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The Bruhat–Chevalley–Renner Order on the Set Partitions

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Abstract. We define combinatorially a partial order on the set partitions and show that it is equivalent to the Bruhat–Chevalley–Renner order on the upper triangular matrices. By considering subposets consisting of set partitions with a fixed number of blocks, we introduce and investigate "Stirling posets". As we show, the Stirling posets have a hierarchy and they glue together to give the whole set partition poset. Moreover, we show that they (Stirling posets) are graded and EL-shellable. We offer various reformulations of their length functions and determine the recurrences for their length generating series.

Keywords: Borel monoid, Stirling numbers

This extended abstract is based on our article [1], where one can find the complete proofs of our theorems.

Let n be a nonnegative integer. A collection $S_1, \ldots, S_r$ of non-empty subsets of an n-element set S is said to be a set partition of S if S_i's ($i = 1, \ldots, r$) are mutually disjoint and $\cup_{i=1}^r S_i = S$. In this case, S_i's are called the blocks of the partition. If $n > 0$ and $S = \{1, \ldots, n\}$, the collection of all set partitions of S is denoted by Π_n. We will often drop set parentheses and commas and just put vertical bars between blocks. If $B_1, \ldots, B_k$ are the blocks of a set partition π from Π_n, then the *standard form* of π is defined as $B_1|B_2|\cdots|B_k$, where we assume that $\min B_1 < \cdots < \min B_k$ and the elements of each block are listed in increasing order. For example, $\pi = 136|2459|78$ is a set partition from Π_9.

The set Π_n is known to be a host to many interesting algebraic and combinatorial structures. Among these structures is the following well-studied partial ordering: let A and A' be two set partitions of S. A is said to *refine* A' if each block of A is contained in some block of A'. This "refinement ordering" makes Π_n into a lattice, called the partition lattice, and by a result of Pudlak and Tuma [9] it is known that every lattice is isomorphic to a sublattice of Π_n for some n.

A property that is shared by all partition lattices is that their order complexes have the homotopy type of a wedge of spheres. This important combinatorial topological

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property is seen by analyzing the labelings of the covering relations of the refinement ordering. Indeed, it follows as a consequence of the fact that the refinement ordering is an "edge lexicographically shellable" (EL-shellable for short) poset as shown by Gessel (mentioned in [2]) and by Wachs in [13]. We postpone the proper definition of EL-shellability to our preliminaries section but let us only mention very briefly that the property of EL-shellability of a graded poset is a way of linearly ordering the maximal faces of the associated order complex, say $F_1, \cdots, F_m$, in such a way that $F_k \cap \left(\cup_{i=1}^{k-1} F_i \right)$ is a nonempty union of maximal proper faces of F_k ($k = 2, \ldots, m$). Having this property immediately implies a plethora of results on the topology of the underlying poset, such as Cohen–Macaulayness. It is also helpful for better understanding the Möbius function of the poset.

Our purpose in this paper is to present another natural partial ordering on Π_n and to show that our poset is EL-shellable as well. To define our ordering, we start with defining its most basic ingredient, namely the "arc-diagram." It is customary to call a linearly ordered poset a *chain*. We identify chains by their Hasse diagrams, and we draw them in an unorthodox way, horizontally, by placing the smallest entry on the left and connecting the vertices by arcs. For example, in Figure 1, we depict the chain on 9 vertices, where each arc represents a covering relation.

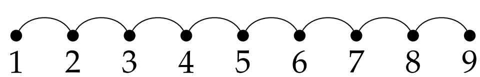

Figure 1: A chain on 9 vertices.

Definition 0.1. By a *labeled chain* we mean a chain whose vertices are labeled by distinct numbers. An *arc-diagram on n vertices* is a disjoint union of labeled chains where the labels are from $\{1, \ldots, n\}$ and each label $i \in \{1, \ldots, n\}$ is used exactly once. We depict an example in Figure 2.

