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Crystal structures for canonical and dual weak symmetric Grothendieck functions

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Abstract. We give a $U_q(\mathfrak{sl}_n)$-crystal structure on multiset-valued tableaux, hook-valued tableaux, and valued-set tableaux, whose generating functions are the weak symmetric, canonical, and dual weak symmetric Grothendieck functions, respectively. We show the result is isomorphic to a (possibly infinite) direct sum of highest weight crystals, and we provide an explicit bijection for multiset-valued tableaux. As a consequence, these generating functions are Schur positive; in particular, the canonical Grothendieck functions are Schur positive, which was not previously known. We extend Hecke insertion to express a dual stable Grothendieck function as a sum of Schur functions.

Keywords: canonical Grothendieck function, crystal, quantum group, multiset-valued tableau, hook-valued tableau, valued-set tableau

1 Introduction

The Grassmannian $\mathrm{Gr}(n,k)$ is the set of k-dimensional hyperplanes in $\mathbb{C}^n$. K-theory classes of structure sheaves of Schubert varieties form a basis of the K-theory ring of the Grassmannian and are represented by Grothendieck polynomials. We can take the stable limit to define stable Grothendieck functions $\mathfrak{G}_w$ (often denoted G_w in the literature), where w is a permutation. Stable Grothendieck functions have been well-studied using a variety of methods; see for example [4, 1, 3, 8, 9, 10, 11, 12, 13] and references therein.

The functions $\mathfrak{G}_w$ indexed by Grassmannian permutations w are called symmetric Grothendieck functions, and they form a basis $\{\mathfrak{G}_\lambda\}_\lambda$, where λ is a partition, for (an appropriate completion of) the ring of symmetric functions Λ over $\mathbb{Z}[\beta]$. A well-known basis of Λ is the Schur functions $\{s_\lambda\}_\lambda$, and $\mathfrak{G}_\lambda$ is Schur positive [9] with a finite expansion in each degree β. So we can apply the involution ω that sends $s_\mu \mapsto s_{\mu'}$, where μ' is the conjugate shape of μ. The resulting basis $\{\mathfrak{J}_\lambda\}_\lambda$ is known as the weak stable Grothendieck functions. Since $\{\mathfrak{J}_\lambda\}_\lambda$ is a filtered basis, we can consider its dual basis $\{\mathfrak{g}_\lambda\}_\lambda$ under the Hall inner product called the dual symmetric Grothendieck functions.

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By also applying ω, we obtain the dual weak symmetric Grothendieck functions $\{\mathfrak{j}_\lambda\}_\lambda$. Each of these bases have combinatorial interpretations with tableaux-like objects [3, 8].

In an effort to unify the bases $\{\mathfrak{G}_\lambda\}_\lambda$ and $\{\mathfrak{J}_\lambda\}_\lambda$, Yeliussizov introduced in [16] the canonical Grothendieck functions $\{\mathfrak{H}_\lambda\}_\lambda$ and fused the corresponding combinatorics in hook-valued tableaux by combining those of set-valued tableaux for $\mathfrak{G}_\lambda$ and multiset-valued tableaux for $\mathfrak{J}_\lambda$. Furthermore, we have $\mathfrak{H}_\lambda \mathfrak{H}_\mu = \sum_\nu (\alpha + \beta)^{k_\nu} c_{\lambda\mu}^\nu \mathfrak{H}_\nu$ for some k_ν if and only if $\mathfrak{G}_\lambda \mathfrak{G}_\mu = \sum_\nu c_{\lambda\mu}^\nu \mathfrak{G}_\nu$, and similarly for the coproduct. Likewise, he defined the dual canonical Grothendieck functions as the corresponding dual basis, described them combinatorially using rim border tableaux, and showed they are Schur positive.

Since a Schur function is a character for the special-linear Lie algebra $\mathfrak{sl}_n$, the Schur positivity implies that (multi)set-valued tableaux, reverse plane partitions, and rim border tableaux should have $U_q(\mathfrak{sl}_n)$-crystal structures. Indeed, this was done for set-valued tableaux [10] and for reverse plane partitions [6]. In this work, we obtain similar results by constructing a $U_q(\mathfrak{sl}_n)$-crystal structure on multiset-valued tableaux and valued-set tableaux. Furthermore, we show that many results from [10] for set-valued tableaux have analogs for multiset-valued tableaux. More specifically, we extend the notion of the uncrowding crystal isomorphism from [3, Section 6] to an explicit crystal isomorphism from multiset-valued tableaux to the usual crystal on semistandard tableaux $B(\lambda)$. We also extend Hecke insertion [4] to give a crystal structure on weakly decreasing factorizations and give a positive Schur expansion of general weak stable Grothendieck functions.

Our other main result constructs a $U_q(\mathfrak{sl}_n)$-crystal structure on hook-valued tableaux. This implies that $\mathfrak{H}_\lambda$ is Schur positive as a corollary, which was not previously known. Our crystal structure on hook-valued tableaux is a combination of the crystal structures on set-valued tableaux and multiset-valued tableaux. However, we are not able to provide an explicit isomorphism with a highest weight crystal and instead must rely on the Stembridge axioms [15]. Indeed, the set-valued (resp. multset-valued) tableaux crystal structure preserves rows (resp. columns), each of which is isomorphic to hook shape, and so the crystal structures are not directly compatible.

