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Coxeter groups, graphs and Ricci curvature

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Abstract. We consider a notion of curvature for graphs introduced in 1998 by Schmuckenschläger which is an analogue of Ricci curvature. First of all we see some general results on the discrete Ricci curvature of any locally finite graph. We then focus on graphs associated with Coxeter groups: Bruhat graphs, weak order graphs and Hasse diagrams of the Bruhat order. In particular we see that the discrete Ricci curvature for the Bruhat order is always 2 and that the discrete Ricci curvature of the weak order graph of a strictly linear Coxeter system (W, S) is $-2cos(\pi/|S|)$.

Keywords: Coxeter group, Bruhat order, discrete Ricci curvature

1 Introduction

In the last decades there have been various attempts of defining an analogue of Ricci curvature for graphs, see for example [10] and [3]. The one we will consider here was given in 1998 by Schmuckenschläger [11] and was inspired by a previous definition given in [2] of curvature for probability spaces. Both these notions of curvature are based on the following fact in Riemannian geometry. Given a Riemannian manifold (M, g) and L the Laplacian operator we define the following two functions on $C^\infty(M) \times C^\infty(M)$:

- $\Gamma(f,g) = \frac{1}{2}[L(fg) - L(f)g - L(g)f]$; (1)
- $\Gamma_2(f,g) = \frac{1}{2}[L(\Gamma(f,g)) - \Gamma(L(f),g) - \Gamma(L(g)f)]$. (2)

Then we define

$$R(L) := sup\{r|\Gamma_2(f,f)(x) \geq r\Gamma(f,f)(x), \forall f \in C^\infty(M), x \in M\}.$$

This value turns out to be the maximum lower bound for the Ricci curvature of M.

In 1985 Backry and Emery defined a Ricci curvature for probability spaces using this fact. They considered a probability space (Ω, μ) and L an operator on $L^2(\Omega)$ such that $L(fg)$ is defined for any $f, g \in L^2(\Omega)$. Through L they defined the two operators Γ and Γ_2 on $L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ using formulas (1) and (2). The Ricci curvature of L at $x \in \Omega$ is defined as:

$$R(L)(x) := sup\{r|\Gamma_2(f,f)(x) \geq r\Gamma(f,f)(x), \forall f\}.$$

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The same idea of Bakry and Emery is used by Schmuckenschläger for the definition of the discrete Ricci curvature which is presented in Section 3.

In 2015 Klartag et al. [8] studied some inequalities that hold for this discrete Ricci curvature and computed it for various graphs such as the hypercube, the complete graph K_n and some finite Cayley graphs. The computations here were carried in a direct fashion working on the explicit formulas for Γ and Γ_2 (see (2.4)).

In this extended abstract we present some general results about Ricci curvature of locally finite graphs. We then focus on the curvature of graphs associated to Coxeter groups. The main motivation for this is the lack of examples of these kind of computation, the hope is in the long term to develop an intuition on what kind of graphs have small or big values for such a curvature.

The main results we obtain are the value of the Ricci curvature of any finite Bruhat graph with a general proof, and the Ricci curvature of weak orders with a case-by-case proof. Furthermore, to study the curvature of Hasse diagrams of the Bruhat order of case B_n we see an analogue of a result from [1].

This work is divided in five sections. in Section 2 we introduce Coxeter groups and discrete Ricci curvature. First we recall some basic facts in Coxeter theory, in particular given a Coxeter group W we describe its Bruhat graph ($B(W)$), its weak order graph ($V(W)$) and its Hasse diagram associated to the Bruhat order ($H(W)$). We end with the definition of discrete Ricci curvature

In Section 3 we state some results about the computation of discrete Ricci curvature, the main one is Theorem 3.2). We end with some corollaries of Theorem 3.2 and together with some application.

In Section 4 we compute the Ricci curvature of Bruhat graphs of finite Coxeter groups and see that it is always 2.

