Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #55, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

On linearization coefficients of q-Laguerre polynomials

Byung-Hak Hwang[*1], Jang Soo Kim[†2], Jaeseong Oh[‡1], and Sang-Hoon Yu[§1]

[1] *Department of Mathematics, Seoul National University, Seoul, South Korea*
[2] *Department of Mathematics, Sungkyunkwan University, Suwon, South Korea*

Abstract. The linearization coefficient $\mathcal{L}(L_{n_1}(x)\dots L_{n_k}(x))$ of classical Laguerre polynomials $L_n(x)$ is well known to be equal to the number of $(n_1,\dots,n_k)$-derangements, which are permutations with a certain condition. Kasraoui, Zeng and Stanton found a q-analog of this result using q-Laguerre polynomials with two parameters q and y. Their formula expresses the linearization coefficient of q-Laguerre polynomials as the generating function for $(n_1,\dots,n_k)$-derangements with two statistics counting weak excedances and crossings. In this paper their result is proved by constructing a sign-reversing involution on marked perfect matchings.

Keywords: orthogonal polynomials, Laguerre polynomials, linearization coefficient, sign-reversing involution

1 Introduction

A family of polynomials $P_n(x)$ are called *orthogonal polynomials* with respect to a linear functional $\mathcal{L}$ if $\deg P_n(x) = n$ for $n \ge 0$ and $\mathcal{L}(P_m(x)P_n(x)) = 0$ if and only if $m \ne n$. The nth *moment* μ_n of the orthogonal polynomials is defined by $\mu_n = \mathcal{L}(x^n)$. It is well known that monic orthogonal polynomials $P_n(x)$ satisfy a three-term recurrence of the form

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x). \tag{1.1}$$

Viennot [6] developed a combinatorial theory to study orthogonal polynomials. In particular, he showed that orthogonal polynomials $P_n(x)$ and the moments μ_n are expressed as weighted sums of certain lattice paths. There are several classical orthogonal polynomials whose moments have simple combinatorial meanings. For example, the nth moment of the Hermite (respectively, Charlier and Laguerre) polynomials

*xoda@snu.ac.kr.
†jangsookim@skku.edu. Supported by NRF grants #2016R1D1A1A09917506 and #2016R1A5A1008055.
‡jaeseong_oh@snu.ac.kr.
§ysh4017@snu.ac.kr.

is the number of perfect matchings (respectively, set partitions and permutations) on $[n] := \{1, 2, \dots, n\}$.

By definition of orthogonal polynomials, it is easily seen that

$$P_m(x)P_n(x) = \sum_{\ell} c_{m,n}^{\ell} P_\ell(x), \qquad c_{m,n}^{\ell} = \mathcal{L}(P_\ell(x)P_m(x)P_n(x))/\mathcal{L}(P_\ell(x)^2).$$

Thus the coefficients $c_{m,n}^{\ell}$ can be computed using the quantities $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$. We call $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$ a *linearization coefficient*.

For the above mentioned classical orthogonal polynomials, the linearization coefficients also have nice combinatorial interpretations. If $P_n(x)$ are the Hermite (respectively, Charlier and Laguerre) polynomials, then $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$ is the number of inhomogeneous perfect matchings (respectively, set partitions and permutations) on $[n_1] \sqcup \cdots \sqcup [n_k]$, see [2, 7] and references therein. Here, $[n_1] \sqcup \cdots \sqcup [n_k]$ is the disjoint union of $[n_i]$'s and a perfect matching $\mathfrak{m}$ (respectively, set partition π and permutation σ) is *inhomogenous* if there are no edges (respectively, two elements in the same block and two elements j and $\sigma(j)$) that are contained in the same set $[n_i]$.

There are q-analogs of the above combinatorial formulas for linearization coefficients of Hermite, Charlier and Laguerre polynomials due to Ismail, Stanton and Viennot [3], Anshelevich [1] and Kasraoui, Stanton and Zeng [4], respectively. We refer the reader to the survey [2] for more details on these linearization coefficients.

Suppose that $P_n(x)$ are orthogonal polynomials whose moments $\mathcal{L}(x^n)$ have a combinatorial model as in the case of Hermite, Charlier or Laguerre polynomials. Since $P_n(x)$ satisfy a simple recurrence (1.1), one may also give a combinatorial model for $P_n(x)$ with possibly negative signs involved. These combinatorial models for $P_n(x)$ and $\mathcal{L}(x^n)$ naturally yield a combinatorial meaning to $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$, which may have negative signs. Therefore, if there is a combinatorial formula for $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$ with only positive terms, the most satisfying combinatorial proof of this formula would be finding a sign-reversing involution on the combinatorial models for $\mathcal{L}(P_{n_1}(x) \dots P_{n_k}(x))$ whose fixed points give the positive terms in the formula.

Indeed, the formulas for linearization coefficients of q-Hermite [3] and q-Charlier polynomials [1] have been proved in this way by Ismail, Stanton and Viennot [3] and Kim, Stanton and Zeng [5]. However, such a proof is missing in the case of q-Laguerre polynomials. In this paper, we prove the formula for linearization coefficients of q-Laguerre polynomials due to Kasraoui, Stanton and Zeng [4] by finding a sign-reversing involution. We now describe their result below.

The *q-Laguerre polynomials* $L_n(x; q, y)$ are defined by the three-term recurrence relation

$$L_{n+1}(x;q,y) = (x - y[n+1]_q - [n]_q)L_n(x;q,y) - y[n]_q^2 L_{n-1}(x;q,y) \tag{1.2}$$

with $L_0(x;q,y) = 1$ and $L_1(x;q,y) = x - y$. Here, we use the notation $[n]_q = 1 + q + \cdots + q^{n-1}$. From now on $\mathcal{L}$ denotes the linear functional with respect to which the q-Laguerre polynomials are orthogonal.

