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Random t-cores and hook lengths in random partitions

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Abstract. Fix $t \geq 2$. We first give an asymptotic formula for certain sums of the number of t-cores. We then use this result to compute the distribution of the size of the t-core of a uniformly random partition of an integer n. We show that this converges weakly to a gamma distribution after appropriate rescaling. As a consequence, we find that the size of the t-core is of the order of $\sqrt{n}$ in expectation. We then apply this result to show that the probability that t divides the hook length of a uniformly random cell in a uniformly random partition equals $1/t$ in the limit. Finally, we extend this result to all modulo classes of t using abacus representations for cores and quotients.

Keywords: t-core, t-quotient, uniformly random partition, gamma distribution, hook length, random cell, abacus

1 Introduction and statement of results

The irreducible representation of the symmetric group S_n are indexed by partitions of n. While studying modular representations of S_n, one naturally encounters special partitions called *t-cores* [6], which are defined for any integer $t \geq 2$. The t-core of a partition λ, denoted $\text{core}_t(\lambda)$, can be obtained from λ by a sequence of operations. See Section 2.2 for the precise definitions. A partition λ is itself called a t-core if $\text{core}_t(\lambda) = \lambda$. Let $c_t(n)$ be the number of t-cores of size n. We will be interested in finding the asymptotic behavior of certain sums involving $c_t(n)$. Anderson [1] has obtained detailed asymptotic results for $c_t(n)$ using the circle method, but these will not suffice for our purposes. We define the number of partitions obtained by taking t-cores of all partitions of n by

$$C_t(n) := \#\{\text{core}_t(\lambda) \mid \lambda \quad \text{a partition of } n\}. \tag{1.1}$$

When t is a prime number, $C_t(n)$ can be defined to be the number of t-blocks, that is, the number of connected components of the Brauer graph, in the t-modular representation theory of the symmetric group S_n [5].

Our first main result is the following.

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Theorem 1.1. *Fix $t \geq 2$. Then*

$$C_t(n) = \frac{(2\pi)^{(t-1)/2}}{t^{(t+2)/2}\,\Gamma(\frac{t+1}{2})}\left(n + \frac{t^2-1}{24}\right)^{(t-1)/2} + O(n^{(t-2)/2}). \tag{1.2}$$

Corollary 1.2. *Fix $t \geq 2$. Then $c_t(n) = O(n^{(t-2)/2})$.*

Let n be a positive integer and $t \geq 2$ be a fixed positive integer as before. Let λ be a uniformly random partition of n. Let Y_n be a random variable on $\mathbb{N}_{\geq 0}$ given by

$$Y_n \equiv Y_{n,t} = |\,\mathrm{core}_t(\lambda)|, \tag{1.3}$$

where $|\cdot|$ denotes the size of the partition. We will be interested in the convergence of Y_n. The probability mass function of Y_n is given by

$$\mu_n(k) \equiv \mu_{n,t}(k) = \frac{\#\{\lambda \vdash n : |\,\mathrm{core}_t(\lambda)| = k\}}{p(n)}, \tag{1.4}$$

where $p(n)$ is the number of partitions of n. It can be shown (see Corollary 2.6) that μ_n can be written as

$$\mu_n(k) = \frac{c_t(k)d_t(n-k)}{p(n)}, \tag{1.5}$$

where $d_t(m)$ is the number of partitions of m with empty t-core. Let $X_n \equiv X_{n,t}$ be continuous random variables defined on $[0,\infty)$ with the probability density function $f_n \equiv f_{n,t}$ given by

$$f_n(x) = \frac{\sqrt{n}\,c_t(\lfloor x\sqrt{n}\rfloor)\,d_t(n - \lfloor x\sqrt{n}\rfloor)}{p(n)}. \tag{1.6}$$