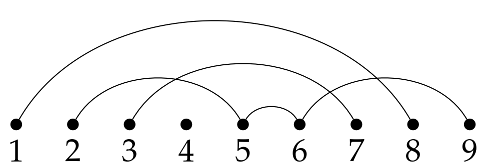

Figure 2: An arc-diagram on 9 vertices

It is easy to see that the arc-diagrams on n vertices are in bijection with the elements of Π_n. Indeed, the map that is defined by grouping the labels of a chain into a set extends to define a bijection from arc-diagrams to the set partitions. For example, under this bijection, the arc-diagram in Figure 2 corresponds to the set partition $18|2569|37|4$ in Π_9. In the light of this bijection, from now on, we will work with the arc-diagrams instead of set partitions. Let us use the notation $\mathcal{A}_n$ to denote the set of all arc-diagrams on n vertices. The goal of our article is to endow $\mathcal{A}_n$ with a partial order and to use it to investigate certain subposets of $\mathcal{A}_n$. In particular, we will focus on the subposets $\mathcal{A}_{n,k} \subset \mathcal{A}_n$, where the elements of $\mathcal{A}_{n,k}$ have exactly k chains. We will call these subposets as the title of our paper [1], namely, the Stirling posets.

Next we proceed to define the partial order that we will use throughout the paper. Let A be an arc-diagram. We will identify the vertices of A with their labels. An *arc* in A is a covering relation in any of the labeled chains in A. If the arc denoted by α is a covering relation between the vertices i and j, then we write $\alpha = \{i, j\}$. In practice (while drawing the diagrams) we will always think of an arc as the graph of a connected concave down path in $\mathbb{R}^2$. From this point of view, one of our most crucial conventions is that the arcs of A do not intersect each other if they do not have to. We illustrate what we mean here in Figure 3. If there is no possibility of continuously deforming two arcs α_1 and α_2 so that they do not intersect in $\mathbb{R}^2$, then they are said to *cross* each other. Otherwise, we call them *non-crossing* arcs.

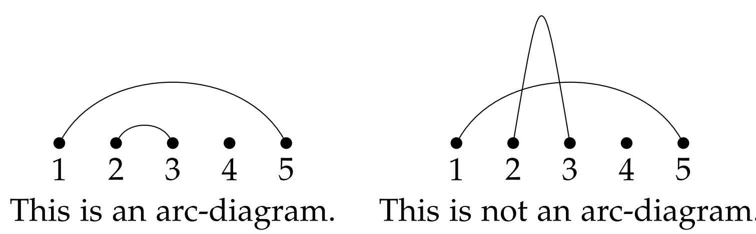

Figure 3: Conventions.

Before we proceed to explain our ordering on the arc-diagrams we will introduce a very useful function which will eventually lead us to a grading on our poset. This function is defined on all of the set of vertices, arcs, and chains of the arc-diagram. We will occasionally call a pair of non-crossing arcs nested if both of the starting and the ending vertices of one of the arcs stay below the other arc.

Definition 0.2. Let A be an arc-diagram and let α be a vertex, or an arc, or a chain from A. The depth of α, denoted by $depth(\alpha)$ is the total number of arcs "above" α.

Let us be more specific about what we mean by the word "above" in Definition 0.2: If α is a chain where i is its leftmost vertex and j is its rightmost vertex, then an arc $\{r, s\}$ is said to be above α if $r < i$ and $s > j$. For an example, see Figure 4, where every arc is

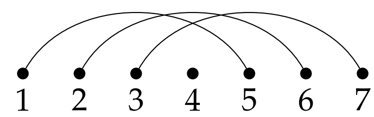

Figure 4: $depth(\{2,6\}) = 0$.

of depth 0 and the vertex 4 has depth 3. Obviously, for every arc-diagram the depths of the first and the last vertices are zero, that is, $depth(1) = depth(n) = 0$. Another simple observation that will be useful in what follows is that if an arc-diagram A on n vertices has k arcs, then A has exactly $n-k$ chains. In this regard, let us point out that the number of set partitions in Π_n with k blocks, hence the number of arc-diagrams in $\mathcal{A}_n$ with k chains, is given by the Stirling numbers of the second kind; it is easy to calculate these numbers by using the simple recurrence, $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

Let A and B be two arc-diagrams on n vertices. B is said to cover A, and denoted by $A \prec B$, if it is obtained from A by one of the following three operations:

Rule 1. The shortening of an arc of A. With this operation, we move exactly one endpoint of an arc to another vertex so that the resulting arc is shortened as minimally as possible but the number of crossings does not change. For example, see Figure 5, where we depict two examples. On the left, the left endpoint of the arc $\{1,4\}$ is moved to the nearest available position, which is the vertex 3. Indeed, there is already an arc which emanates to the right from the vertex 2.

Figure 5: Two examples for shortening.