Since dual canonical Grothendieck are Schur positive [16, Theorem 9.8], there should exist a $U_q(\mathfrak{sl}_n)$-crystal structure on rim border tableaux with an additional marking of all interior boxes by either α or β as the exponent of $(\alpha + \beta)$ corresponds to the number of interior boxes. However, the crystal structure appears to be more complicated than combining the crystal structures on reverse plane partitions and valued-set tableaux. Thus, it is an open problem to construct a $U_q(\mathfrak{sl}_n)$-crystal on marked rim border tableaux.

This paper is organized as follows. In Section 2, we provide the necessary background. In Section 3 (resp. Section 4, Section 5), we give our results on multiset-valued (resp. hook-valued, valued-set) tableaux. This is an extended abstract version of [7].

2 Background

We use English convention for partitions and tableaux. Let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting indeterminates. Let $U_q(\mathfrak{sl}_n)$ be the quantum group of the special linear Lie algebra over $\mathbb{C}$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition, also written as $\sum_{i=1}^n c_i \Lambda_i$, where c_i is the number of columns of height i, under the identification with the $U_q(\mathfrak{sl}_n)$ weight lattice.

A *(multi)set-valued tableau of shape* λ is a filling T of the boxes of Young diagram of λ by finite nonempty (multi)sets of positive integers such that rows are weakly increasing and columns are strictly increasing in the following sense: For every (multi)set A to the left of a (multi)set B in the same row, we have $\max A \leq \min B$, and for C below A in the same column, we have $\max A < \min C$. A *semistandard tableau* is a set-valued tableau with all sets of size 1. Let $\mathrm{SVT}^n(\lambda)$ (resp. $\mathrm{SST}^n(\lambda)$, $\mathrm{MVT}^n(\lambda)$) denote the set-valued (resp. semistandard, multiset-valued) tableaux of shape λ with entries at most n.

In [10], a $U_q(\mathfrak{sl}_n)$-crystal structure was given on $\mathrm{SVT}^n(\lambda)$ whose *crystal operators* $e_i, f_i \colon \mathrm{SVT}^n(\lambda) \to \mathrm{SVT}^n(\lambda) \sqcup \{0\}$, where $i \in I := \{1, \dots, n-1\}$, are defined as follows.

Definition 2.1. Fix $T \in \mathrm{SVT}^n(\lambda)$ and $i \in I$. Write $+$ (resp. $-$) above columns of T containing an i but not an $i+1$ (resp. an $i+1$ but not an i). Cancel signs in ordered pairs $-+$ until obtaining a sequence of the form $+\cdots+-\cdots-$ called the *i-signature*.

The action of f_i is given as follows. If there is not a $+$ in the resulting sequence, then $f_i T = 0$. Otherwise let b correspond to the box of the rightmost uncanceled $+$. Then $f_i T$ either given by removing the i from $\mathsf{b}^{\rightarrow}$ and adding an $i+1$ to b if there exists a box $\mathsf{b}^{\rightarrow}$ immediately to the right of b that contains an i and otherwise replacing the i in b with an $i+1$. The action of e_i is the reverse: Let b be the box for the leftmost uncanceled $-$. Move the $i+1$ from $\mathsf{b}^{\leftarrow}$, the box immediately to the left of b, into b as an i if changing the $i+1 \in \mathsf{b}$ is not a valid set-valued tableau.

The *weight* of a set-valued tableau $T \in \mathrm{SVT}^n(\lambda)$ is $\mathrm{wt}(T) := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where m_i is the number of occurrences of i in T. Let $|T| := \sum_{i=1}^n m_i$. This gives a $U_q(\mathfrak{sl}_n)$-crystal structure on $\mathrm{SVT}^n(\lambda)$.[1] We say a $T \in \mathrm{SVT}^n(\lambda)$ is *highest weight* if $e_i T = 0$ for all $i \in I$. A crystal morphism is called *strict* if it commutes with all e_i and f_i. For more details on crystals, we refer the reader to [5]. This crystal structure on $\mathrm{SST}^n(\lambda)$ is the crystal $B(\lambda)$ of the irreducible $U_q(\mathfrak{sl}_n)$-module of highest weight λ. Furthermore, this gives a $U_q(\mathfrak{sl}_n)$-crystal structure on words of length ℓ by equating with $B(\Lambda_1)^{\otimes \ell}$.

Theorem 2.2 ([10, Theorem 3.9])**.** *Let λ be a partition. Then $\mathrm{SVT}^n(\lambda) \cong \bigoplus_{\lambda \subseteq \mu} B(\mu)^{\oplus S_\lambda^\mu}$, where the S_λ^μ is the highest weight elements of weight μ in $\mathrm{SVT}^n(\lambda)$.*

From [3], we can define a *symmetric Grothendieck function* as

$$\mathfrak{G}_\lambda(\mathbf{x}; \beta) := \sum_{T \in \mathrm{SVT}^\infty(\lambda)} \beta^{|T| - |\lambda|} \mathrm{wt}(T),$$

[1] We give the weights as a multiplicative group as it is useful for defining polynomials in the sequel.