In Section 5 we study the curvature of weak order graphs of finite and affine Coxeter groups.

In Section 6 we study the Ricci curvature of some Hasse diagrams associated to the Bruhat order.

2 Preliminaries

2.1 Orders on Coxeter groups

We recall Coxeter systems as pairs (W, S) where W is a group generated by the elements in S and $S = \{s\}_{i \in I}$ is a finite set with the following relations:

$$(s_i s_j)^{m_{ij}} = e.$$

with $m_{ii} = 1$ for all $i \in I$ and $m_{ij} \geq 2$ (including $m_{ij} = \infty$) for all $(i, j) \in I \times I$. The values m_{ij} are usually seen as the entries of a symmetric matrix called Coxeter matrix.

The group W so defined is called a Coxeter group, S is a minimal set of generators whose elements are called Coxeter generators. We define strictly linear Coxeter groups following [9]:

Definition 2.1. Given a Coxeter system $(W, \{s_1, \dots, s_n\})$, this is called strictly linear if:

- $m_{ij} \geq 3$ if $|i-j| = 1$;
- $m_{ij} = 2$ if $1 < |i-j| \leq n-1$.

Notice that these groups are the ones whose Coxeter graph is a path. We go on with some classical definitions in Coxeter theory. Given an element w in W, this can be written as a product of elements in S

$$w = s_1 \dots s_k.$$

If k is the minimal length of all the possible expressions for w, we say that k is the length of w and we write $l(w) = k$. We define in W the set of reflections as the union of all the conjugates of S,

$$T := \cup_{w \in W} wSw^{-1}.$$

The definitions of length and reflections allow us to define two partial orders on the set W:

Definition 2.2. Given $w \in W$ and $t \in T$, if $w' = tw$ and $l(w') \geq l(w)$ we write $w' \leftarrow w$. Given two elements $v, w \in W$ we say that $v \geq w$ according to the Bruhat order if there are $w_0 \dots w_k \in W$ such that

$$v = w_0 \leftarrow w_1 \dots w_{k-1} \leftarrow w_k = w.$$

This defines a partial order on W called the Bruhat order. Through this order two different graphs can be associated to a given Coxeter group. The first one is called the Bruhat graph (denoted $B(W)$): its set of vertices is W, two vertices v, w are connected by an edge if $w \leftarrow v$ holds. The second graph is the Hasse diagram associated to the Bruhat order on W, we denote it as $H(W)$. Again the set of vertices of $H(W)$ is W; two vertices are connected if one covers the other according to the Bruhat order. Saying that $'x$ covers y' we mean that $x > y$ and there are no elements z such that $x > z > y$.

The second order we define is the weak order on W:

Definition 2.3. Given $u, v \in W$, we say that: $u \leq_R w$ if $w = us_1 \dots s_k$, for some $s_i \in S$ such that $l(us_1 \dots s_i) = l(u) + i$ for any $0 \leq i \leq k$. This is the right weak order on W, analogously one defines the left weak order, multiplying the $s_1, \dots, s_k$ on the left. These two orders are isomorphic.