Rule 2. Deleting a crossing. With this operation, we interchange the rightmost endpoints of two crossing arcs so that they become a pair of non-crossing and nested arcs; we require in this operation that only one crossing is deleted as a result of this operation. For example, in Figure 6, the endpoints of $\{1,5\}$ and $\{2,6\}$ are interchanged.

As a non-example, we consider $A = \{1,4\}\{2,5\}\{3,6\}$, which has three crossings. The removal of the crossing between $\{1,4\}$ and $\{3,6\}$ according to the rule that we described in the previous paragraph gives $A' = \{1,6\}\{2,5\}\{3,4\}$, which has no crossings.

Rule 3. Adding a new arc. With this operation, a new arc is introduced between two vertices in such a way that the new arc is not under any other (older) arcs and the endpoints of the new arc are as far from each other as possible. In Figure 7 we depict

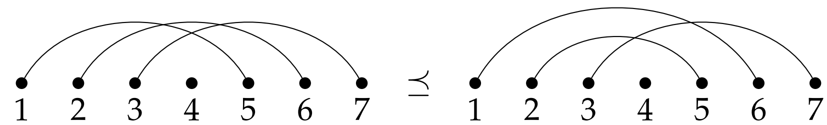

Figure 6: Interchanging two endpoints.

two examples. In the former one the new arc is $\{1,6\}$ and in the latter the new arc is $\{3,6\}$.

Figure 7: Two examples of adding a new arc.

From now on we will call the set $\mathcal{A}_n$ together with the transitive closure of the covering relations we just defined the *arc-diagram poset* and denote it by $(\mathcal{A}_n, \preceq)$.

Next, we define our first combinatorial statistic.

Definition 0.3. Let A be an arc-diagram on n vertices $v_1, \ldots, v_n$ and with k arcs α_1, α_2,...,α_k. We define the depth-index of A, denoted by $\mathtt{t}(A)$ by the formula

$$\mathtt{t}(A) = \sum_{i=1}^{k}(n-i) - \sum_{j=1}^{n} depth(v_j) + \sum_{m=1}^{k} depth(\alpha_m).$$

One of the main results of our paper is the following statement.

Theorem 0.4. *For every positive integer n, the arc-diagrams poset $(\mathcal{A}_n, \preceq)$ is a bounded, graded, and an EL-shellable poset. The depth-index function is the grading of $\mathcal{A}_n$.*

The proof of our theorem is at least as interesting as its statement. To explain it, we venture outside of combinatorics. Here we assume some familiarity with elementary algebraic geometry. Let Mat_n denote the linear algebraic monoid of $n \times n$ matrices defined over $\mathbb{C}$. The group of invertible elements, also called the *unit group*, of Mat_n is the general linear group of invertible $n \times n$ matrices. The (standard) Borel subgroup of GL_n, denoted by B_n, is the subgroup $\mathrm{B}_n \subset \mathrm{GL}_n$ consisting of upper triangular matrices only. Then the doubled Borel group $\mathrm{B}_n \times \mathrm{B}_n$ acts on matrices via

$$(b_1, b_2) \cdot x = b_1 x b_2^{-1} \qquad (b_1, b_2 \in \mathrm{B}_n,\ x \in \mathrm{Mat}_n) \tag{0.1}$$

Clearly, GL_n is stable under this action. By the special case of an important result of Renner [11], it is known that the action (0.1) has finitely many orbits and moreover the orbits of the action are parametrized by a finite inverse semigroup: $\mathrm{Mat}_n = \bigsqcup_{\sigma \in R_n} \mathrm{B}_n \sigma \mathrm{B}_n$, where R_n is the finite monoid consisting of $n \times n$ 0/1 matrices with at most one 1 in each row and each column. The monoid R_n is called the rook monoid; its elements are called rooks. (The nomenclature comes from the fact that the elements of R_n are in bijection with the non-attacking rook placements on an $n \times n$ chessboard.) The Bruhat–Chevalley–Renner ordering on R_n is the partial ordering that is defined by

$$\sigma \leq \tau \iff \mathrm{B}_n \sigma \mathrm{B}_n \subseteq \overline{\mathrm{B}_n \tau \mathrm{B}_n} \tag{0.2}$$

for $\sigma, \tau \in R_n$. This poset structure on R_n is well studied, [7]. It is known that $(R_n, \leq)$ is a graded, bounded, EL-shellable poset, see [3].