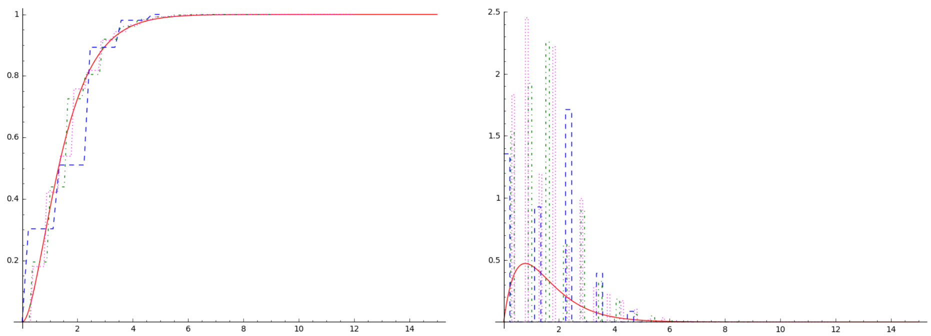
Note that the integral of f_n being 1 is equivalent to $\sum_k \mu_n(k) = 1$. Recall that the *gamma distribution* with *shape parameter* $\alpha > 0$ and *rate parameter* $\beta > 0$ is a continuous random variable on $[0,\infty)$ with density given by

$$\gamma(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x), & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1.7}$$

where Γ is the standard gamma function.

Theorem 1.3. *The random variable X_n converges weakly to a gamma-distributed random variable X with shape parameter $\alpha = (t-1)/2$ and rate parameter $\beta = \pi/\sqrt{6}$.*

See Figure 1 for an illustration of Theorem 1.3 for $t = 5$. Notice that while the distribution seems to converge pointwise in Figure 1(a), the density in Figure 1(b) does not. An immediate consequence of Theorem 1.3 is the following result.



(a) (b)

Figure 1: Comparison of the limiting CDFs and densities for small values of n with $t = 5$. A red solid line is used for the limiting distribution, dashed blue for $n = 20$, dash-dotted green for $n = 62$ and dotted magenta for $n = 103$. In (a) the CDFs, and in (b) the densities, are plotted for X and these X_n's.

Corollary 1.4. *The expectation of the size of the t-core for a uniformly random partition of size n is asymptotic to* $(t-1)\sqrt{6n}/2\pi$.

We illustrate Corollary 1.4 with the example of $t = 3$ in Figure 2. Theorem 1.3 and Corollary 1.4 will be proved in Section 4.

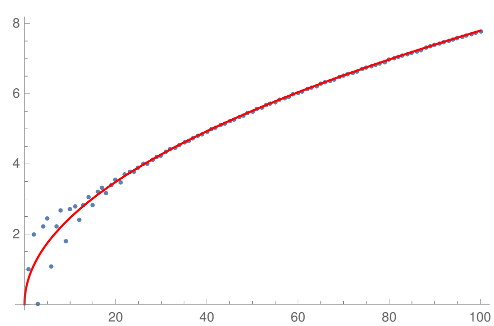


Figure 2: The average size of the 3-core for partitions of size 1 to 100 in blue circles, along with the result from Corollary 1.4, $\sqrt{6x}/\pi$, as a red line.

Let λ be a partition of n and c denote a cell in the Young's diagram of λ. Then we define h_c be the hook length associated to the cell c. See Section 2 for the precise definitions. Our final major result is a statement about the remainder of hook lengths of cells when divided by t.

Theorem 1.5. *For a uniformly random cell c of a uniformly random partition λ of n, the probability that the hook length of c in λ is congruent to i modulo t is asymptotic to* $1/t$ *for any* $i \in \{0, 1, \ldots, t-1\}$.

We will devote Section 2 to state some preliminary results. We will sketch the proofs of Theorems 1.1, 1.3 and 1.5 in Sections 3 to 5 respectively. The complete proofs of these results can be found in the extended version of this paper [2], which will be published elsewhere.

2 Preliminaries

Recall that an *(integer) partition* λ of a nonnegative integer n is a nonincreasing tuple of nonnegative integers which sum up to n. If λ is a partition of n, we write $\lambda \vdash n$ and say that the *size* of λ, denoted $|\lambda|$, is n. Let $p(n)$ denote the number of partitions of n. We write $\mathcal{P}$ for the set of all partitions.