The set of permutations of $[n]$ is denoted by S_n. For $\sigma \in S_n$, a *weak excedance* of σ is an integer $i \in [n]$ such that $\sigma(i) \geq i$. A *crossing* of σ is a pair (i, j) of integers $i, j \in [n]$ such that $i < j \leq \sigma(i) < \sigma(j)$ or $\sigma(i) < \sigma(j) < i < j$. We denote by $\mathrm{wex}(\sigma)$ (respectively, $\mathrm{cross}(\sigma)$) the number of weak excedances (respectively, crossings) of σ.

Kasraoui, Stanton and Zeng [4] showed that the nth moment is given by

$$\mu_n(q, y) = \mathcal{L}(x^n) = \sum_{\sigma \in S_n} y^{\mathrm{wex}(\sigma)} q^{\mathrm{cross}(\sigma)}. \tag{1.3}$$

They also proved the following formula for the linearization coefficients of q-Laguerre polynomials. Note that a permutation $\sigma \in S_{n_1+\cdots+n_k}$ is called an $(n_1, \ldots, n_k)$-*derangement* if there is no i such that

$$n_1 + \cdots + n_{r-1} + 1 \leq i, \sigma(i) \leq n_1 + \cdots + n_r$$

for some $0 \leq r \leq k$ where $n_0 = 0$. Denote $\mathcal{D}(n_1, \ldots, n_k)$ by the set of $(n_1, \ldots, n_k)$-derangements.

Theorem 1.1. *[4] The linearization coefficients of q-Laguerre polynomials are given by*

$$\mathcal{L}(L_{n_1}(x; q, y) \cdots L_{n_k}(x; q, y)) = \sum_{\sigma \in \mathcal{D}(n_1, \ldots, n_k)} y^{\mathrm{wex}(\sigma)} q^{\mathrm{cross}(\sigma)}.$$

In [4] they proved Theorem 1.1 using a recurrence relation for $\mathcal{L}(L_{n_1}(x; q, y) \cdots L_{n_k}(x; q, y))$ and induction. The purpose of this paper is to give a proof of Theorem 1.1 by constructing a sign-reversing involution. Our fundamental combinatorial objects are matchings instead of permutations.

2 q-Laguerre polynomials and their moments

In this section we give combinatorial interpretations for the q-Laguerre polynomials $L_n(x; q, y)$ and their moments $\mu_n(q, y)$ using matchings and perfect matchings. The results in this section generalize the combinatorial models for Laguerre polynomials and their moments due to Viennot [6, Chapter 6]. We start with basic definitions.

Definition 2.1. Let $K_{n,n}$ be the complete bipartite graph with $2n$ vertices, i.e., the graph with vertex set $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$ and edge set $\{(i, \bar{j}) : 1 \leq i, j \leq n\}$. A *matching* of degree n is a subgraph π of $K_{n,n}$ such that π contains every vertex of $K_{n,n}$ and no two distinct edges of π have common vertices. A matching π of degree n is called a *perfect matching* if π has exactly n edges. Denote the set of all matchings (respectively, perfect matchings) of degree n by M_n (respectively, PM_n). For $\pi \in \mathrm{M}_n$, we denote by $E(\pi)$ the set of edges in π and let $e(\pi) = |E(\pi)|$.

We visualize a matching π of degree n by placing the *upper vertices* $1, 2, \dots, n$ in the upper row and the *lower vertices* $\overline{1}, \overline{2}, \dots, \overline{n}$ in the lower row. If there is no possible confusion, we will simply write j instead of $\overline{j}$. For example, since every edge of a matching is of the form $(i, \overline{j})$, we will also write this edge as (i, j).

For $\pi \in \mathrm{M}_n$, if $(i, j) \in \pi$, we denote $\pi(i) = j$ and $e_i = (i, \pi(i))$. An upper vertex i of π is said to be *unmatched* if there is no edge of the form (i, j). Similarly, a lower vertex j of π is *unmatched* if there is no edge of the form (i, j). Note that if $\pi \in \mathrm{PM}_n$, there are no unmatched vertices and we can identify π with the permutation $\sigma \in S_n$ given by $\sigma(i) = \pi(i)$ for all $i \in [n]$. We will often use this identification in this paper.

Let $\pi \in \mathrm{PM}_n$. An edge $e = (i, \pi(i))$ of π is called a *weak excedance* if $i \le \pi(i)$. A pair (e, e') of edges $e = (i, \pi(i))$ and $e' = (j, \pi(j))$ is said to be *overlapping* if $i < j \le \pi(i) < \pi(j)$ or $\pi(i) < \pi(j) < i < j$. Let $\mathrm{wex}(\pi)$ and $\mathrm{ov}(\pi)$ denote the number of weak excedances and overlapping pairs of π.

By the identification of PM_n and S_n we can rewrite (1.3) as follows:

$$\mu_n(q, y) = \mathcal{L}(x^n) = \sum_{\pi \in \mathrm{PM}_n} y^{\mathrm{wex}(\pi)} q^{\mathrm{ov}(\pi)}. \tag{2.1}$$

For the remainder of this section we will find a combinatorial model for $L_n(x; q, y)$ in Theorem 2.4 and give yet another expression for $\mu_n(q, y)$ in (2.3). To do this, we define some statistics for matchings. Given a matching $\pi \in \mathrm{M}_n$, let $P = (B_1, \dots, B_l)$ be the unique ordered set partition of the upper vertices of π satisfying the following conditions:

- Each block B_r consists of consecutive elements. In other words, B_r is of the form $B_r = \{i, i+1, \dots, j\}$.
- For each $i \in [n]$, i is the largest element in some block B_r if and only if i is an unmatched vertex or $i = n$.