Towards a proof of Theorem 0.4, we make use of an important algebraic submonoid of Mat_n; it is the closure in Zariski topology of the Borel subgroup B_n in Mat_n. We will call $\overline{\mathrm{B}}_n$ the (standard) Borel submonoid. The first systematic study of the theory of Borel submonoids as a part of more general but interrelated theory of parabolic monoids is undertaken by Putcha in [10]. Here we are focusing on one extreme case only.

The Borel submonoid $\overline{\mathrm{B}}_n$ consists of all upper triangular $n \times n$ matrices with complex entries. To see this, we use the standard (semidirect product) decomposition $\mathrm{B}_n = \mathrm{T}_n \mathrm{U}_n$, where T_n is the maximal torus consisting of invertible diagonal matrices and U_n is the unipotent subgroup consisting of upper triangular unipotent matrices. It is easy to check that U_n is already closed in Mat_n, therefore, the Borel submonoid is determined (generated) by its submonoids $\overline{\mathrm{T}}_n$ and U_n. Here, $\overline{\mathrm{T}}_n$ is the diagonal submonoid consisting of all diagonal matrices. Note that $\overline{\mathrm{T}}_n$ is an affine toric variety and there is a one-to-one correspondence between the cones of its defining "fan" and its set of idempotents. (An idempotent in a monoid is an element e such that $e^2 = id$.)

Let M be a monoid and let 1_M denote its neutral element. For us, a submonoid N in a monoid M is a subsemigroup $N \subset M$ such that $1_M \in N$. In particular, 1_M is the identity element in N. Now, $\overline{\mathrm{B}}_n$ is a submonoid of Mat_n. Moreover, since it is closed under the two sided action of B_n, it has the induced Bruhat–Chevalley–Renner decomposition

$$\overline{\mathrm{B}}_n = \bigsqcup_{\sigma \in B_n} \mathrm{B}_n \sigma \mathrm{B}_n. \tag{0.3}$$

Here, B_n is the set of all $n \times n$ rooks which are upper triangular in shape. Note that B_n is a submonoid of R_n according to our definition. We call it the *upper triangular rook monoid* (on n letters). In Figure 8, we depict the induced Bruhat–Chevalley–Renner ordering on B_3.

Another subsemigroup that is very useful for our purposes is the semigroup of all nilpotent rooks from B_n, which we call the *standard nilpotent rook semigroup*, denoted by B_n^{nil}. In fact, for $n > 0$, it is not difficult to see that $(B_n^{nil}, \leq)$ is isomorphic, as a poset,