Let $c \equiv (i, j)$ be a cell in the Young diagram of λ. The *hook* of c is a subset of the cells in the Young diagram containing the cells to the right of c in the same row, those below c in the same column, and c itself. The *hook length* is the cardinality of the cells in the hook of c and is denoted by h_c^λ.

2.1 Abacus representation

Definition 2.1. An *abacus* or *(1-runner)* is a function $w : \mathbb{Z} \to \{0, 1\}$ such that there exist $m, n \in \mathbb{Z}$ such that $w_i = 1$ (resp. $w_i = 0$) for all $i \leq m$ (resp. $i \geq n$).

Starting from an abacus w, consider the up-right path formed by replacing 1's by vertical steps and 0's by horizontal steps. This path will form the outer boundary of a partition. However, note that this is not a bijective correspondence because any shift of the abacus will lead to the same partition.

An abacus w is called *justified at position p* if $w_i = 1$ (resp. $w_i = 0$) for $i < p$ (resp. $i \geq p$). Any abacus can be transformed to a justified one by moving the 1's to the left past the 0's start from the leftmost movable 1. An abacus is called *balanced* if, after this transformation, it is justified at position 0. Note that balanced abaci are in bijection with partitions.

We now summarize properties of abaci that will be relevant to us. Readers interested in the details can look at [6, 9, 8].

Proposition 2.2. *Let λ be a partition with corresponding balanced abacus w and c be a cell in the Young diagram of λ. Then the following properties hold.*

1. *Cells in the Young diagram of λ are in bijection with pairs (i, j) such that $i < j$, $w_i = 0$ and $w_j = 1$. The hook length of the cell c corresponding to the pair (i, j) in w is $j - i$.*

2. *Suppose c has hook length t. Then c corresponds to a pair $(i, i + t)$ in w. Then, removing a t-rim hook for the cell c (see the beginning of Section 2.2) from the Young diagram of λ amounts to exchanging the i'th and $(i + t)$'th entries in w.*

2.2 Cores and quotients

To describe cores and quotients, we will need some notation. Associated to a partition λ and a cell c in its Young diagram, the set of cells joining the two corners in the hook of c along the boundary is known as the *rim hook* or *ribbon* of c.

The *t-core of a partition* λ, denoted $\mathrm{core}_t(\lambda)$, is the partition obtained by removing as many rim hooks of size t from λ as possible. Using Proposition 2.2, we can define $\mathrm{core}_t(\lambda)$ as the partition corresponding to abacus obtained by exchanging as many $(0,1)$ pairs at positions $(i, i+t)$ as possible.

A partition λ is called a *t-core* if $\mathrm{core}_t(\lambda) = \lambda$, or equivalently, if none of the hook numbers in the Young diagram of λ is divisible by t. From Proposition 2.2, it follows that the partition λ is a t-core if and only if there is no $i \in \mathbb{Z}$ such that $w_i = 0$ and $w_{i+t} = 1$. Let $\mathcal{C}_t$ be set of all t-cores. Let $c_t(n)$ be the number of t-cores of size n. Then it is known [7] that

$$\sum_{n=0}^{\infty} c_t(n) x^n = \prod_{k=1}^{\infty} \frac{(1-x^{tk})^t}{1-x^k}. \tag{2.1}$$

Definition 2.3. A partition is said to be *t-divisible* if it has empty t-core.

Let $\mathcal{D}_t(n)$ be the set of t-divisible partitions of n and $d_t(n)$ be its cardinality. Given a partition λ with abacus w, construct t abaci by letting $\lambda^i = (w_{nt+i})_{n\in\mathbb{Z}}$ for $0 \le i \le t-1$. The *t-runner abacus* of λ is then $(\lambda^0, \dots, \lambda^{t-1})$ of 1-runner abaci, where the zeroth position in λ^0 is underlined. We define the *t-quotient* of λ to be a t-tuple of partitions corresponding to the entries of the t-runner $(\lambda^0, \dots, \lambda^{t-1})$. We will use the same notation for the i'th entry of the t-quotient and the t-runner. It turns out that t-cores and t-divisible partitions have a natural interpretation in terms of t-runner abaci.