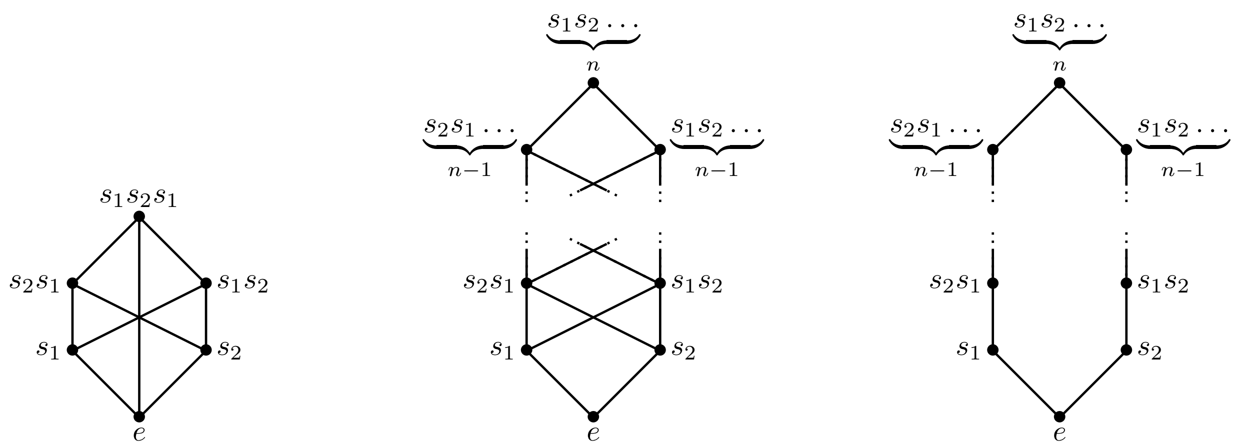

Figure 1: $B(I_2(3)), H(I_2(n)), V(I_2(n))$.

Given a Coxeter group (W, S) we denote by $V(W)$ the Hasse diagram associated to the weak right order. This has set of vertices W, two elements in the graph are connected by an (undirected) edge if one covers the other according to the weak order.

Example 2.4. Dihedral groups are among the easiest examples of Coxeter groups. The dihedral groups $I_2(n)$ is the one generated by two elements $S = \{s_1, s_2\}$ with the following relations:

$$(s_1)^2 = (s_2)^2 = e \quad (s_1s_2)^n = e.$$

Figure 1 shows a picture of the Bruhat graph of $I_2(3)$, the Hasse diagram associated to the Bruhat order of $I_2(n)$ and the weak order graph of $I_2(n)$.

We end this section recalling two results about Coxeter groups that will be useful in Sections 4 and 6. The first one is Theorem 3.1 from [1] which describes the maximal degree in $H(A_n)$:

Theorem 2.5. *For $n \geq 2$, the maximal degree of a vertex in the Hasse diagram of the strong Bruhat order on A_{n-1} is*

$$\lfloor \frac{n^2}{4} \rfloor + n - 2.$$

The second one is a result from [5] about dihedral subgroups of Coxeter groups:

Lemma 2.6. *Suppose t_1, t_2, t_3, t_4 are reflections in W and $t_1t_2 = t_3t_4 \neq 1$. Then $W' =< t_1, t_2, t_3, t_4 >$ is a dihedral reflection subgroup of (W, R).*

2.2 Discrete Ricci curvature

Let G be a graph, we denote by $\mathcal{V}(G)$ the set of vertices of G and by $\delta(x,y)$ the function $\delta : \mathcal{V}(G) \times \mathcal{V}(G) \to \mathbb{N}$ that gives the distance between two vertices. For $x \in \mathcal{V}(G)$ and $i \in \mathbb{N}$ we define the set:

$$B(i,x) := \{u \in \mathcal{V}(G) | \delta(x,u) = i\};$$

we denote by $d(x)$ the cardinality of $B(1,x)$ and call it the degree of x. We will only deal with locally finite graphs meaning that $d(x) < \infty$ for all $x \in \mathcal{V}(G)$.

Given f, g real functions on $\mathcal{V}(G)$ and $x \in \mathcal{V}(G)$ define the operator $L(f)(x) = \sum_{v \in B(1,x)} (f(v) - f(x))$. Following formulas (1) and (2), Γ and Γ_2 one defines:

- $\Gamma(f,g)(x) := \frac{1}{2} \sum_{v \in B(1,x)} (f(x) - f(v))(g(x) - g(v))$;
- $\Gamma_2(f,f)(x) := \frac{1}{2}[L(\Gamma(f,f))(x) - \Gamma(f, L(f))(x)]$.