We define the *upper block index* $\mathrm{bindex}^U_\pi(i)$ of a vertex i to be the integer r such that $i \in B_r$. Note that $\mathrm{bindex}^U_\pi(i)$ is equal to one more than the number of unmatched vertices appearing before i in the upper row. The *lower block index* $\mathrm{bindex}^L_\pi(i)$ is defined similarly by considering the ordered set partition of the lower vertices of π.

Definition 2.2. For a matching $\pi \in \mathrm{M}_n$, the *block difference* $\mathrm{bdiff}_\pi(e)$ of an edge $e = (i, \pi(i))$ is the difference between $\mathrm{bindex}^L_\pi(\pi(i))$ and $\mathrm{bindex}^U_\pi(i)$, that is,

$$\mathrm{bdiff}_\pi(e) = \mathrm{bindex}^L_\pi(\pi(i)) - \mathrm{bindex}^U_\pi(i).$$

An edge $e \in E(\pi)$ is called a *block weak excedance* if $\mathrm{bdiff}_\pi(e) \ge 0$. Denote the number of block weak excedances in π by $\mathrm{bwex}(\pi)$. The *block weight* $\mathrm{bwt}(\pi)$ of $\pi \in \mathrm{M}_n$ is defined by

$$\mathrm{bwt}(\pi) = \sum_{\mathrm{bdiff}_\pi(e) \ge 0} \mathrm{bdiff}_\pi(e) + \sum_{\mathrm{bdiff}_\pi(e) < 0} (-\mathrm{bdiff}_\pi(e) - 1).$$

A *crossing* of π is a pair (e, e') of edges $e = (i, \pi(i))$ and $e' = (j, \pi(j))$ in π such that $i < j$ and $\pi(i) > \pi(j)$. The number of crossings of π is denoted by $\mathrm{cr}(\pi)$.

We note that the notion of crossing for a matching $\pi \in \mathrm{M}_n$ is different from that for a permutation $\sigma \in S_n$. If $\pi \in \mathrm{PM}_n$ corresponds to $\sigma \in S_n$ using the identification, we have $\mathrm{cross}(\sigma) = \mathrm{ov}(\pi)$ but $\mathrm{cross}(\sigma) \neq \mathrm{cr}(\pi)$. A crossing of $\pi \in \mathrm{M}_n$ can be understood as a pair of edges that intersect in the visualization of π.

Example 2.3. Let π be the matching given by $\pi(1) = 4, \pi(2) = 6, \pi(3) = 2, \pi(5) = 1$ and $\pi(7) = 3$. Then the ordered set partition for the upper row is $(\{1,2,3,4\}, \{5,6\}, \{7\})$ and the ordered set partition for the lower row is $(\{1,2,3,4,5\}, \{6,7\})$. Let $e = (7, \overline{3})$. The block indices of its two endpoints are $\mathrm{bindex}_\pi^U(7) = 3$ and $\mathrm{bindex}_\pi^L(3) = 1$, so we have $\mathrm{bdiff}_\pi(e) = -2$. The number of block weak excedances in π is $\mathrm{bwex}(\pi) = 3$, the block weight of π is $\mathrm{bwt}(\pi) = 0$, and the number of crossings of π is $\mathrm{cr}(\pi) = 7$.

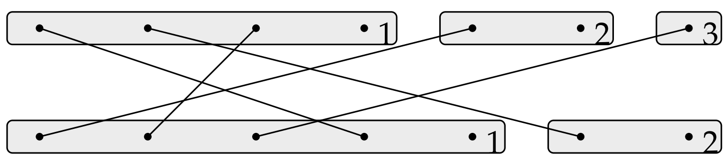

Figure 1: A matching π with its blocks. The block numbers are shown.

We are now ready to express the q-Laguerre polynomials combinatorially.

Theorem 2.4. *For $n \geq 0$, we have*

$$L_n(x; q, y) = \sum_{\pi \in \mathrm{M}_n} (-1)^{\mathrm{e}(\pi)} y^{\mathrm{bwex}(\pi)} q^{\mathrm{bwt}(\pi)+\mathrm{cr}(\pi)} x^{n-\mathrm{e}(\pi)}. \tag{2.2}$$

Example 2.5. There are 7 matchings of degree 2 as shown in Figure 2.

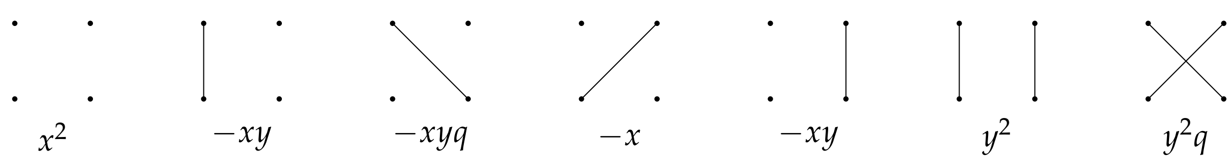

Figure 2: The matchings of degree 2 and their corresponding terms.