Proposition 2.4. *Let λ be a partition and $(\lambda^0, \dots, \lambda^{t-1})$ be its t-runner abacus. Then*

1. *λ is a t-core if and only if λ^i is justified for $0 \le i \le t-1$, and*
2. *λ is t-divisible if and only if λ^i is balanced for $0 \le i \le t-1$.*

A fundamental result on cores and quotients is the *partition division theorem* [8, Theorem 11.22], which is an analogue of the division algorithm for integers. We restate it in slightly different terminology more suited to our purposes.

Theorem 2.5. *Let $t \ge 2$ be an integer. Then there is a natural bijection*

$$\Delta_t : \mathcal{P} \to \mathcal{C}_t \times \mathcal{D}_t$$

defined by $\Delta_t(\lambda) = (\rho, \nu)$, where ρ is the t-core of λ and ν is a t-divisible partition whose t-quotient, $(\nu^0, \dots, \nu^{t-1})$, is same as that of λ. Moreover $|\lambda| = |\rho| + |\nu|$.

The following corollary is then immediate.

Corollary 2.6. *Let i, n be positive integers such that $i < n$. Then*

$$\#\{\lambda \vdash n \mid |\mathrm{core}_t(\lambda)| = i\} = d_t(n-i) c_t(i).$$

3 Asymptotics of the number of t-cores

The asymptotics of the number of t-cores, $c_t(n)$, was obtained using the circle method by Anderson [1, Theorem 2] for $t \geq 6$. We demonstrate a new method to obtain these asymptotics for $t \geq 2$.

The following theorem describe the quantity $c_t(n)$ as the number of integer solutions of a particular quadratic equation. This is a reformulation of [4, Bijection 2], and we give a new proof using the abacus representation which can be found in the [2]. Let H_t denote the hyperplane in $\mathbb{R}^t$ given by $H_t = \{(x_0, \ldots, x_{t-1}) \mid x_0 + x_1 + \ldots x_{t-1} = 0\}$.

Theorem 3.1. *The number of t-cores of n, $c_t(n)$, is equal to the number of integer solutions of $F_t(p) = n$ on the hyperplane H_t, where F_t is defined by*

$$F_t(p) = \frac{t}{2} \sum_{i=0}^{t-1} p_i^2 + \sum_{i=0}^{t-1} i p_i. \tag{3.1}$$

Observe that subtracting $(t-1)\left(\sum_{i=0}^{t-1} p_i\right)/2$ from (3.1) yields the equation of a t-dimensional sphere centered at a point in the hyperplane H_t given by

$$\sum_{i=0}^{t-1} \left(p_i - \frac{t-1-2i}{2t}\right)^2 = \frac{2}{t}\left(n + \frac{t^2-1}{24}\right). \tag{3.2}$$

Denote the $(t-1)$-dimensional ball cut out by the hyperplane H_t and the sphere given by (3.2) as $B_{t-1}(n)$. The radius of $B_{t-1}(n)$ is unchanged since the center $1/(2t)(t-1, t-3, \ldots, -(t-3), -(t-1))$ lies on H_t. Then the volume $V_t(n)$ of $B_{t-1}(n)$, under the induced measure on the hyperplane H_t, is

$$V_t(n) = \frac{1}{\Gamma(\frac{t+1}{2})}\left(\frac{2\pi}{t}\left(n + \frac{t^2-1}{24}\right)\right)^{(t-1)/2}. \tag{3.3}$$

Define the lattice $\Lambda_t = H_t \cap \mathbb{Z}^t$. It is clear that Λ_t is a full lattice of the codimension one subspace H_t with $\mathbb{Z}$-basis $\{e_0 - e_1, e_0 - e_2, \ldots, e_0 - e_{t-1}\}$. We use standard techniques to note that the volume $V(\Lambda_t)$ of the fundamental domain of Λ_t in H_t is $\sqrt{t}$.

We obtain the next result by working over the module $(\mathbb{Z}/t\mathbb{Z})^t$. To avoid confusion, we will denote points in $(\mathbb{Z}/t\mathbb{Z})^t$ with a tilde, e.g. $\widetilde{P}, \widetilde{Q}$.

