$

From Lemma 2.5 we obtain the following interpretation for the nested basis

Theorem 2.6. *Let M be a loopless matroid of rank $r = d + 1$. For each $0 \le c \le d$, the cap product map*

$$A^c_\nabla(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Delta_M$$

induces a bijection between the monomial basis for $A^c_\nabla(M)$ given in Proposition 2.1 and the set of Bergman classes $\Delta_{M'}$ of loopless relative nested quotients $M' \twoheadleftarrow M$ with $\mathrm{rk}(M') = \mathrm{rk}(M) - c$.

3 Applications to Hodge theory for matroids

By refining and applying Theorem 2.6, we are able to recover the portion of the Hodge theory for matroids used for combinatorial applications. Before proceeding, we review the main results of [1], which establish Hodge theory for matroids, and establish terminology to be used in subsequent sections. The following result is sometimes referred to as the **Kähler package** for matroids.

Theorem 3.1. *[1, Theorems 6.19 and 8.9] If M is a matroid of rank $r = d+1$ on ground set E, then there is an isomorphism of vector spaces $\int_M : A^d(M) \to \mathbb{R}$ and a cone of elements $\ell \in A^1(M)$ such that for $0 \le i \le \left\lfloor \frac{d}{2} \right\rfloor$,*

Poincare duality *The pairing $A^i(M) \times A^{d-i}(M) \to \mathbb{R}$ defined by $(a,b) \mapsto \int_M ab$ is nondegenerate.*

Hard Lefschetz *The map $L^i_\ell : A^i \to A^{d-i}$ given by $a \mapsto \ell^{d-2i}a$ is an isomorphism.*

Hodge–Riemann *The quadratic form $Q^i_\ell : A^i \times A^i \to \mathbb{R}$ given by $(x,y) \mapsto \int xL^i_\ell(y)$ is such that $(-1)^i Q^i_\ell$ is positive definite on $P^i_\ell := \{x \in A^i : x\ell^{d-2i+1} = 0\}$.*

The isomorphism $\int_M$ is the **degree map** of $A^\bullet(M)$.

The authors of [1] establish Theorem 3.1 by adapting a double-inductive argument of McMullen [21] to the setting of matroids. To carry out their induction, they introduce a generalization of Bergman fans, which are associated to order filters on the lattice of flats, and proceed by a double induction on the rank of a matroid and the cardinality of an order filter.

Using the simplicial presentation, we are able to provide independent proofs of Poincaré duality (Theorem 3.2) and both the Hard Lefschetz and Hodge–Riemann relations in degrees $i = 0, 1$ (Theorem 3.9). In geometric contexts, some authors refer to this result as the **Hodge index theorem**.

Our proof of Poincaré duality is quite direct, and when establishing the Hard Lefschetz and Hodge–Riemann relations, Theorem 3.8 provides us a key step in the induction that allows us to avoid the use of order filters. Hence our proof is a single induction on rank which involves only the classical Bergman fans of matroids.

We remark that while [1] establishes the Hodge–Riemann relations in all degrees, the Hodge–Riemann relation in degree 1 accounts for all currently known combinatorial applications—Huh has posed the discovery of combinatorial applications for the Hodge–Riemann relations in higher degrees as an open problem [15].

3.1 Poincaré duality

A modification of Hampe's argument in [12, Proposition 3.2] implies that the images of the nested basis elements in the Chow ring of a free matroid are linearly independent. Combining this fact with some algebraic manipulations, we obtain:

Theorem 3.2. *Let M be a loopless matroid on $E = \{0, 1, \dots, n\}$. Let Δ_M be the Bergman class of the matroid M considered as an element of $A^\bullet(U_{E,|E|})$. Then we have*

$$A^\bullet(M) \simeq A^\bullet(U_{E,|E|}) / \operatorname{ann}(\Delta_M), \quad \text{where } \operatorname{ann}(\Delta_M) = \{f \in A^\bullet(U_{E,|E|}) \mid f \cdot \Delta_M = 0\}.$$

The theorem is modeled after the geometric behavior of the pullback maps of Chow rings over certain closed embeddings of smooth projective varieties As a consequence, we recover [1, Theorem 6.19] that the ring $A^\bullet_\nabla(M)$ satisfies the Poincaré duality property.