Then by Theorem 2.4, we have

$$L_2(x; q, y) = x^2 - (yq + 2y + 1)x + y^2 + y^2 q.$$

Now we modify the combinatorial expression (2.1) for the moment $\mu_n(q,y)$ so that the new expression is more suitable for our approach. For $\pi \in \mathrm{PM}_n$, the *weight* $\mathrm{wt}(\pi)$ of π is defined by

$$\mathrm{wt}(\pi) = \sum_{\pi(i)\geq i} (\pi(i) - i) + \sum_{\pi(i)<i} (i - \pi(i) - 1).$$

In fact, this definition is obtained from the definition of the block weight by replacing block differences $\mathrm{bdiff}_\pi(e)$ by $\pi(i) - i$. The following lemma gives a relation between $\mathrm{ov}(\pi)$, $\mathrm{wt}(\pi)$ and $\mathrm{cr}(\pi)$.

Lemma 2.6. *For $\pi \in \mathrm{PM}_n$, $\mathrm{ov}(\pi) = \mathrm{wt}(\pi) - \mathrm{cr}(\pi)$.*

By Lemma 2.6 we can rewrite the moment $\mu_n(q,y)$ using $\mathrm{wt}(\pi)$ and $\mathrm{cr}(\pi)$ instead of $\mathrm{ov}(\pi)$:

$$\mu_n(q,y) = \sum_{\pi\in\mathrm{PM}_n} y^{\mathrm{wex}(\pi)} q^{\mathrm{wt}(\pi)-\mathrm{cr}(\pi)}. \tag{2.3}$$

In the next section we will use Theorem 2.4 and (2.3) to give a combinatorial meaning to the linearization coefficients of q-Laguerre polynomials.

3 Linearization coefficients and a sign-reversing involution

3.1 A combinatorial interpretation of linearization coefficients

In this section we give a combinatorial interpretation of the linearization coefficient $C(n_1,\dots,n_k) := \mathcal{L}(L_{n_1}\cdots L_{n_k})$ of the q-Laguerre polynomials $L_n = L_n(x;q,y)$. First we recall the expression of L_n in terms of matchings in Theorem 2.4:

$$L_n = \sum_{\pi\in\mathrm{M}_n} (-1)^{\mathrm{e}(\pi)} y^{\mathrm{bwex}(\pi)} q^{\mathrm{bwt}(\pi)+\mathrm{cr}(\pi)} x^{n-\mathrm{e}(\pi)}. \tag{3.1}$$

To give a description of the product $L_{n_1}\cdots L_{n_k}$, we embed $\mathrm{M}_{n_1}\times\cdots\times\mathrm{M}_{n_k}$ in M_N, where $N = \sum_{i=1}^k n_i$, by horizontally concatenating the k matchings $\pi_1,\dots,\pi_k$ for each $(\pi_1,\dots,\pi_k) \in \mathrm{M}_{n_1}\times\cdots\times\mathrm{M}_{n_k}$. Let $\mathrm{M}_{n_1,\dots,n_k} \subset \mathrm{M}_N$ denote the embedded image of $\mathrm{M}_{n_1}\times\cdots\times\mathrm{M}_{n_k}$.

Let $\pi \in \mathrm{M}_N$. We say that an edge $(i,\pi(i))$ of π is *homogeneous* with respect to $(n_1,\dots,n_k)$ if

$$n_1+\cdots+n_{r-1}+1 \leq i, \pi(i) \leq n_1+\cdots+n_r$$

for some $0 \leq r \leq k$, where $n_0 = 0$, and *inhomogeneous* otherwise. For simplicity, we omit the expression 'with respect to $(n_1,\dots,n_k)$' when there is no confusion. Note that $\mathrm{M}_{n_1,\dots,n_k}$ is the set of matchings in M_N such that every edge is homogeneous. We will write $E^H(\pi)$ for the set of homogeneous edges of π.

Note that if $\pi \in \mathrm{M}_{n_1,\dots,n_k}$ is the concatenation of $\pi_1, \dots, \pi_k$, then each statistic in (3.1) satisfies the relation $\mathrm{stat}(\pi) = \sum_{i=1}^{k} \mathrm{stat}(\pi_i)$. Thus the product $L_{n_1} \cdots L_{n_k}$ is written as

$$L_{n_1} \cdots L_{n_k} = \sum_{\pi \in \mathrm{M}_{n_1,\dots,n_k}} (-1)^{\mathrm{e}(\pi)} y^{\mathrm{bwex}(\pi)} q^{\mathrm{bwt}(\pi)+\mathrm{cr}(\pi)} x^{N-\mathrm{e}(\pi)}. \tag{3.2}$$

Applying $\mathcal{L}$ to (3.2), we have

$$\mathcal{L}\left(L_{n_1} \cdots L_{n_k}\right) = \sum_{\pi \in \mathrm{M}_{n_1,\dots,n_k}} (-1)^{\mathrm{e}(\pi)} y^{\mathrm{bwex}(\pi)} q^{\mathrm{bwt}(\pi)+\mathrm{cr}(\pi)} \mathcal{L}\left(x^{N-\mathrm{e}(\pi)}\right).$$

Here we recall the formula of the nth moment in (2.3):

$$\mu_n(q,y) = \mathcal{L}(x^n) = \sum_{\pi \in \mathrm{PM}_n} y^{\mathrm{wex}(\pi)} q^{\mathrm{wt}(\pi)-\mathrm{cr}(\pi)}.$$

Note that $N - \mathrm{e}(\pi)$, the power of x in (3.2), represents the number of unmatched vertices in the upper (or lower) row, or equivalently, the number of edges we need to add to make it a perfect matching. Thus, applying the functional $\mathcal{L}$ to $x^{N-\mathrm{e}(\pi)}$ is interpreted as summing up all possible ways to complete π into a perfect matching, by adding edges on the unmatched vertices, allowing inhomogeneous edges.