Lemma 3.2. *For any $n \in \mathbb{N}$, the cardinality of the set $\{\widetilde{Q} \in \widetilde{H}_t \mid F_t(\widetilde{Q}) \equiv n \mod t\}$ is t^{t-2}, where F_t is defined in (3.1) and $\widetilde{H}_t$ is the hyperplane of $(\mathbb{Z}/t\mathbb{Z})^t$ defined by $q_0 + q_1 + \cdots + q_{t-1} = 0$.*

We are now in a position to prove the main result of this section.

Proof of Theorem 1.1. Recall that the number of partitions obtained by taking t-cores of all partitions of n is denoted by $C_t(n)$ and is defined in(1.1). It is easy to prove the relation

$$C_t(n) = \sum_{i=0}^{\lfloor \frac{n}{t} \rfloor} c_t(n - i\,t).$$

Using Theorem 3.1, we conclude that

$$C_t(n) = \#\big\{Q \in \Lambda_t \cap B_{t-1}(n) \mid F_t(Q) \equiv n \mod t\big\}. \tag{3.4}$$

Splitting the set of points in the set on the right hand side according to their remainder modulo t, we arrive at

$$C_t(n) = \sum_{\substack{\widetilde{P} \in \Lambda_t / t\Lambda_t \\ F(\widetilde{P}) \equiv n \mod t}} \#\left((\widetilde{P} + t\Lambda_t) \cap B_{t-1}(n)\right), \tag{3.5}$$

where $\widetilde{P} + t\Lambda_t$ is the lattice $t\Lambda_t$ shifted by $\widetilde{P}$. Clearly, the fundamental domain has volume $V(\widetilde{P} + t\Lambda_t) = t^{t-1}V(\Lambda_t)$. We use Lemma 3.2 and standard results on the asymptotic of the number of lattice points inside a $(t-1)$ dimensional sphere to obtain the required approximation in terms of $V_t(n)$ defined in (3.3). □

4 Distribution of core sizes

In this section we will sketch the proof of Theorem 1.3. We prove the weak convergence by calculating moments. As usual, we fix $t \geq 2$. Let $\epsilon(n) \in \{0, 1, \ldots, t-1\}$ be the remainder of n is divided by t. For convenience, define $\ell_n(y) = t\lfloor y\sqrt{n}\rfloor + \epsilon(n)$, and we assume that all the functions defined below are supported on $y \geq 0$. We will make crucial use of the functions

$$\psi_n(y) = \frac{c_t(\ell_n(y))}{n^{(t-3)/4}}, \tag{4.1}$$

and

$$g_{n,k}(y) = \ell_n(y)^k n^{(t-1)/4-k/2} \frac{d_t(n - \ell_n(y))}{p(n)}. \tag{4.2}$$

Recall that the function $f_n(x)$ is defined in (1.6). The functions $g_{n,k}$ and ψ_n are defined so that $\psi_n(y)g_{n,0}(y) = f_n(\ell_n(y)/\sqrt{n})$ is roughly like $f_n(ty)$ for large n. We now define

$$g_k(y) = \kappa y^k e^{-\beta t y}, \tag{4.3}$$

where $\kappa = t^{(t+2+2k)/2} 2^{-3(t-1)/4} 3^{-(t-1)/4}$ and we recall that we have defined $\beta = \pi/\sqrt{6}$.

Lemma 4.1. *For any natural number k, the sequence of functions $g_{n,k}$ converges uniformly to g_k as $n \to \infty$. Moreover,*

$$\lim_{n\to\infty} \int_0^\infty dy\ \psi_n(y) g_{n,k}(y) = \lim_{n\to\infty} \int_0^\infty dy\ \psi_n(y) g_k(y). \tag{4.4}$$

Lemma 4.1 is lengthy and technical, and so we do not sketch the proof. A complete proof can be found in [2].