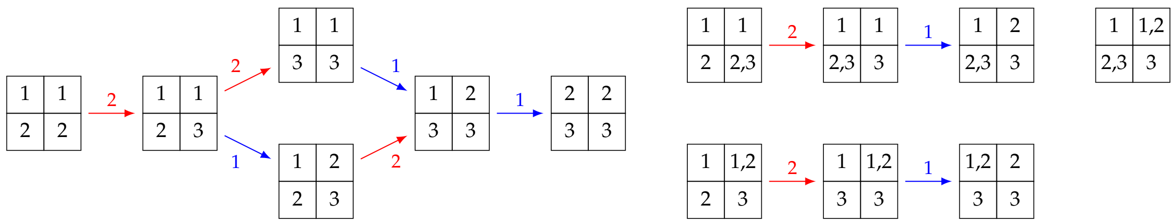


Figure 1: The $U_q(\mathfrak{sl}_3)$-crystal structure on $\mathrm{SVT}^3\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) = \mathrm{SVT}^3(2\Lambda_2)$.

where $|\lambda|$ denotes the size of λ (*i.e.*, the number of boxes in λ). When $\beta = 0$, we recover the *Schur function*, where the sum is instead over all $T \in \mathrm{SST}^\infty(\lambda)$. The *weak symmetric Grothendieck function* is defined by

$$\mathfrak{J}_\lambda(\mathbf{x};\alpha) := \sum_{T\in\mathrm{MVT}^\infty(\lambda)} \alpha^{|T|-|\lambda|}\,\mathrm{wt}(T) = \mathfrak{G}_\lambda\left(\frac{x_1}{1-\alpha x_1}, \frac{x_2}{1-\alpha x_2}, \frac{x_3}{1-\alpha x_3}, \dots; \alpha\right),$$

which recovers the definition given in [12, Theorem 6.11] when $\alpha = -1$ and $x_i \mapsto -x_i$. Another equivalent way to define a weak symmetric Grothendieck function is by using the involution ω on symmetric functions given by $\omega s_\lambda(\mathbf{x}) = s_{\lambda'}(\mathbf{x})$, where λ' is the conjugate partition of λ. Indeed, we have $\mathfrak{J}_{\lambda'}(\mathbf{x};\alpha) = \omega\mathfrak{G}_\lambda(\mathbf{x};\alpha)$ [8, Proposition 9.22].

The bases $\{\mathfrak{G}_\lambda\}_\lambda$ and $\{\mathfrak{J}_\lambda\}_\lambda$ have a common generalization given by Yeliussizov [16]. A *hook tableau* is a semistandard Young tableau T of the form

$$\begin{array}{|c|c|c|c|}\hline h & A_1 & \cdots & A_k \\ \hline L_1 \\ \cline{1-1} \vdots \\ \cline{1-1} L_\ell \\ \cline{1-1}\end{array}\,, \qquad \text{where} \qquad \begin{array}{l} h \text{ is the } \textit{hook entry}, \\ A(T) := (A_1, \dots, A_k) \text{ is the } \textit{arm}, \\ L(T) := (L_1, \dots, L_\ell) \text{ is the } \textit{leg}. \end{array}$$

Let $L^+(T) := \{h\} \cup L(T)$ denote the *extended leg*. A *(semistandard) hook-valued tableau of shape* λ is a filling T of the boxes of λ by hook tableaux such that the rows are weakly increasing and the columns are strictly increasing the in the same sense as for (multi)set-valued tableaux. Let $\mathrm{HVT}^n(\lambda)$ denote the set of hook-valued tableau of shape λ with max entry n. The *canonical Grothendieck polynomial* is defined in [16] as

$$\mathfrak{H}_\lambda(\mathbf{x};\alpha,\beta) := \sum_{T\in\mathrm{HVT}^\infty(\lambda)} \alpha^{|A(T)|}\beta^{|L(T)|}\,\mathrm{wt}(T).$$

Note that $\mathfrak{H}_\lambda(\mathbf{x};\alpha,0) = \mathfrak{J}_\lambda(\mathbf{x};\alpha)$ and $\mathfrak{H}_\lambda(\mathbf{x};0,\beta) = \mathfrak{G}_\lambda(\mathbf{x};\beta)$.

A *reverse plane partition (RPP) of shape* λ is a filling of λ by positive integers such that rows and columns are weakly increasing. Define the weight of a RRP P to be $\mathrm{wt}(P) := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where here m_i is the number of *columns* that contain an i in P. Denote $|P| := \sum_{i=1}^{n} m_i$. Let $\mathrm{RPP}^n(\lambda)$ be the set of reverse plane partitions with maximum entry n. The *dual symmetric Grothendieck function* $\mathfrak{g}_\lambda(\mathbf{x}; \beta)$ is defined

$$\mathfrak{g}_\lambda(\mathbf{x}; \beta) = \sum_{P \in \mathrm{RPP}^\infty(\lambda)} \beta^{|\lambda| - |P|} \,\mathrm{wt}(P).$$

The basis $\{\mathfrak{g}_\lambda\}_\lambda$ is dual to $\{\mathfrak{G}_\lambda\}_\lambda$ under the Hall inner product and is Schur positive [8].