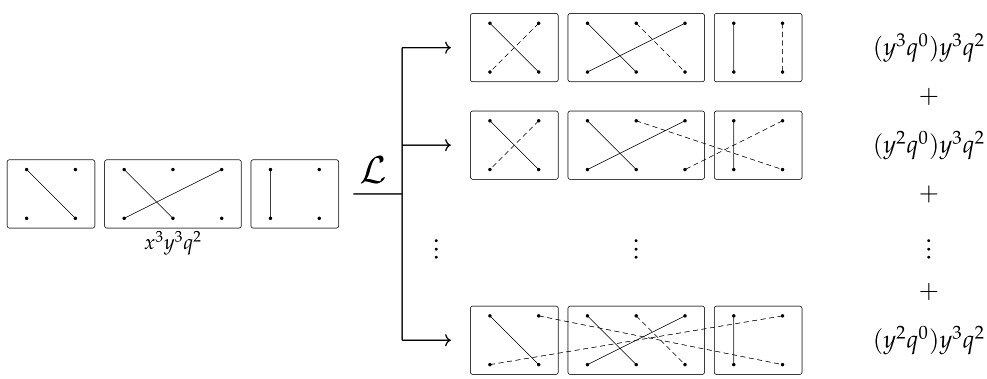

Figure 3: An example of applying $\mathcal{L}$ to a term $x^3y^3q^2$ in the product $L_2L_3L_2$. There are 3!=6 terms in $\mathcal{L}(x^3)$ corresponding to all possible completions of the original matching.

Example 3.1. Figure 3 describes an example of the application of $\mathcal{L}$. The matching on the left side represents a term $x^3y^3q^2$ in $L_2L_3L_2$, which is the product of three terms $-xyq$, xyq and $-xy$ in L_2, L_3 and L_2, respectively. Applying $\mathcal{L}$ gives an equation

$$\left(\sum_{\pi \in \mathrm{PM}_3} y^{\mathrm{wex}(\pi)} q^{\mathrm{wt}(\pi)-\mathrm{cr}(\pi)}\right) y^3q^2,$$

where each summand corresponds to a way to add edges to remaining vertices, represented in dashed lines.

In order to describe the expansion of $\mathcal{L}(L_{n_1} \cdots L_{n_k})$, we introduce a perfect matching model containing the information of which edges are newly added by applying $\mathcal{L}$. Let $\mathrm{PM}^*_{n_1,\dots,n_k}$ be the set of pairs $\mathfrak{m} = (\pi, S)$ such that

- $\pi \in \mathrm{M}_N$ is a perfect matching of degree $N = \sum_{i=1}^{k} n_i$,
- S is a subset of edges in π, which contains all inhomogeneous edges of π, i.e., $E(\pi) \setminus E^H(\pi) \subseteq S$.

We call an element $\mathfrak{m} = (\pi, S)$ of $\mathrm{PM}^*_{n_1,\dots,n_k}$ a *marked perfect matching*. An edge e of π is said to be *marked* if $e \in S$. In other words, S is the set of marked edges. With marks on edges, we can distinguish new edges added by applying $\mathcal{L}$ from the original edges from $L_{n_1} \cdots L_{n_k}$. The condition $E(\pi) \setminus E^H(\pi) \subseteq S$ is needed since inhomogeneous edges cannot be presented in the original matching coming from $L_{n_1} \cdots L_{n_k}$.

Now we give a bijective correspondence between the terms in the expansion of $\mathcal{L}(L_{n_1} \cdots L_{n_k})$ and $\mathrm{PM}^*_{n_1,\dots,n_k}$. To do this, we extend our former definitions of statistics on M_n and PM_n to marked perfect matchings. In detail, we consider the decomposition of $\mathfrak{m}$ into unmarked and marked portions. For $\mathfrak{m} = (\pi, S) \in \mathrm{PM}^*_{n_1,\dots,n_k}$, define $\pi \setminus S$ and $\pi|_S$ as follows:

- $\pi \setminus S$ (*unmarked portion* of $\mathfrak{m}$) is the matching in $\mathrm{M}_{n_1,\dots,n_k}$ with $n_1 + \cdots + n_k - |S|$ edges obtained from π by deleting the $|S|$ marked edges but leaving their incident vertices not deleted.
- $\pi|_S$ (*marked portion* of $\mathfrak{m}$) is the perfect matching in $\mathrm{PM}_{|S|}$ obtained from π by deleting all unmarked edges and their adjacent vertices.

Definition 3.2. For $\mathfrak{m} = (\pi, S) \in \mathrm{PM}^*_{n_1,\dots,n_k}$, define statistics $\mathrm{e}(\mathfrak{m}), \mathrm{bwex}(\mathfrak{m}), \mathrm{cr}(\mathfrak{m})$ and $\mathrm{wt}(\mathfrak{m})$ as follows:

$$\begin{aligned} \mathrm{e}(\mathfrak{m}) &= \mathrm{e}(\pi \setminus S), & \mathrm{bwex}(\mathfrak{m}) &= \mathrm{bwex}(\pi \setminus S) + \mathrm{wex}(\pi|_S), \\ \mathrm{cr}(\mathfrak{m}) &= \mathrm{cr}(\pi \setminus S) - \mathrm{cr}(\pi|_S), & \mathrm{wt}(\mathfrak{m}) &= \mathrm{bwt}(\pi \setminus S) + \mathrm{wt}(\pi|_S). \end{aligned}$$

Example 3.3. Figure 4 shows a marked perfect matching $\mathfrak{m}$ in $\mathrm{PM}^*_{2,3,2}$ and block indices of its vertices. The block difference of each edge is indicated above its upper endpoint. The statistics $\mathrm{bwex}(\mathfrak{m}) = 5$ and $\mathrm{wt}(\mathfrak{m}) = 2$ can be computed directly by the notion of block difference in $\mathfrak{m}$, or summing the statistics defined on each $\pi \setminus S$ and $\pi|_S$. For the other statistics of $\mathfrak{m}$, we have $\mathrm{e}(\mathfrak{m}) = 4$ and $\mathrm{cr}(\mathfrak{m}) = 0$.