Instead of $\Gamma(f,f)(x)$ (resp.$\Gamma_2(f,f)(x)$) we write $\Gamma(f)(x)$ (resp. $\Gamma_2(f)(x)$). Notice that all the sums are finite because we assume G to be locally finite.

We define the Ricci curvature of a graph following [11]:

Definition 2.7. The discrete Ricci curvature of a graph G, denoted $\mathrm{Ric}(G)$, is the maximum value $K \in \mathbb{R} \cup \{-\infty\}$ such that for any function f on $\mathcal{V}(G)$ and any vertex x, $\Gamma_2(f)(x) \geq K\Gamma(f)(x)$ holds.

Notice that $L(f), \Gamma(f)$ and $\Gamma_2(f)$ are unchanged by the addition of a constant to f, for this reason we always assume $f(x) = 0$ when computing $L(f)(x), \Gamma(f)(x)$ and $\Gamma_2(f)(x)$. Furthermore, whenever G has more than one vertex an easy computation shows that we can assume $\Gamma(f)(x) \neq 0$. This brings us to the following expressions for $\mathrm{Ric}(G), \Gamma(f)(x)$ and $\Gamma_2(f)(x)$ see also [8, Section 1.1]:

$$\mathrm{Ric}(G) = \inf_{x,f} \frac{\Gamma_2(f)(x)}{\Gamma(f)(x)}; \tag{2.1}$$

$$\Gamma(f)(x) = \frac{1}{2} \sum_{v \in B(1,x)} f(v)^2 \tag{2.2}$$

$$2\Gamma_2(f)(x) = \frac{1}{2} \sum_{u \in B(2,x)} \sum_{v \in B(1,u) \cap B(1,x)} (f(u) - 2f(v))^2 + \left(\sum_{v \in B(1,x)} f(v) \right)^2 + \tag{2.3}$$

$$+ \sum_{t(x,v,v')} \left(2(f(v) - f(v'))^2 + \frac{1}{2}(f(v)^2 + f(v')^2) \right) + \sum_{v \in B(1,x)} \frac{4 - d(x) + d(v)}{2} f(v)^2. \tag{2.4}$$

Where $t(x, v, v')$ denotes the triplets in $\mathcal{V}(G)$ that are vertices of a triangle in G. If a graph is triangle-free we have this corollary of [8, Theorem 1.2]:

Corollary 2.8. *If G is a triangle-free graph then* $\mathrm{Ric}(G) \leq 2$.

Sometimes we need to consider the curvature on a single vertex $x \in \mathcal{V}(G)$ this is

$$\mathrm{Ric}(G)_x = \inf_f \frac{\Gamma_2(f)(x)}{\Gamma(f)(x)}; \quad (*)$$

and is called local Ricci curvature at x.

Remark 2.9. Looking at the expressions for Γ and Γ_2 we notice that the local Ricci curvature only depends on the distance-two neighbourhoods of x. By distance-two neighbourhood we mean the induced subgraph of G whose vertices are the ones with $d(x,y) \leq 2$. This implies that if two vertices have isomorphic distance-two neighbourhoods then they have the same local Ricci curvature.

3 Ricci curvature of locally finite graphs

In this section we see some general result about the Ricci curvature of any locally finite graph. First we notice that graph automorphisms respect the Ricci curvature:

Lemma 3.1. *If G is any locally finite graph and χ is a graph automorphism then the following holds:*

$$Ric(G)_x = Ric(G)_{\chi(x)}.$$

Now we introduce our main result about the computation of discrete Ricci curvature:

Theorem 3.2. *Given G a locally finite graph, it is possible to associate to any of its vertices x a matrix M_x such that:*

$$\mathrm{Ric}(G)_x = min\{eigenvalues\ of\ M_x\};$$

Therefore

$$\mathrm{Ric}(G) = inf\{\lambda | \lambda \text{ is eigenvalue of some } M_x, x \in \mathcal{V}(G)\}.$$