Let $\mathrm{j}_\lambda(\mathbf{x}; \alpha)$ denote the *dual weak symmetric Grothendieck function*, which define by $\mathrm{j}_\lambda(\mathbf{x}; \alpha) = \omega \mathfrak{g}_{\lambda'}(\mathbf{x}; \alpha)$. The dual weak symmetric Grothendieck functions form the dual basis of $\{\mathfrak{J}_\lambda\}_\lambda$ [8, Theorem 9.15] with the following combinatorial interpretation. Define a *valued-set tableau of shape* λ to be a semistandard Young tableau of shape λ along with a grouping of boxes such that each *group* is composed of adjacent boxes with the same content (our description is conjugate to [8]). See Examples 5.2 and 5.4 for examples. The weight of a valued-set tableau V is $\mathrm{wt}(V) := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where here m_i is the number of *groups* that contain an i in V. Denote $|P| := \sum_{i=1}^{n} m_i$. Thus, we have

$$\mathrm{j}_\lambda(\mathbf{x}; \alpha) = \sum_{V \in \mathrm{VST}^\infty(\lambda)} \alpha^{|\lambda| - |V|} \,\mathrm{wt}(V),$$

where $\mathrm{VST}^n(\lambda)$ is the set of all valued-set tableaux of shape λ with max entry n. We call the leftmost entry in a group the *anchor*. We will also consider groups constructed by adding a vertical *divider* between certain pairs of entries i in the same row.

3 Crystal structure on multiset-valued tableaux

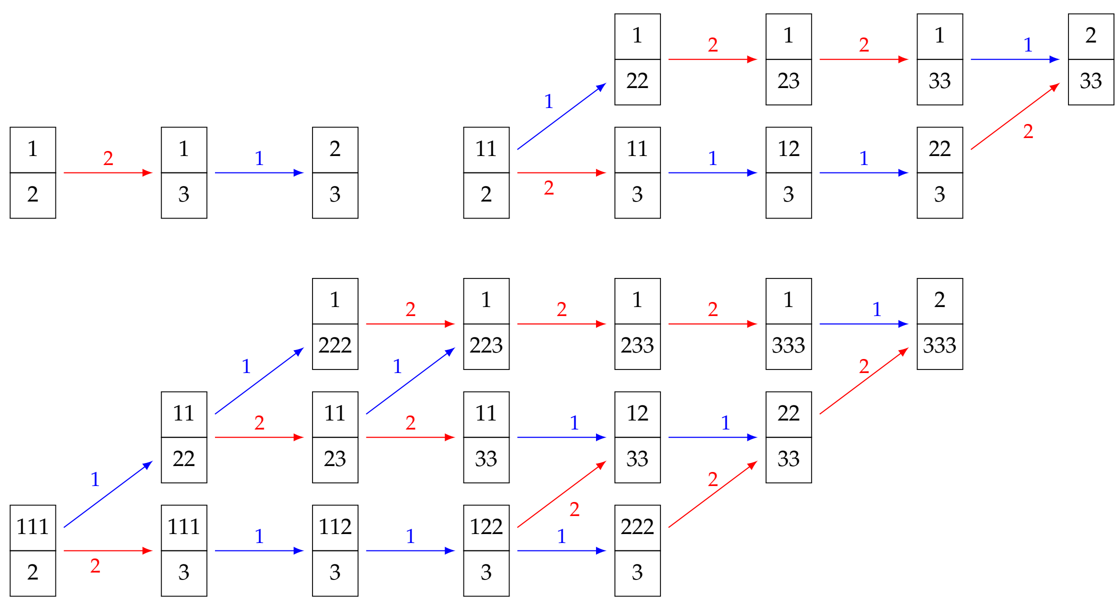
Definition 3.1 (Reading word)**.** Let C be a column of $T \in \mathrm{MVT}^n(\lambda)$. Define the *column reading word* $\mathrm{rd}(C)$ by reading the smallest entry of each box from bottom-to-top in C and then the remaining entries from smallest to largest in each box from top-to-bottom in C. Define the *reading word* $\mathrm{rd}(T) = \mathrm{rd}(C_1)\,\mathrm{rd}(C_2) \cdots \mathrm{rd}(C_k)$, where $C_1, C_2, \ldots, C_k$ are the columns of T from left-to-right.

Example 3.2. For the multiset-valued (column) tableau

$$C = \begin{array}{|c|} \hline \mathbf{1}13 \\ \hline \mathbf{4}445 \\ \hline \mathbf{6} \\ \hline \mathbf{7}899 \\ \hline \end{array} \longrightarrow \mathrm{rd}(C) = \mathbf{7641}13445899.$$

Definition 3.3 (Crystal operators)**.** Fix $T \in \mathrm{MVT}^n(\lambda)$ and $i \in I$. Consider the i-signature as in Definition 2.1 using the reading word. The action of f_i is given as follows. If there is not a $+$ in the resulting sequence, then $f_i T = 0$. Otherwise let b correspond to the box of the rightmost uncanceled $+$. Then $f_i T$ is given by removing an i from b and adding an $i+1$ to $\mathsf{b}^{\downarrow}$ if there exists a box $\mathsf{b}^{\downarrow}$ immediately below b that contains an $i+1$ and otherwise replacing the i in b with an $i+1$. The action of e_i is the reverse: Let b be the box for the leftmost uncanceled $-$. Move the $i+1$ from b into $\mathsf{b}^{\uparrow}$, the box immediately above b, as an i if changing the $i+1 \in \mathsf{b}$ is not a valid multiset-valued tableau.