Lemma 4.2. *For all positive integer k,*

$$\lim_{n\to\infty} \mathbb{E}(X_n^k) = \left(\frac{\sqrt{6}}{\pi}\right)^k \prod_{i=1}^k \left(\frac{t-1}{2} + k - i\right). \tag{4.5}$$

Sketch of proof. We first note that the required limit of expectation equals the expression in (4.4). Using Lemma 4.1 and integration by parts, we reduce the problem to finding the integral of $\psi_n(y)$, with other quantities well understood in terms of g_k. We use Theorem 1.1 to obtain an estimate for the integral of $\psi_n(y)$, which enables us to obtain the required expression for the k'th moment. □

Proof of Theorem 1.3. From the formula for the moments of the gamma distribution, one can verify that the moment generating function of the gamma distribution is determined by its moments (see [3, Theorem 30.1], for example). Using Lemma 4.2, we have shown that all the moments of X_n exist and converge to those of X, which is gamma distributed. Again appealing to a standard result on weak convergence [3, Theorem 30.2], we see that X_n converges weakly to X. □

5 Hook lengths of random cells of random partitions

The main result of this section is that for large enough n, the modulo class of the hook length of a uniformly random cell of a uniformly random partition is approximately the uniform distribution on the modulo classes $\{0, 1, \ldots, t-1\}$.

We obtain the following proposition as an easy application of Corollary 1.4 by noting that removing a t-rim hook results reduces the number of hooks h_c with $t|h_c$ by exactly one. The same is not true for non-zero modulo classes.

Proposition 5.1. *For a uniformly random cell c of a uniformly random partition λ of n, the probability that the hook length of c in λ is divisible by t is $1/t + O(n^{-1/2})$.*

Using the results in Section 4, we are only able to prove the results in Proposition 5.1. To obtain the stronger result stated in Theorem 1.5, we will need to appeal to the theory of t-runner abaci.

We will now estimate the number of cells of the region with small hook lengths. The following lemma gives us a crucial bound in proving the main theorem of this section.

Lemma 5.2. *For $\lambda \vdash n$ and an integer m, the cardinality of the set $\mathcal{B} = \{c \in \lambda \mid h_c < m\}$ is less than $m\sqrt{2n}$.*

Proof. It is enough to construct an injection from the set of unordered pairs of distinct cells in $\mathcal{B}$ to $\lambda \times [m] \times [m]$. We describe such a construction below.

Suppose c_1 and c_2 are two cells in $\mathcal{B}$, and say that c_1 is to the west of c_2 and if both are in the same column let c_1 to be south of c_2. Let a_2 be the arm length of c_2 and l_1 be the leg length of c_1. Then map the pair $\{c_1, c_2\}$ to (c, a_2, l_1), where c is the unique cell in the intersection of the column containing c_1 and the row containing c_2.

This map is clearly injective since the cell c describes the column (or row) in which the cell c_1 (resp. c_2) lives and l_1 (resp a_2) gives the exact location of c_1 (resp c_2) in the partition. □

5.1 A natural action of S_t

We define the action of S_t on $\mathcal{D}_t$, the set of t-divisible partitions, as follows. For any $\sigma \in S_t$ and a t-divisible partition ν with t-quotient $(\nu^0, \nu^1, \dots, \nu^{t-1})$, define $\sigma\nu$ be the t-divisible partition corresponding to the t-quotient $(\nu^{\sigma_0}, \nu^{\sigma_1}, \dots, \nu^{\sigma_{t-1}})$. Note that the above action preserves the size of the t-divisible partition.

Definition 5.3. The *b-smoothing* of a t-divisible partition ν, denoted C_ν^b, is the union of cells in the Young diagram of ν whose corresponding $(0,1)$ pairs are at least $(b+1)$ columns apart in the t-runner abacus of ν.