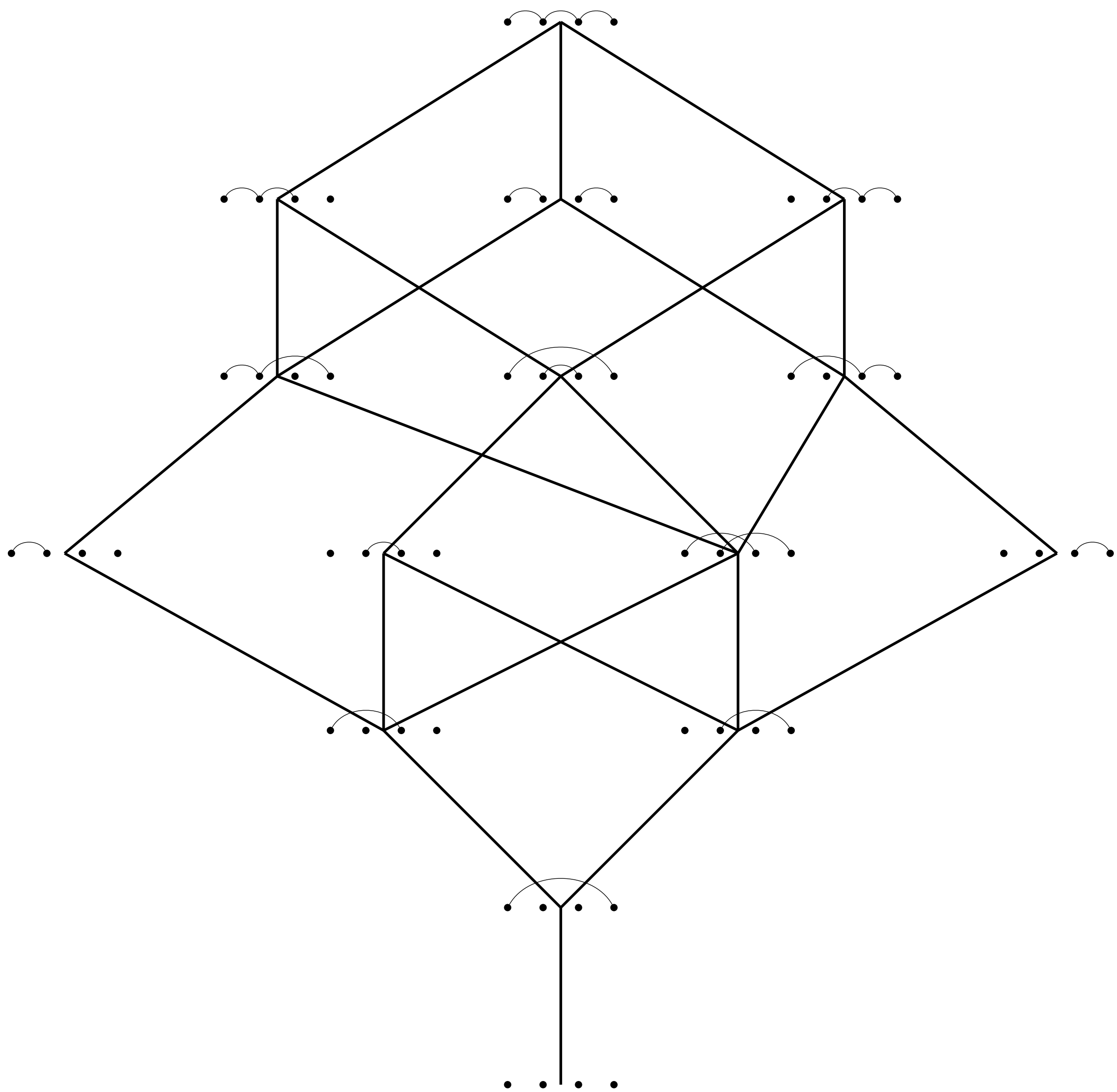

Figure 8: Bruhat–Chevalley–Renner order on $\mathcal{A}_4$.

to the upper triangular rook monoid $(B_{n-1}, \leq)$. By going through the same vein, we observe that the semigroup of nilpotent elements in $\overline{\mathrm{B}}_n$ is isomorphic, as an algebraic variety, to $\overline{\mathrm{B}}_{n-1}$.

The sets of idempotents of the monoids $\overline{\mathrm{B}}_n$ and $\overline{\mathrm{T}}_n$ are the same; they consist of $n \times n$ diagonal matrices with 0/1 entries. Let us denote this common set of idempotents by E_n. It is not difficult to see that E_n is a Boolean lattice with respect to the ordering

$$e \leq f \iff ef = fe = e \qquad (e, f \in E_n).$$

In particular, E_n has 2^n elements. We denote by $E_{n,k}$ the set of idempotents from E_n whose matrix rank is k and we define the following subvariety the Borel monoid:

$$\mathrm{B}_{n,k} := \bigcup_{e \in E_{n,k}} \overline{\mathrm{B}_n e \mathrm{B}_n}. \tag{0.4}$$

Notice that, except for $k \in \{0, n\}$, the variety $\mathrm{B}_{n,k}$ is not irreducible. Obviously, $\mathrm{B}_{n,n}$ is equal to $\overline{\mathrm{B}}_n$ and $\mathrm{B}_{n,0} = \overline{\mathrm{B}_n \cdot \mathbf{0} \cdot \mathrm{B}_n} = \{\mathbf{0}\}$.

Next, we list some important properties of our varieties $\mathrm{B}_{n,k}$ and the corresponding posets $B_{n,k}$.

1. For $k = 0, \dots, n$, the number of irreducible components of $\mathrm{B}_{n,k}$ is $\binom{n}{k}$ and they are all equal dimensional.

2. $\mathrm{B}_{n,k}$'s form a flag $\{\mathbf{0}\} = \mathrm{B}_{n,0} \subset \mathrm{B}_{n,1} \subset \cdots \subset \mathrm{B}_{n,n-1} \subset \mathrm{B}_{n,n} = \overline{\mathrm{B}}_n$.