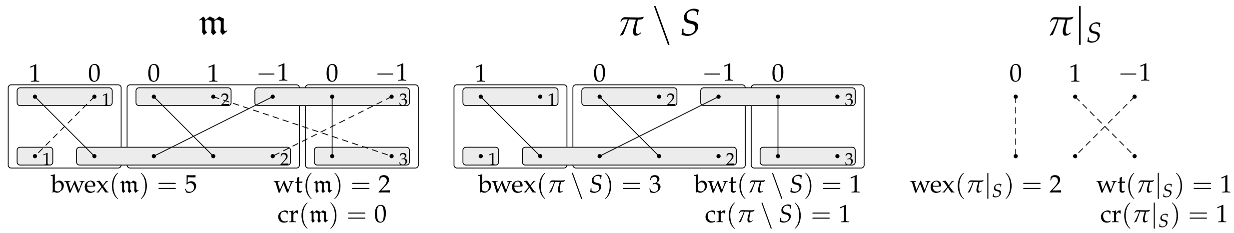

Figure 4: An example of a marked perfect matching $\mathfrak{m}$ in $\mathrm{PM}^*_{2,3,2}$ and its unmarked and marked portions.

Under this construction, the linearization coefficient $C(n_1, \ldots, n_k)$ is expressed in terms of marked perfect matchings by

$$C(n_1, \ldots, n_k) = \sum_{\mathfrak{m} \in \mathrm{PM}^*_{n_1,\ldots,n_k}} (-1)^{\mathrm{e}(\mathfrak{m})} y^{\mathrm{bwex}(\mathfrak{m})} q^{\mathrm{wt}(\mathfrak{m})+\mathrm{cr}(\mathfrak{m})}. \tag{3.3}$$

There are many cancellations in this summation. Our goal is to cancel all negative terms by finding a sign-reversing involution on $\mathrm{PM}^*_{n_1,\ldots,n_k}$.

Recall that $\mathcal{D}(n_1, \ldots, n_k) \subset S_N$ is the set of $(n_1, \ldots, n_k)$-derangements. The set $\mathcal{D}(n_1, \ldots, n_k)$ can be naturally identified with the set of marked perfect matchings whose edges are all inhomogeneous (necessarily marked). To be more precise, let $\sigma \in \mathcal{D}(n_1, \ldots, n_k)$. Then we will identify σ with the marked perfect matching $\mathfrak{m} = (\pi, E(\pi)) \in \mathrm{PM}^*_{n_1,\ldots,n_k}$, where $\pi \in \mathrm{PM}_N$ is given by $\pi(i) = \sigma(i)$ for all $i \in [N]$. Under this identification one can easily check that $\mathrm{wex}(\sigma) = \mathrm{wex}(\pi) = \mathrm{bwex}(\mathfrak{m})$ and $\mathrm{cross}(\sigma) = \mathrm{ov}(\pi) = \mathrm{wt}(\pi) - \mathrm{cr}(\pi) = \mathrm{wt}(\mathfrak{m}) + \mathrm{cr}(\mathfrak{m})$. By abuse of notation from now on we will write

$$\mathcal{D}(n_1, \ldots, n_k) = \{(\pi, S) \in \mathrm{PM}^*_{n_1,\ldots,n_k} : E^H(\pi) = \varnothing\}.$$

Using the above discussion we can rewrite Theorem 1.1 as follows.

Theorem 3.4. *We have*

$$C(n_1, \ldots, n_k) = \sum_{\mathfrak{m} \in \mathcal{D}(n_1,\ldots,n_k)} y^{\mathrm{bwex}(\mathfrak{m})} q^{\mathrm{wt}(\mathfrak{m})+\mathrm{cr}(\mathfrak{m})}.$$

3.2 Construction of a sign-reversing involution

In order to prove Theorem 3.4, we give a sign-reversing involution Φ on $\mathrm{PM}^*_{n_1,\ldots,n_k}$ that preserves the statistics bwex and wt +cr. Indeed, Φ will be a map that marks or unmarks a single homogeneous edge, or does not change anything. First we introduce some facts and definitions that we need to describe the map Φ.

For $\mathfrak{m} = (\pi, S) \in \mathrm{PM}^*_{n_1,\dots,n_k}$, let us observe a change in the block difference of an edge e_j while marking or unmarking a homogeneous edge e_i. If we mark e_i that was unmarked before, the upper (respectively, lower) index $\mathrm{bindex}^U_{\mathfrak{m}}(j)$ (respectively, $\mathrm{bindex}^L_{\mathfrak{m}}(\pi(j))$) increases by 1 if and only if $j > i$ (respectively, $\pi(j) > \pi(i)$). Therefore the block difference $\mathrm{bdiff}_{\mathfrak{m}}(e_j) = \mathrm{bindex}^L_{\mathfrak{m}}(\pi(j)) - \mathrm{bindex}^U_{\mathfrak{m}}(j)$ changes if and only if e_j crosses e_i. More precisely, if $\mathfrak{m} = (\pi, S)$ with $e_i \notin S$ turns into $\mathfrak{m}' = (\pi, S \cup \{e_i\})$, then we have

$$\mathrm{bdiff}_{\mathfrak{m}'}(e_j) = \begin{cases} \mathrm{bdiff}_{\mathfrak{m}}(e_j) & \text{if } e_j = e_i, \text{ or } e_j \text{ and } e_i \text{ do not cross each other,} \\ \mathrm{bdiff}_{\mathfrak{m}}(e_j) + 1 & \text{if } j < i \text{ and } \pi(j) > \pi(i), \\ \mathrm{bdiff}_{\mathfrak{m}}(e_j) - 1 & \text{if } j > i \text{ and } \pi(j) < \pi(i). \end{cases} \tag{3.4}$$