Theorem 3.3. *Let M be a loopless matroid of rank $r = d + 1$. Then $A^\bullet_\nabla(M)$ is a Poincaré duality algebra of dimension d with a degree map $\int_M$.*

When M is the Boolean matroid, the relative nested quotients of M are known as nested matroids or (loopless) Schubert matroids, and a main theorem of Hampe in [12] states that the nested matroids define a basis for the homology groups of the permutohedral variety. Combining Theorem 2.6 with Theorem 3.2 we obtain the following generalization.

Theorem 3.4. *The Bergman classes of relative nested quotients of M form a basis for $\mathrm{MW}_\bullet(\Sigma_M)$, the space of Minkowski weights supported on the Bergman fan Σ_M of M.*

3.2 Volume polynomial

The geometric utility of the simplicial presentation is particularly visible in the intersection numbers of divisors. We give the following formula for the intersection numbers of the simplicial generators h_F.

Theorem 3.5. *For M a loopless matroid of rank $r = d + 1$, and a multiset of nonempty flats $\{F_1, \dots, F_d\}$, we have*

$$\int_M h_{F_1} \cdots h_{F_d} = \begin{cases} 1 & \text{if } \operatorname{rk}_M(\bigcup_{j \in J} F_j) \geq |J| + 1 \text{ for every } \emptyset \subsetneq J \subseteq \{1, \dots, d\} \\ 0 & \text{otherwise} \end{cases}.$$

We call the condition that $\operatorname{rk}_M(\bigcup_{j \in J} F_j) \geq |J| + 1$ the **dragon Hall–Rado condition**, as it provides a common generalization of Rado's theorem [24] and Postnikov's dragon marriage condition [23].

When the matroid is realizable, the generators h_F are nef divisors on the associated wonderful compactification, and thus have non-negative intersection numbers. For general matroids, the simplicial generators are combinatorially nef, and our formula displays that the same non-negativity behavior holds. Moreover, that these intersection numbers are always either 0 or 1 stands in stark contrast to calculations in the classical presentation of the Chow ring of the matroid; for the calculation of intersection numbers in the classical presentation we point to the second author's work [8].

A presentation of a Poincaré duality algebra is encoded by its volume polynomial. In classical algebraic geometry, the volume polynomial of the cohomology ring of a smooth projective variety measures the degrees of its ample (nef) line bundles [6]. The intersection numbers computed in the Theorem 3.5 stated above yield immediately the following formula for the volume polynomial of $A^\bullet_\nabla(M)$.

Corollary 3.6. *Let M be a loopless matroid on E of rank $d+1$. The volume polynomial $VP^\nabla_M(\underline{t}) \in \mathbb{Q}[t_F \mid F \subseteq E \text{ nonempty flat in } M]$ of $A^\bullet_\nabla(M)$ is*

$$VP^\nabla_M(\underline{t}) := \int_M \Big(\sum_F t_F h_F\Big)^d = \sum_{(F_1,\ldots,F_d)} t_{F_1}\cdots t_{F_d}$$

where the sum is over ordered collections of flats $(F_1,\ldots,F_d)$ of M satisfying $\operatorname{rk}_M(\bigcup_{j\in J} F_j) \geq |J|+1$ for every $\emptyset \subsetneq J \subseteq \{1,\ldots,d\}$.

One recovers a central result [23, Corollary 9.4] of Postnikov on the formula for volumes of generalized permutohedra by setting $M = U_{E,|E|}$.