Idea of the proof. First we describe the entries of the matrix associated to x. We denote the d elements of $B(1,x)$ as $v_1, \ldots, v_d$.

$$M_{ij}(x) = \begin{cases} \sum_{u \in \mathcal{U}_{v_i}} \frac{2(n_u-1)}{n_u} + 1 + \frac{4-d(x)-d(v_i)}{2} + \frac{3}{2} t_{v_i} & \text{if } i = j \\ \sum_{\mathcal{U}_{v_i} \cap \mathcal{U}_{v_j}} -\frac{2}{n_u} + 1 + 2T(v_i, v_j) & \text{if } i \neq j \end{cases}.$$

Where $\mathcal{U}_{v_i} := B_{2,x} \cap B_{1,v_i}$; t_{v_i} is the number of triangles containing vertices v_i and x. For any $u \in B(2,x)$ we denote by n_u the cardinality of $B(1,u) \cap B(1,x)$. The function $T : B(1,x) \times B(1,x) \Rightarrow \{0,1\}$ is defined as follows:

$$T(v_i, v_j) = \begin{cases} 1 & \text{if there is a triangle with vertices } x, v_i, v_j' \\ 0 & \text{otherwise} \end{cases}.$$

The proof is based on the following main points:

- Formula (*) for $\mathrm{Ric}(G)_x$ can be rewritten obtaining a formulation that depends only on the elements in $B(1,x)$ (and not on the elements in $B(2,x)$);
- such a formula expresses a quadratic form on a sphere in the real space of dimension $d(x) = |B(1,x)|$. The minimum of a quadric form on a sphere is equal to the minimum eigenvalue of a matrix.

□

We end this subsection with some corollaries and consequences of Theorem 3.2, this allows us to compute the discrete curvature of some Hasse diagrams (see Section 6).

First we consider the following Theorem which holds for triangle free graphs and gives an inequality for the discrete curvature:

Theorem 3.3. *Let G be a triangle free graph then the following inequality holds:*

$$\mathrm{Ric}(G) \geq 4 - \frac{3d(x) + d(y)}{2}. \tag{3.1}$$

where (x,y) *is a pair of connected vertices that maximizes* $3d(x) + d(y)$ *. If furthermore G has no length four cycles the following upper bound holds:*

$$\mathrm{Ric}(G) \leq min(2, \frac{2 + d(x') - d(y')}{2}).$$

with $d(x')$ $d(y')$ *are connected vertices that minimize* $d(x') - d(y')$.

Idea of the proof. Theorem 3.2 associates to any vertex of G a matrix. To give a bound on the curvature of G it is sufficient to bound the eigenvalues of the generic matrix associated to one of its vertices. We use some classical results of numerical analysis to obtain the statement. The restriction to graphs with no triangles nor quadrilaterals simplifies the description of the matrices. □

This result can be used to bound the Ricci curvature of any tree.

A corollary follows, this one gives the exact Ricci curvature on a subfamily of graphs with no triangles nor quadrilaterals.

Corollary 3.4. *Let G be a graph with no triangles and quadrilaterals and with the property that all the distance-two neighbourhoods are isomorphic. Then*

$$\mathrm{Ric}(G) = 2 - d$$

where d is the degree of any vertex in G.

Though the hypothesis is quite strong, this corollary shows that any integer number smaller than 2 is the curvature of a graph. In particular, any negative integer z is the curvature of the Cayley graph of the free group generated by $2 - z$ elements.

4 Discrete curvature of Bruhat graphs

In this section we study the Ricci curvature of the Bruhat graphs associated to Coxeter groups. Discrete Ricci curvature is defined only for locally finite graphs, therefore we will consider only finite Coxeter groups.