Conversely, if we unmark a marked edge $e_i \in E^H(\pi)$ so that $\mathfrak{m} = (\pi, S)$ turns into $\mathfrak{m}' = (\pi, S \setminus \{e_i\})$, then we have

$$\mathrm{bdiff}_{\mathfrak{m}'}(e_j) = \begin{cases} \mathrm{bdiff}_{\mathfrak{m}}(e_j) & \text{if } e_j = e_i, \text{ or } e_j \text{ and } e_i \text{ do not cross each other,} \\ \mathrm{bdiff}_{\mathfrak{m}}(e_j) - 1 & \text{if } j < i \text{ and } \pi(j) > \pi(i), \\ \mathrm{bdiff}_{\mathfrak{m}}(e_j) + 1 & \text{if } j > i \text{ and } \pi(j) < \pi(i). \end{cases} \tag{3.5}$$

From now on, let us adopt an expression *e_j crosses e_i from the left*, or equivalently *e_i crosses e_j from the right* for the relation $j < i$ and $\pi(j) > \pi(i)$. With this observation, we define the *convertibility* of a homogeneous edge, which is a key ingredient of the map Φ.

Definition 3.5. Let $\mathfrak{m} = (\pi, S) \in \mathrm{PM}^*_{n_1,\dots,n_k}$. An edge $e \in E^H(\pi)$ is said to be *convertible* (in $\mathfrak{m}$) if it satisfies the following conditions.

For every edge e' that crosses e, either

- e' crosses e from the left and $\mathrm{bdiff}_{\mathfrak{m}}(e') \geq 0$, or
- e' crosses e from the right and $\mathrm{bdiff}_{\mathfrak{m}}(e') \leq -1$,

where the inequalities are strict if e is marked, i.e. $e \in S$

Note that if an edge $e \in E^H(\pi)$ is convertible, then the status of other edges being block weak excedances does not change under the map $\mathfrak{m} = (\pi, S) \mapsto \mathfrak{m}' = (\pi, S \triangle \{e\})$, where $X \triangle Y$ denotes the symmetric difference $(X \cup Y) \setminus (X \cap Y)$. In particular, marking or unmarking a convertible edge preserves the statistic bwex. Note also that an edge e is convertible in $\mathfrak{m} = (\pi, S)$ if and only if it is convertible in $\mathfrak{m}' = (\pi, S \triangle \{e\})$.

Remark 3.6. Suppose that $e' = (i, \pi(i))$ is an inhomogeneous edge of $\mathfrak{m} = (\pi, S) \in \mathrm{M}_{n_1,\dots,n_k}$. Then $n_1 + \cdots + n_{r-1} + 1 \leq i \leq n_1 + \cdots + n_r$ and $n_1 + \cdots + n_{s-1} + 1 \leq \pi(i) \leq n_1 + \cdots + n_s$ for some $r \neq s$. It is easy to check that the block difference $\mathrm{bdiff}_{\mathfrak{m}}(e')$ is nonzero, and its sign is determined by r and s. Thus, marking or unmarking a homogeneous edge $e \in E^H(\mathfrak{m})$ does not change the status of whether e' is a block weak excedance or not. Therefore, when we are concerned with the change of the statistic bwex, it is sufficient to consider the changes of block differences of homogeneous edges when we toggle e.

We are now ready to define the involution Φ.

Definition 3.7. (The involution Φ) For $\mathfrak{m} = (\pi, S) \in \mathrm{PM}^*_{n_1,\ldots,n_k}$, we define $\Phi(\mathfrak{m})$ as follows.

Case 0 If $\mathfrak{m}$ has no homogeneous edges, then define $\Phi(\mathfrak{m}) = \mathfrak{m}$. In other words, Φ is the identity map on $\mathcal{D}(n_1, \ldots, n_k)$.

Case 1 Suppose $\mathfrak{m}$ has homogeneous edges and $\mathrm{bdiff}_{\mathfrak{m}}(e) \geq 0$ for all $e \in E^H(\pi)$. Define $\Phi(\mathfrak{m}) = (\pi, S \triangle \{e_i\})$, where i is the integer satisfying $\pi(i) = \min\{\pi(j) : e_j \in E^H(\pi)\}$. In other words, we mark or unmark the homogeneous edge whose lower endpoint is the leftmost one among the homogeneous edges.

Case 2 Suppose $\mathfrak{m}$ has homogeneous edges and $\mathrm{bdiff}_{\mathfrak{m}}(e) < 0$ for some $e \in E^H(\pi)$. Let $i = \min\left\{j : e_j \in E^H(\pi), \mathrm{bdiff}_{\mathfrak{m}}(e_j) < 0\right\}$. Depending on the convertibility of the edge e_i, we consider two subcases.

Subcase 2-(a) If e_i is convertible, then define $\Phi(\mathfrak{m}) = (\pi, S \triangle \{e_i\})$.