Example 3.4. The connected components in $\mathrm{MVT}^3(\Lambda_2)$ with the crystal operators from Definition 3.3 that correspond to α^0, α^1, and α^2 are



Lemma 3.5. *Let $T \in \mathrm{MVT}^n(\lambda)$ and suppose $f_i T \neq 0$, then the i changed to $i+1$ in $f_i T$ does not change its position in the reading word. That is to say, we have $\mathrm{rd}(f_i T) = f_i \,\mathrm{rd}(T)$. Moreover, this defines a strict crystal embedding from $\mathrm{MVT}^n(\lambda)$.*

Remark 3.6. We note that our reading word is the column version of the reading word from [1, Definition 2.5]. Furthermore, when we consider the reading word from [1, Definition 2.5] applied to $\mathrm{SVT}^n(\lambda)$, but otherwise keep the same crystal operators, then the analog of Lemma 3.5 holds in that setting. In addition, our reading word and crystal structure for a single column is similar to the one for the minimaj crystal from [2].

Theorem 3.7. *Let λ be a partition. We have*

$$\mathrm{MVT}^n(\lambda) \cong \bigoplus_{\mu \supseteq \lambda} B(\mu)^{\oplus M_\lambda^\mu}, \qquad \mathfrak{J}_\lambda(\mathbf{x};\alpha) = \sum_{\mu \supseteq \lambda} \alpha^{|\mu|-|\lambda|} M_\lambda^\mu s_\mu(\mathbf{x}),$$

where M_λ^μ is the number of highest weight elements of weight μ in $\mathrm{MVT}^n(\lambda)$.

Our proof is similar to that from [10]: we show the isomorphism for a building block, here these are single columns (in [10], these are single rows), and using general properties of the tensor product rule. Note that Lemma 3.5 also yields Theorem 3.7 as every $B(\mu) \subseteq B(\Lambda_1)^{\otimes|\mu|}$, where the strict embedding is given by the reading word.

Proposition 3.8. *Suppose $T \in \mathrm{MVT}^n(\lambda)$ is a highest weight element. Then the i-th row of T contains only instances of the letter i.*

Let $\mathcal{F}^c_{\mu/\lambda}$ denote the set of *column flagged tableaux*, semistandard tableaux that strictly increase across rows and whose i-th column has maximum entry strictly less than i. Let $T \xleftarrow{\mathrm{RSK}} T'$ denote the Robinson–Schensted–Knuth (RSK) insertion of $\mathrm{rd}(T')$ into T.

Next, we construct an explicit crystal isomorphism

$$\Upsilon\colon \mathrm{MVT}^n(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} B(\mu) \times \mathcal{F}^c_{\mu/\lambda},$$

where the crystal structure on the codomain is given by $f_i(b \times F) = (f_i b) \times F$ for all $b \times F \in B(\mu) \times \mathcal{F}^c_{\mu/\lambda}$ for any fixed μ. We call the map Υ *uncrowding* as it is similar to the uncrowding map for set-valued tableaux (see [3, Section 6], [1, Section 5], and [10, Theorem 3.12]), but working column-by-column and measuring the growth of the diagram along columns. More specifically, for any $T \in \mathrm{MVT}^n(\lambda)$ we define $\Upsilon(T)$ recursively starting with $b_{\lambda_1+1} \times F_{\lambda_1+1} = \emptyset \times \emptyset$. Suppose we are at step i with the current state being $b_i \times F_i$, and let C_j denote the j-th column of T. Construct

$$b_{i-1} := \mathrm{rd}(C_{\lambda_1}) \xleftarrow{\mathrm{RSK}} \cdots \xleftarrow{\mathrm{RSK}} \mathrm{rd}(C_i) \xleftarrow{\mathrm{RSK}} \mathrm{rd}(C_{i-1}).$$

Construct F_{i-1} by starting first with F_i of shape μ_i but shifting the necessary elements to the right one step to partially fill in the (skew) shape of b_{i-1}. Then add entries in the unfilled boxes in column j with entry $j-1$ until F_{i-1} has been filled in. Thus, we constructed the $(i-1)$-th step $b_{i-1} \times F_{i-1}$. The final result is $\Upsilon(T) = b_1 \times F_1$.