Theorem 4.1. *Given a finite Coxeter system, the discrete Ricci curvature of its Bruhat graph is* 2*.*

Idea of the proof. The proof relies on four main facts:

- By Lemma 3.1 it is sufficient to study the local Ricci curvature at one vertex, say the one associated to the trivial element.
- By Remark 2.9 we have to study the structure of the distance-two neighbourhood of e in $B(W)$.
- One can prove that such a distance-two neighbourhood can be written as union of Bruhat graphs of dihedral groups. This was achieved defining an equivalence relation on $B(2, e)$ and using Lemma 2.6.
- One can see that Ricci curvature of $B(I_2(m))$ is 2 for any $m \geq 3$.

□

The Ricci curvature for the Bruhat graph in type A_n have already been studied in [8, Section 2.6].

5 Discrete curvature of Weak orders

In this section we compute the Ricci curvature of weak orders associated to Coxeter groups. Unlike the case of Bruhat graphs the values of the curvatures are different, giving quite a different statement:

Theorem 5.1. *Given (W, S) an irreducible finite Coxeter system and $V(W)$ the weak order graph associated to it, the following holds:*

- $\mathrm{Ric}(V(W)) = -2cos(\frac{\pi}{|S|})$ *if W is a strictly linear Coxeter group;*
- $-4 \leq \mathrm{Ric}(V(W)) \leq -1$ *if $W = D_n$;*
- $\mathrm{Ric}(V(E_6)) \simeq -2.30$, $\mathrm{Ric}(V(E_7)) \simeq -2.33$ *and* $\mathrm{Ric}(V(E_8)) \simeq -2.34$.

Idea of proof. The proof relies on the following facts:

- We can apply Lemma 3.1 so we can consider again the local Ricci curvature of a single vertex in $V(W)$;
- We apply Theorem 3.2 to compute the local Ricci curvature at a given point. For all the exceptional groups we conclude by computing the smallest eigenvalue of the matrices so obtained;
- For A_n and B_n we have to study the eigenvalues of two families of matrices, namely $\{M_{A_n}\}_{n\geq 2}$ and $\{M_{B_n}\}_{n\geq 3}$. Both the families of matrices are tridiagonal with of dimension n. In [12] the eigenvalues of these matrices are described as functions of the dimension of the matrices. .
- For case D_n we obtained a family of matrices $\{M_{D_n}\}_{n\geq 3}$ but we did not find a formula in n for the eigenvalues. We obtained therefore an inequality using Gershgorin's Theorem (see [6]) which is a classical result to bound eigenvalues in numerical linear algebra .

□

For any finite Coxeter group the Ricci curvature for the weak order graph can be computed through this result:

Theorem 5.2. *Let (W, S) be any finite Coxeter group with $W = W_1 \times \ldots \times W_k$ and $S = S_1 \cup \ldots \cup S_k$ where (W_i, S_i) is an irreducible Coxeter group for all $i = 1, \ldots, k$. Let $V(W)$ be the weak order graph associated to (W, S), then:*

$$\mathrm{Ric}(V(W)) = min_{1\leq i\leq k}\,\mathrm{Ric}(V(W_i)).$$

In the last part of this subsection we apply the same procedure used to compute the Ricci curvature of weak orders to affine Weyl groups. As already noticed the generators of an affine Weyl group are finite, therefore the Hasse diagram associated to the weak order is a locally finite graph.

Theorem 5.3. *Let (W, S) be an affine Weyl group, $V(W)$ the Hasse graph of the weak order. Then we have*

- $\mathrm{Ric}(V(\tilde{A}_n)) = 2\cos(\frac{2\pi}{n}\lceil\frac{n}{2}\rceil)$;
- $\mathrm{Ric}_(\, V(\tilde{C}_n)) = 2cos(\frac{\pi}{n})$ *if* $W = \tilde{C}_n$;
- $-4 \leq \mathrm{Ric}(V(W)) \leq -1$ *if* $W = \tilde{B}_n, \tilde{D}_n$;
- $\mathrm{Ric}(V(\tilde{E}_6)) \sim -2.414$, $\mathrm{Ric}(V(\tilde{E}_7)) \sim -2.36$, $\mathrm{Ric}(V(\tilde{E}_8)) \sim -2.34$;
- $\mathrm{Ric}(V(\tilde{F}_4)) = -2cos(\frac{\pi}{5})$;
- $\mathrm{Ric}(V(\tilde{G}_2)) = -\sqrt{3}$;
- $\mathrm{Ric}(V(\tilde{A}_1)) = -\sqrt{2}$.