Proposition 5.4. *The set of cells C_ν^b is a connected subpartition of ν. Moreover $C_\nu^b = C_{\sigma\nu}^b$ for all $\sigma \in S_t$ and $b \geq 0$.*

For any cell $c \in C_\nu^b$, define h_c^ν to be the hook length of the cell c in the partition ν. For the corresponding $(0,1)$ pair in the t-runner abacus of ν, where $0 \in \nu^i$ and $1 \in \nu^j$, we see that $h_c^\nu \equiv j - i \mod t$. Using the action of S_t defined above, we get the following

Lemma 5.5. *Let ν be a uniformly random t-divisible partition of n and $b \geq 0$. For a uniformly random cell $c \in C_\nu^b$, the probability that the hook length h_c^ν is congruent to i modulo t, where $i \neq 0$, is independent of i.*

Definition 5.6. We define the action of $\sigma \in S_t$ on $\mathcal{P}$ by $\sigma\lambda = \Delta_t^{-1}(\rho, \sigma\nu)$, where $\Delta_t(\lambda) = (\rho, \nu)$ is defined in Theorem 2.5.

Let $\lambda \in \mathcal{P}$ with $\Delta_t(\lambda) = (\rho, \nu)$, and ρ be determined by the t-tuple $(p_0, \dots, p_{t-1})$, where p_i is the position of justification of abacus ρ^i and let $b_\lambda = \max_{1 \leq i < j \leq t-1} |p_i - p_j|$. We will denote $C_\nu^{b_\lambda}$ by C_λ for brevity and call it the *canonical smoothing* of the t-quotient of λ.

Proposition 5.7. *Let λ be a partition with $\Delta_t(\lambda) = (\rho, \nu)$. Then there exists an injective map ϕ that takes the cells in C_λ to the cells in λ such that for any cell $c \in C_\lambda$, the hook lengths $h^\lambda_{\phi(c)} \equiv h^\nu_c \mod t$.*

Proof. Let $c \in C_\lambda \subset \nu$ be given by the pair $(\nu^a_i, \nu^b_j) = (0, 1)$ where $j - i > b_\lambda$ by definition. Let ρ be determined by the justification positions $(p_0, \ldots, p_{t-1})$, where $\sum_{i=0}^{t-1} p_i = 0$ as explained before. Then a required map can be defined as

$$\phi(c) := (\nu^a_{i+p_a}, \nu^b_{j+p_b}).$$

□

5.2 Proof of the main theorem

The main idea is to show that the image of the canonical smoothing $\phi(C_\lambda)$ comprise of most of the cells in a uniformly random partition λ of size n. We first obtain a bound $|b_\lambda| \leq 2\sqrt{|\rho|}$ using elementary counting. Then we apply Lemma 5.2 to get the following

Proposition 5.8. *For any partition $\lambda \vdash n$ with $\Delta_t(\lambda) = (\rho, \nu)$,*

$$|\nu| - |C_\lambda| < \#\{c \in \nu \mid h_c < t(b_\lambda + 1)\} = O(t(b_\lambda + 1)\sqrt{n}). \tag{5.1}$$

We are now in a position to prove the main result of this section.

Proof of Theorem 1.5. Let $x_i(n)$ be the probability that a uniformly random cell c of a uniformly random partition of n has hook length congruent to i modulo t. Using Proposition 5.7 and Lemma 5.5, we obtain by conditioning for any nonzero classes i and j modulo t, the difference between $x_i(n)$ and $x_j(n)$ in absolute value is upper bounded by the probability that $c \notin \phi(C_\lambda)$. Therefore, by Theorem 2.5, we have that

$$|x_i(n) - x_j(n)| \leq \sum_{\substack{\lambda \vdash n \\ \Delta_t(\lambda) = (\rho, \nu)}} \frac{|\nu| - |C_\lambda| + |\rho|}{np(n)}.$$

Now, by Proposition 5.8 and Corollary 1.4,

$$|x_i(n) - x_j(n)| < O(n^{-1/2}\mathbb{E}(\sqrt{|\operatorname{core}_t(\lambda)|})) + O(n^{-1/2}).$$

Using the standard fact that $\mathbb{E}\sqrt{Y} \leq \sqrt{\mathbb{E}Y}$ for any nonnegative random variable Y, we see that the right hand side is $O(n^{-1/4})$. Since $x_0(n) + x_1(n) + \cdots + x_{t-1}(n) = 1$ and $x_0(n) = 1/t + O(n^{-1/4})$, we obtain the required result. □

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