Example 3.9. Applying uncrowding to

$$T = \begin{array}{|c|c|c|} \hline 112 & 22 & 256 \\ \hline 33 & 444 & 7 \\ \hline 568 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array},$$

where $\mathrm{rd}(T) = 953112368\ 42244\ 7256$, we start with $b_4 \times F_4 = \emptyset \times \emptyset$ and then obtain

$$b_3 \times F_3 = \begin{array}{|c|c|c|} \hline 2 & 5 & 6 \\ \hline 7 \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \cdot & \mathbf{1} & \mathbf{2} \\ \hline \cdot \\ \cline{1-1} \end{array},$$

$$b_2 \times F_2 = \begin{array}{|c|c|c|c|c|c|}\hline 2&2&2&4&5&6\\\hline 4&4&7\\\cline{1-3}\end{array} \times \begin{array}{|c|c|c|c|c|c|}\hline \cdot&\cdot&1&2&\mathbf{4}&\mathbf{5}\\\hline \cdot&\cdot&\mathbf{2}\\\cline{1-3}\end{array},$$

$$\Upsilon(T) = b_1 \times F_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&2&2&2&2&4&5&6\\\hline 3&3&4&4&7\\\cline{1-5} 5&6&8\\\cline{1-3} 9\\\cline{1-1}\end{array} \times \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&1&2&4&5&\mathbf{7}&\mathbf{8}\\\hline \cdot&\cdot&\cdot&2&\mathbf{4}\\\cline{1-5} \cdot&\mathbf{1}&\mathbf{2}\\\cline{1-3} \cdot\\\cline{1-1}\end{array}.$$

Theorem 3.10. *Under the isomorphism by the uncrowding map* Υ, *we have*

$$\mathrm{MVT}^n(\lambda) \cong \bigoplus_{\mu \supseteq \lambda} B(\mu)^{\oplus |\mathcal{F}^c_{\mu/\lambda}|}, \qquad \mathfrak{J}_\lambda(\mathbf{x};\alpha) = \sum_{\mu \supseteq \lambda} \alpha^{|\mu|-|\lambda|} |\mathcal{F}^c_{\mu/\lambda}| s_\mu(\mathbf{x}).$$

As a consequence of Theorem 3.10, we have $M_\lambda^\mu = |\mathcal{F}_{\mu/\lambda}|$. Furthermore, these are the conjugate of the flagged increasing tableaux of Lenart [9], and so

$$\mathfrak{G}_\lambda(\mathbf{x};\beta) = \sum_{\mu \supseteq \lambda} M_\lambda^{\mu'} s_\mu = \sum_{\mu' \supseteq \lambda} M_\lambda^{\mu'} \omega s_{\mu'} = \omega \mathfrak{J}_\lambda(\mathbf{x};\beta),$$

yielding a crystal-theoretic proof of [8, Proposition 9.22] (recall that ω is an involution).

The 0-*Hecke monoid* is the monoid of all finite words in the alphabet $\{1,2,\dots,n\}$ subject to the relations $ij \equiv ji$ if $|i-j| > 1$, $iji \equiv jij$ if $|j-i| = 1$, and $ii \equiv i$. For any $w \in S_n$, let $\mathcal{H}_w^k$ denote the words of length k that are equivalent to a reduced expression of w in the 0-Hecke monoid (*i.e* $w = s_{i_1} \cdots s_{i_\ell}$ is considered as $i_1 \cdots i_\ell$ and does not depend on the reduced expression of w). Let $\widehat{\mathcal{H}}_{w,m}^k$ denote the set of two-line arrays

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & 2 & \cdots & 2 & 2 & \cdots & m & \cdots & m \\ a_{1\ell_1} & \cdots & a_{12} & a_{11} & a_{2\ell_2} & \cdots & a_{22} & a_{21} & \cdots & a_{m\ell_m} & \cdots & a_{m1} \end{bmatrix}$$

such that $1 \leq a_{p1} \leq \cdots \leq a_{p\ell_p} < n$, $(a_{1\ell_1} \cdots a_{11}) \cdots (a_{m\ell_m} \cdots a_{m1}) \equiv w$, and $\sum_{p=1}^m \ell_p = k$.

Let $\mathcal{P}_w(\lambda)$ denote the set of increasing tableaux of shape λ such that reading the entries of P from top-to-bottom, right-to-left is equivalent to w in the 0-Hecke monoid. Let $\mathrm{MVT}(\lambda)_k$ denote the set of multiset-valued tableaux T such that $|\mathrm{wt}\, T| = k$.

Proposition 3.11. *Hecke insertion defined in [4] is a bijection* $\widehat{\mathcal{H}}_w^k \to \bigsqcup_\lambda \mathcal{P}_w(\lambda) \times \mathrm{MVT}(\lambda)_k$.