3. Each $\mathrm{B}_{n,k}$ $(k = 0, \dots, n)$ has the structure of an algebraic semigroup.

4. Each $\mathrm{B}_{n,k}$ $(k = 0, \dots, n)$ has a Renner decomposition

$$\mathrm{B}_{n,k} = \bigsqcup_{\sigma \in B_{n,k}} \mathrm{B}_n \sigma \mathrm{B}_n, \tag{0.5}$$

   where $B_{n,k}$ is a finite subsemigroup of B_n and it consists of rooks whose matrix rank is at most k. Moreover, with respect to induced Bruhat–Chevalley–Renner ordering the poset $(B_{n,k}, \leq)$ is a union of lower intervals of equal lengths in B_n.

5. The subsemigroups $B_{n,k} \subset B_n$ form a flag $\{\mathbf{0}\} \subset B_{n,1} \subset \cdots \subset B_{n,n} = B_n$ and moreover the number of elements of $B_{n,k} \setminus B_{n,k-1}$ is given by the Stirling number $S(n+1, n+1-k)$.

6. The Bruhat–Chevalley–Renner ordering restricted to the subsets of the form $B_{n,k} \setminus B_{n,k-1}$ (for $k = 1, \dots, n$) is graded with a minimum and there are $\binom{n}{k}$ maximal elements. Each maximal interval in this poset is an interval in B_n, therefore, it is an EL-shellable poset.

As an application of our study of the Bruhat–Chevalley–Renner ordering on $B_{n,k}$'s we will prove the following theorem, which, in turn, will give us the proof of Theorem 0.4. Indeed, the poset $(B_n^{nil}, \leq)$ is a lower interval in the rook monoid, and R_n is known to be an EL-shellable poset.

Theorem 0.5. *The arc-diagram poset $(\mathcal{A}_n, \preceq)$ is isomorphic to $(B_n^{nil}, \leq)$, hence, it is a bounded, graded, and EL-shellable poset.*

Next, we show that the arc-diagram poset is a disjoint union of EL-shellable subposets, which are not necessarily intervals. The cardinalities of these subposets will be given by the Stirling numbers of the second kind.

Theorem 0.6. *If $\mathcal{A}_{n,k}$ denotes the set of arc-diagrams with $n-k$ chains, then $(\mathcal{A}_{n,k}, \preceq)$ is a graded EL-shellable poset with a unique minimum and $\binom{n-1}{k}$ maximum elements.*

Definition 0.7. The (n,k)-th Stirling poset is the poset $(\mathcal{A}_{n,k}, \preceq)$. By abusing notation, we will denote it by $\mathcal{A}_{n,k}$.

To contrast $\mathcal{A}_{n,k}$ with the corresponding subposet in the refinement ordering on set partitions, let us mention that any two unequal set partitions of $\{1,\ldots,n\}$ with the same number of blocks are not comparable. In other words, the collection of arc-diagrams with the same number of chains do not form an interesting poset with respect to refinement ordering. On the other hand, similarly to the refinement ordering, in $(\mathcal{A}_n, \preceq)$, the Stirling subposets have a hierarchy in the sense that $\mathcal{A}_{n,k}$ lies above $\mathcal{A}_{n,k-1}$. Indeed, if x and y are two maximal elements from $\mathcal{A}_{n,k}$ and $\mathcal{A}_{n,k-1}$, respectively, then $\mathtt{t}(x) - \mathtt{t}(y) = n-k$. From a similar vein, if x_0 and y_0 denotes, respectively, the minimum elements of $\mathcal{A}_{n,k}$ and $\mathcal{A}_{n,k-1}$, then $\mathtt{t}(x_0) - \mathtt{t}(y_0) = k$. It is not difficult to verify that when $k=1$, $\mathcal{A}_{n,1}$ is the "fish net" as in Figure 9, hence every interval in $\mathcal{A}_{n,1}$ is a lattice. As k increases, $\mathcal{A}_{n,k}$ becomes more complicated. Nevertheless, it is a pleasantly surprising fact that $\mathcal{A}_{n,2}$ is a lattice as well. The smallest integer n for which $\mathcal{A}_{n,k}$ has a non-lattice subinterval is $n=5$ (and $k=3$).