It is a classical statement in algebraic geometry that a volume polynomial arising from nef divisors is log-concave [18, Corollary 1.6.3]. The analogous statement holds for the volume polynomial of the simplicial presentation of the Chow ring of a matroid.

Corollary 3.7. *The volume polynomial $VP^\nabla_M(\underline{t}) \in \mathbb{Q}[t_F \mid F \subseteq E \text{ nonempty flat in } M]$, considered as a real-valued function, is log-concave in the positive orthant.*

We establish this corollary by showing that the volume polynomial is Lorentzian in the sense of [3]. The authors of [3] introduce Lorentzian polynomials of degree d as a certain family of real multivariate homogeneous polynomials of degree d with nonnegative coefficients. These polynomials are characterized by two conditions: one on their supports and one on the signatures of the quadratic forms obtained as $(d-2)$-th partial derivatives of the polynomials.

We demonstrate that the support of VP^∇_M has a rich combinatorial structure as the lattice points of a generalized permutohedron; this is a combinatorial generalization of the fact that the wonderful compactification can be constructed as a closure in a product of projective spaces. Moreover, Lemma 2.5 implies easily that taking partial derivatives of VP^∇_M corresponds to performing principal truncations on the matroid. These properties of VP^∇_M allow us to prove the following.

Theorem 3.8. *The volume polynomial VP^∇_M of a loopless matroid M is Lorentzian.*

3.3 Hodge theory in degree 1

From the log-concavity of VP_M^∇, we recover the combinatorially relevant portion of the Hodge theory of matroids in [1]. Among the main motivations for the development of the Hodge theory of matroids in [1] was to analyze the behavior of the (reduced) characteristic polynomial of a matroid, which is a generalization of the chromatic polynomial of a graph. The reduced characteristic polynomial of a loopless matroid M of rank $d+1$ is defined as

$$\overline{\chi}_M(t) := \frac{1}{t-1} \sum_{F \in \mathscr{L}_M} \mu(\emptyset, F) t^{\mathrm{rk}(M) - \mathrm{rk}(F)} = \sum_{k=0}^{d} (-1)^k \mu^k(M) t^{d-k}$$

where $\mu(\cdot,\cdot)$ is the Möbius function of the lattice $\mathscr{L}_M$ and $\mu^i(M)$ is the absolute value of the ith coefficient of $\overline{\chi}_M(t)$. The Heron–Rota–Welsh conjecture [25, 13, 28] stated that

$$\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2 \quad \text{for } 0 < k < d.$$

We recover the following the portion of the Hodge theory of matroids in [1] which implies the Heron-Rota-Welsh conjecture.

Theorem 3.9. *Let M be a loopless matroid of rank $r = d+1$ on a ground set E, and $\ell \in A^1(M)$ a combinatorially ample divisor obtained from a strictly submodular function on E. Then the Poincaré duality algebra $A^\bullet(M)$ with the degree map $\int_M$ satisfies the Kähler package in degree zero and one. That is, for $i \leq 1$,*

($\mathrm{HL}^{\leq 1}$) *(hard Lefschetz in degree ≤ 1) the multiplication by ℓ*

$$L_\ell^i : A^i(M) \to A^{d-i}(M),\ a \mapsto \ell^{d-2i} a$$

is an isomorphism, and

($\mathrm{HR}^{\leq 1}$) *(Hodge–Riemann relations in degree ≤ 1) the symmetric form*

$$(-1)^i Q_\ell^i : A^i(M) \times A^i(M) \to \mathbb{R},\ (x,y) \mapsto (-1)^i \int_M xy\ell^{d-2i}$$

is non-degenerate on $A^i(M)$ and positive-definite when restricted to $P_\ell^i := \{a \in A^i(M) : \ell^{d-2i+1} z = 0\}$.

Our proof of Theorem 3.9 employs Corollary 3.7 to avoid the technicalities of order filters and flips. The validity of the Hodge–Riemann relations in degree 1 implies the Heron–Rota–Welsh conjecture [1, Section 9].

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