Definition 3.12. The *weak stable Grothendieck polynomial* is defined to be

$$\mathfrak{J}_w(\mathbf{x};\alpha) := \sum_{k=\ell(w)}^{\infty} \alpha^{k-\ell(w)} \sum_{(w,a) \in \widehat{\mathcal{H}}_w^k} \prod_{i=1}^{k} x_{a_i}.$$

The discussion above implies that $\mathfrak{J}_w(\mathbf{x};\alpha) = \sum_{k=\ell(w)}^\infty \alpha^{k-\ell(w)} \sum_{(P,\widehat{Q})} \mathrm{wt}(\widehat{Q})$, where we are summing over all $(P,\widehat{Q}) \in \bigsqcup_\lambda \mathcal{P}_w(\lambda) \times \mathrm{MVT}(\lambda)_k$. Putting this all together we obtain:

Proposition 3.13. *For any $w \in S_n$, we have*

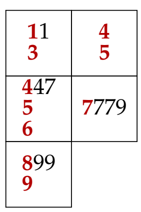
$$\mathfrak{J}_w(\mathbf{x};\alpha) = \sum_{\lambda} \sum_{P\in\mathcal{P}_w(\lambda)} \sum_{k=\ell(w)}^{\infty} \alpha^{k-\ell(w)} \sum_{\widehat{Q}} s_{\mathrm{wt}(\widehat{Q})}, \qquad \mathfrak{J}_w(\mathbf{x};\alpha) = \sum_{\lambda} \alpha^{|\lambda|-\ell(w)} |\mathcal{P}_w(\lambda)| \mathfrak{J}_\lambda(\mathbf{x};\alpha),$$

where we take the sum over all $\widehat{Q} \in \mathrm{MVT}(\lambda)_k$ such that $\widehat{Q}$ is a highest weight element.

4 Crystal structure on hook-valued tableaux

Definition 4.1 (Reading word)**.** Let C be a column of $T \in \mathrm{HVT}^n(\lambda)$. Define the *column reading word* $\mathrm{rd}(C)$ by reading the extended leg from largest to smallest in each box from bottom-to-top in C and then the entries in the arm from smallest to largest in each box from top-to-bottom in C. Define the *reading word* $\mathrm{rd}(T) = \mathrm{rd}(C_1) \cdots \mathrm{rd}(C_k)$, where $C_1, C_2, \ldots, C_k$ are the columns of T from left-to-right.

Example 4.2. For the hook-valued tableau

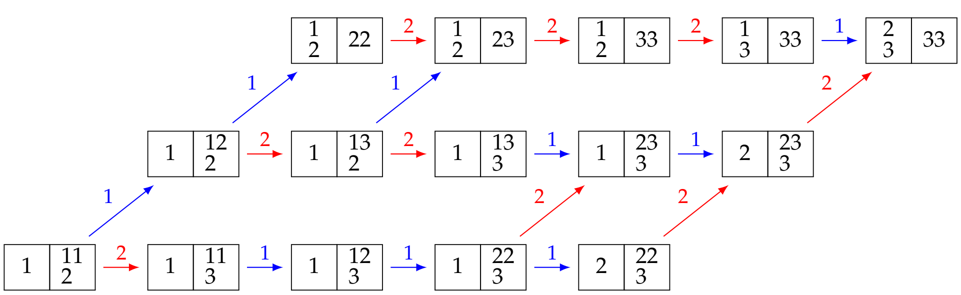
$T =$  $\longrightarrow \mathrm{rd}(T) = 986543114799754779.$

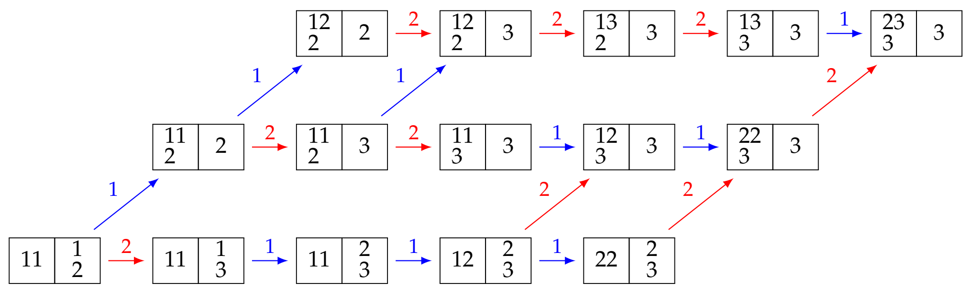
We define crystal operators by combining the set-valued crystal operators and the multiset-valued crystal operators.

Definition 4.3 (Crystal operators)**.** Fix $T \in \mathrm{HVT}^n(\lambda)$ and $i \in I$. We define $f_i T$ (resp. $e_i T$) by applying Definition 3.3 if i (resp. $i+1$) is in the arm of b and Definition 2.1 otherwise.

The hook element is $\min \mathsf{b}$, and so if we move the hook entry, there is no ambiguity in defining the hooks in $f_i T$ and $e_i T$ by the definition of the crystal operators.

Example 4.4. The following connected components in $\mathrm{HVT}^3(2\Lambda_1)$ are those that correspond to $\alpha\beta$ and both are isomorphic to $B(\Lambda_2 + 2\Lambda_1)$:





Lemma 4.5. *e_i and f_i are well-defined and $e_i T = T' \iff T = f_i T'$ for all $T, T' \in \mathrm{HVT}^n(\lambda)$.*