Theorem 0.8. *Let $B(n-1)$ denote the Boolean lattice of all subsets of $\{1,\ldots,n-1\}$. Then, for every integer $n \geq 2$, the $(n,2)$-th Stirling poset $\mathcal{A}_{n,2}$ is isomorphic to the "topless" poset $B(n-1) \setminus \{\{1,\ldots,n-1\}\}$.*

Next, we will discuss the length generating function of the (n,k)-th Stirling poset. Let us denote by $\mathtt{t}_k$ the length function on $\mathcal{A}_{n,k}$. Clearly, $\mathtt{t}_k$ is equal to an appropriate shift of $\mathtt{t}$. More precisely, let A be an element from $\mathcal{A}_{n,k}$. If we view A as an element of $\mathcal{A}_n$, then it is clear that $\mathtt{t}(A) = \mathtt{t}_k(A) + \binom{k}{2}$ since the unique minimum of $\mathcal{A}_{n,k}$ has depth-index $\binom{k}{2}$. To be able to treat all length generating functions $\mathtt{t}_k$ $(k = 0,\ldots,n)$ together, we define

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right] := \sum_{A \in \mathcal{A}_{n,k}} q^{\mathtt{t}(A)}. \tag{0.6}$$

Obviously, (0.6) is a q-analog of the Stirling numbes of the second kind.

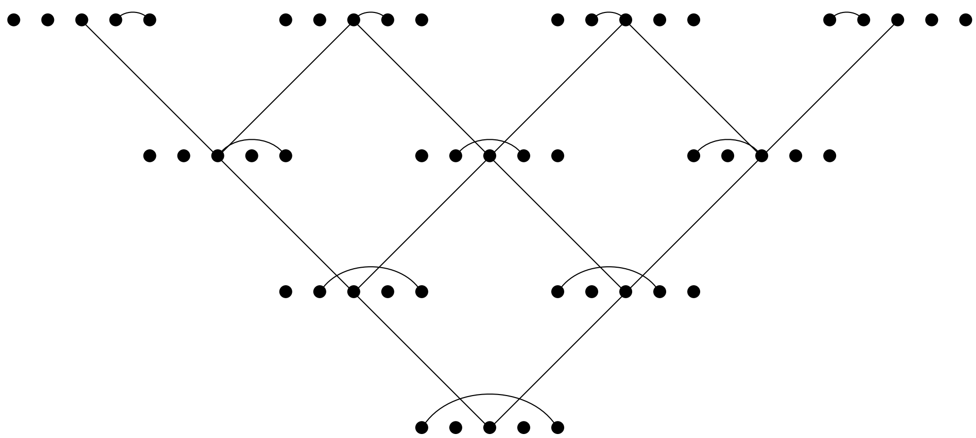

Figure 9: The Stirling poset $\mathcal{A}_{5,1}$.

Theorem 0.9. *For positive integers n and k such that $0 \leq k \leq n+1$ the following recurrence holds true:*

$$\left[\!\!\left[\begin{matrix} n+1 \\ k \end{matrix} \right]\!\!\right] = q^k \left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right] + [n+1-k]_q q^k \left[\!\!\left[\begin{matrix} n \\ k-1 \end{matrix} \right]\!\!\right],$$

where $[k]_q$ is the polynomial $1 + q + \cdots + q^{k-1}$. The initial conditions are $\left[\!\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix} \right]\!\right] = 1$ for all $m \in \mathbb{N}$. In addition, we assume that $\left[\!\left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]\!\right] = 0$ if $k < 0$ or $k > m$.

It turns out that our depth-index function is closely related to another well-studied statistic called the intertwining number of a set partition, see [4]. Also, there are various (p, q)-analogs of Stirling numbers of the second kind, see [12]. It would be interesting to study these (p, q)-analogs as well as other poset theoretic properties of the set-partitions poset under the refinement ordering in relation with our Stirling posets and Borel monoids.

In this paper, we considered the Borel submonoid of Mat_n, which is a $\text{GL}_n \times \text{GL}_n$-equivariant embedding of the symmetric space $(\text{GL}_n \times \text{GL}_n)/\text{diag}(\text{GL}_n)$. By using other embeddings of other symmetric spaces, one can build varieties that are similar to Borel monoids having analogous combinatorial properties. For example, let Sym_n denote the space of $n \times n$ symmetric matrices. We view Sym_n as a GL_n-equivariant embedding of GL_n/O_n in Mat_n. Then we have the Borel monoid-like subvariety $\text{C}_n := \overline{\{bb^T : b \in \text{B}_n\}}$ in Sym_n; it is equal to the Zariski closure of the orbit of the identity element $1 \in \text{Sym}_n$ under the congruence action of B_n on Sym_n. It is known that the inclusion posets of B_n-orbit closures in symmetric matrices, as well as in skew-symmetric matrices are EL-shellable, see [5, 6, 8].

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