Subcase 2-(b) If e_i is not convertible, then define $\Phi(\mathfrak{m}) = (\pi, S \triangle \{e_{i'}\})$, where

$$i' = \max\left\{j < i : e_j \in E^H(\pi), \mathrm{bdiff}_{\mathfrak{m}}(e_j) = 0, e_j \text{ crosses } e_i\right\}.$$

Example 3.8. The applications of the map Φ in Cases 1, 2-(a) and 2-(b) are illustrated in Figure 5. Marked edges are represented in dashed lines, and inhomogeneous edges are colored in gray. The block differences of homogeneous edges are indicated by the numbers above their upper endpoints. The edge chosen by Φ is the thick (dashed) edge.

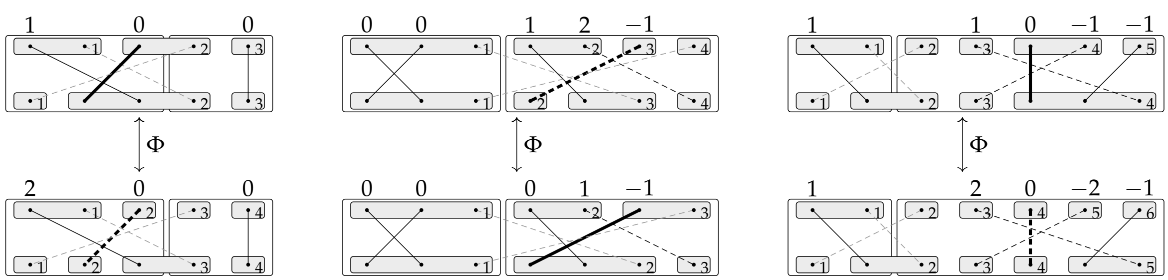

Figure 5

Note that except for Case 0, Φ toggles only one edge's marking status. Hence Φ is sign-reversing. Moreover, it turns out that the map Φ satisfies the followings:

- Φ is well-defined, i.e., $\{j < i : e_j \in E^H(\pi), \mathrm{bdiff}_{\mathfrak{m}}(e_j) = 0, e_j \text{ crosses } e_i\} \neq \emptyset$.
- Φ is an involution, i.e., $\Phi^2 = \mathrm{Id}$.

- Φ preserves the block weak excedances, and the sum of the weight and the number of crossings, i.e.,

$$\mathrm{bwex}(\mathfrak{m}) = \mathrm{bwex}(\Phi(\mathfrak{m})), \qquad \mathrm{wt}(\mathfrak{m}) + \mathrm{cr}(\mathfrak{m}) = \mathrm{wt}(\Phi(\mathfrak{m})) + \mathrm{cr}(\Phi(\mathfrak{m})).$$

In other words, Φ is a sign-reversing and weight-preserving involution on $\mathrm{PM}^*_{n_1,\dots,n_k}$ with fixed point set $\mathcal{D}(n_1,\dots,n_k)$.

Proof of Theorem 3.4. Recall from (3.3) that we have

$$C(n_1,\dots,n_k) = \sum_{\mathfrak{m}\in\mathrm{PM}^*_{n_1,\dots,n_k}} (-1)^{\mathrm{e}(\mathfrak{m})} y^{\mathrm{bwex}(\mathfrak{m})} q^{\mathrm{wt}(\mathfrak{m})+\mathrm{cr}(\mathfrak{m})}.$$

Applying the involution Φ on $\mathrm{PM}^*_{n_1,\dots,n_k}$, the terms corresponding to the matchings $\mathfrak{m} \notin \mathcal{D}(n_1,\dots,n_k)$ are cancelled out and we have

$$C(n_1,\dots,n_k) = \sum_{\mathfrak{m}\in\mathcal{D}(n_1,\dots,n_k)} (-1)^{\mathrm{e}(\mathfrak{m})} y^{\mathrm{bwex}(\mathfrak{m})} q^{\mathrm{wt}(\mathfrak{m})+\mathrm{cr}(\mathfrak{m})}.$$

If $\mathfrak{m} = (\pi, S) \in \mathcal{D}(n_1,\dots,n_k)$, then $S = E(\pi)$ and therefore $\mathrm{e}(\mathfrak{m}) = 0$. Thus we obtain the desired formula. $\square$

References

[1] M. Anshelevich et al. "Linearization coefficients for orthogonal polynomials using stochastic processes". *Ann. Probab.* **33**.1 (2005), pp. 114–136. Link.

[2] S. Corteel, J. Kim, and D. Stanton. "Moments of orthogonal polynomials and combinatorics". 2016. Link.

[3] M. Ismail, D. Stanton, and G. Viennot. "The combinatorics of q-Hermite polynomials and the Askey—Wilson integral". *European J. Combin.* **8**.4 (1987), pp. 379–392. Link.

[4] A. Kasraoui, D. Stanton, and J. Zeng. "The combinatorics of Al-Salam–Chihara q-Laguerre polynomials". *Adv. Appl. Math.* **47**.2 (2011), pp. 216–239. Link.

[5] D. Kim, D. Stanton, and J. Zeng. "The combinatorics of the Al-Salam-Chihara q-Charlier polynomials". *Séem. Lothar. Combin.* **54** (2006), B54i. Link.

[6] G. Viennot. "Une Th'eorie Combinatoire Des Polynômes Orthogonaux Généraux". *Lecture Notes, UQAM, Montréal* (1988).

[7] J. Zeng. "Weighted derangements and the linearization coefficients of orthogonal Sheffer polynomials". *Proc. Lond. Math. Soc.* **3**.1 (1992), pp. 1–22. Link.