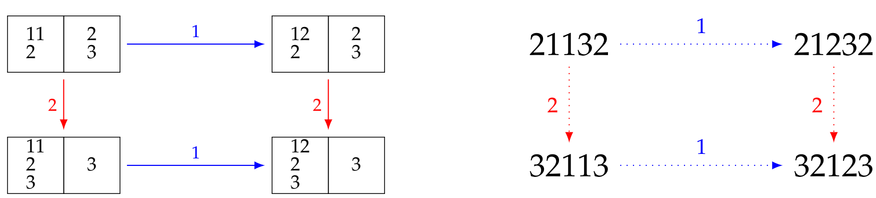
Theorem 4.6. *Let λ be a partition. We have*

$$\mathrm{HVT}^n(\lambda) \cong \bigoplus_{\mu \supseteq \lambda} B(\mu)^{\oplus H_\lambda^\mu}, \qquad \mathfrak{H}_\lambda(\mathbf{x}; \alpha, \beta) = \sum_T \alpha^{\sum_{\mathsf{b} \in T} |A(\mathsf{b})|} \beta^{\sum_{\mathsf{b} \in T} |L(\mathsf{b})|} s_{\mathrm{wt}(T)}(\mathbf{x}),$$

where H_λ^μ is the number of highest weight elements of weight μ in $\mathrm{HVT}^n(\lambda)$ and the sum is taken over all highest weight elements in $\mathrm{HVT}^n(\lambda)$.

We prove Theorem 4.6 using the Stembridge axioms [15].

Example 4.7. We have the following local relation in $\mathrm{HVT}^3(2\Lambda_1)$ on the left and their corresponding reading words on the right:



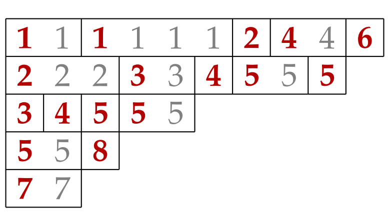
For the the upper-left hook-valued tableau T, we note that while the position of the 1 that is acted on by f_1 in T and $f_2 T$ is the same, it is still the second 1 in the reading word.

5 Crystal structure on valued-set tableaux

Definition 5.1 (Reading word)**.** Let $T \in \mathrm{VST}^n(\lambda)$. Define the *reading word* $\mathrm{rd}(T)$ to be the reading word of the usual reading word of the tableau of the anchors of T.

Example 5.2. For the valued set tableau

$T =$

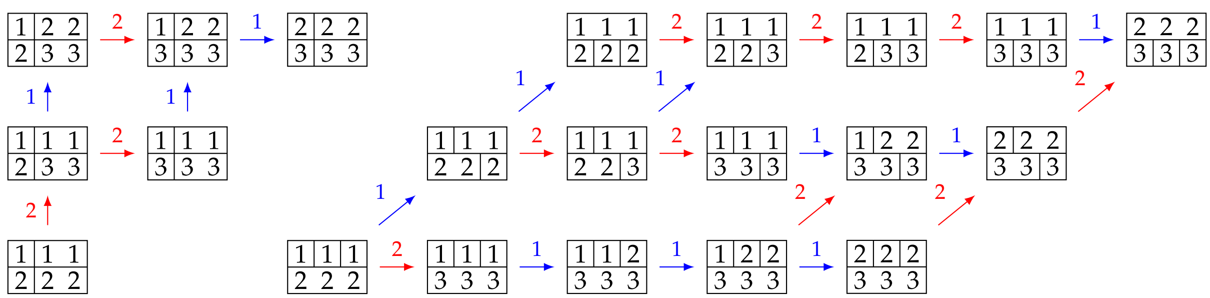


, we have

$$\mathrm{rd}(T) = 75321485153452456,$$
$$\mathrm{wt}(T) = x_1^2 x_2^2 x_3^2 x_4^3 x_5^5 x_6 x_7 x_8.$$

Definition 5.3 (Crystal operators)**.** Fix $T \in \mathrm{VST}^n(\lambda)$ and $i \in I$. Consider the i-signature of $\mathrm{rd}(T)$ as in Definition 2.1. Define $f_i T$ as follows. If there is no $+$ in the resulting sequence, then $f_i T = 0$. Otherwise let g correspond to the group of the rightmost uncanceled $+$. Define $f_i T$ by moving the divider on the left of g one step down if there is an $i+1$ immediately below g and otherwise changing every i to an $i+1$ in g. The definition of e_i is the reverse: Let g be the group of the leftmost uncanceled $-$. Move the divider up if changing all $i+1$'s to i's in g is not a valued-set tableau.

Example 5.4. The following are connected components in $\mathrm{VST}^3(3\Lambda_2)$ corresponding to α^2 that are isomorphic to $B(2\Lambda_2)$ and $B(\Lambda_2 + 2\Lambda_1)$:



Theorem 5.5. *Let λ be a partition. Then* rd *defines strict crystal embedding from* $\mathrm{VST}^n(\lambda)$ *and*

$$\mathrm{VST}^n(\lambda) \cong \bigoplus_{\mu \subseteq \lambda} B(\mu)^{\oplus V_\lambda^\mu}, \qquad \mathfrak{j}_\lambda(\mathbf{x}; \alpha) = \sum_{\mu \subseteq \lambda} \alpha^{|\lambda| - |\mu|} V_\lambda^\mu s_\mu(\mathbf{x}),$$

where V_λ^μ is the number of highest weight elements of weight μ in $\mathrm{VST}^n(\lambda)$.

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