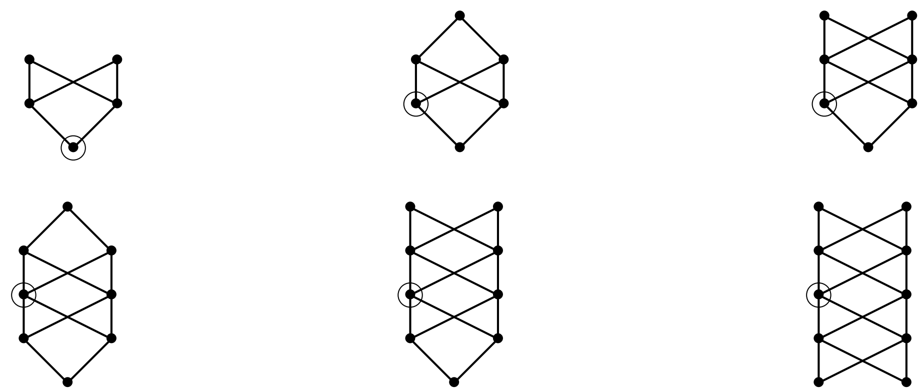

Figure 2: Possible distance-two neighbourhoods in the Hasse diagram of a dihedral group.

6 Discrete curvature of Hasse diagrams of the Bruhat order

We end this article with some results about the Hasse diagram of the Bruhat order of some Coxeter groups. We begin with dihedral groups.

Proposition 6.1. *The following values hold for the Ricci curvature of the Hasse diagram of the dihedral groups:*

- $\mathrm{Ric}(H(I_2(3))) = \frac{21-\sqrt{33}}{12}$,
- $\mathrm{Ric}(H(I_2(4))) = \frac{1}{2}$,
- $\mathrm{Ric}(H(I_2(5))) = \frac{5-\sqrt{17}}{2}$,
- $\mathrm{Ric}(H(I_2(n))) = 0$ *for any* $n > 5$.

Idea of the proof. This proposition is an application of Theorem 3.2. It is sufficient to notice that in $H(I_2(m))$ only six distance-two neighbourhoods appear up to isomorphism (see Figure 2). □

For the Hasse diagrams of A_n and B_n we have the following inequalities:

Proposition 6.2. *If $G = H(A_{n-1})$ then the following holds:*

$$\mathrm{Ric}(G) \geq -\lfloor \frac{n^2}{2} \rfloor - 2n + 8.$$

Proof. This is a consequence of the following corollary of Theorem 3.3:

$$\mathrm{Ric}(G) \geq 4 - 2d_{max}$$

where d_{max} is the maximal degree of the vertices in G. We applied this Theorem 2.5. □

Proposition 6.3. *If $G = H(B_n)$ then the following holds:*

$$\mathrm{Ric}(G) \geq 4(-2n+1) \; for x \leq 5;$$

$$\mathrm{Ric}(G) \geq -2\lfloor \frac{n^2}{2} \rfloor - 2n + 2.$$

To obtain this we followed the reasoning of Proposition 6.2. This used the following analogue of Theorem 2.5 that we proved for case B_n.

Theorem 6.4. *Let $G = H(B_n)$, then $d_{max} = 4(n-1)$ for $n \leq 5$ and $d_{max} = \lfloor \frac{n^2}{2} \rfloor + n - 1$ for $n > 5$. Where d_{max} is the maximal degree of the vertices in G.*

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