Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #1, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

The homology representation of subword order

Sheila Sundaram*[1]

[1] *Pierrepont School, One Sylvan Road North, Westport, CT, USA*

Abstract. We investigate the homology representation of the symmetric group on rank-selected subposets of subword order. We show that the homology module for words of bounded length, over an alphabet of size n, decomposes into a sum of tensor powers of the S_n-irreducible $S_{(n-1,1)}$ indexed by the partition $(n-1,1)$, recovering, as a special case, a theorem of Björner and Stanley for words of length at most k. For arbitrary ranks we show that the homology is an integer combination of positive tensor powers of the reflection representation $S_{(n-1,1)}$, and conjecture that this combination is nonnegative. We uncover a curious duality in homology in the case when one rank is deleted.

We prove that the action on the rank-selected chains of subword order is a nonnegative integer combination of tensor powers of $S_{(n-1,1)}$, and show that its Frobenius characteristic is h-positive and supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\}$.

Our most definitive result describes the Frobenius characteristic of the homology for an arbitrary set of ranks, plus or minus one copy of $S_{(n-1,1)}$, as an integer combination of the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. We conjecture that this combination is nonnegative, establishing this fact for particular cases.

Keywords: Subword order, reflection representation, h-positivity, Whitney homology, Kronecker product, internal product, Stirling numbers.

1 Introduction

Let A^* denote the free monoid of words of finite length in an alphabet A. Subword order is defined on A^* by setting $u \leq v$ if u is a subword of v, that is, the word u is obtained by deleting letters of the word v. This makes $(A^*, \leq)$ into a graded poset with rank function given by the length $|w|$ of a word w, the number of letters in w. The topology of this poset was first studied by Farmer (1979) and then by Björner, who showed in [4, Theorem 3] that any interval of this poset admits a dual CL-shelling. The intervals are thus homotopy Cohen–Macaulay, as well as all rank-selected subposets obtained by considering only words whose rank belongs to a finite set S [2, 5]. Suppose now that the alphabet A is finite, of cardinality n. The symmetric group S_n acts on A, and thus on A^*. To avoid trivialities we will assume $n \geq 2$.

*shsund@comcast.net.

In this paper we examine the homology representation of rank-selected subposets of A^*, using the Whitney homology technique and other methods developed in [11]. We refer the reader to [9] for general facts about rank-selection. Theorem 3.4 shows that for intervals $[r,k]$ of consecutive ranks in A^*, the unique nonvanishing homology decomposes as a direct sum of copies of r consecutive tensor powers of the reflection representation of S_n, that is, the irreducible representation $S_{(n-1,1)}$ indexed by the partition $(n-1,1)$. This generalises a result in [4], conjectured by Björner and proved by Stanley. We establish similar results for the Whitney and dual Whitney homology modules. Both turn out to be permutation modules in each degree, with pleasing orbit stabilisers. Theorem 4.1 establishes the nonnegative decomposition into tensor powers of $S_{(n-1,1)}$ for the case when one rank is deleted from the interval $[1,k]$, and leads to a curious homology isomorphism (Proposition 4.2), suggesting an equivariant homotopy equivalence between the simplicial complexes associated to the rank sets $[1,k]\backslash\{r\}$ and $[1,k]\backslash\{k-r\}$, for fixed $r, 1 \leq r \leq k-1$.

More generally, we show in Theorem 5.1 that for any nonempty subset T of ranks $[1,k]$, the homology representation of S_n may be written as an integer combination of positive tensor powers of the reflection representation. Based on our determination of this and other cases of rank-selection, we propose the following conjecture:

Conjecture 1.1. *Let A be an alphabet of size $n \geq 2$. Then the S_n-action on the homology of any finite nonempty rank-selected subposet of subword order on A^* is a nonnegative integer combination of positive tensor powers of the irreducible indexed by the partition $(n-1,1)$.*

These considerations lead us to examine the tensor powers of the reflection representation, and the question of how many tensor powers are linearly independent characters. In answering these questions, we are led to a decomposition (Theorem 7.2) showing that the kth tensor power of $S_{(n-1,1)}$ plus or minus one copy of $S_{(n-1,1)}$, has Frobenius characteristic equal to a nonnegative integer combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geq 2\}$. It is "almost" an h-positive permutation module. Inspired by this phenomenon, we prove, in Theorem 7.5, that in fact for each rank subset T and associated rank-selected subposet $A_{n,k}(T)$, the homology module $\tilde{H}(A_{n,k}(T))$ has the property that $\tilde{H}(A_{n,k}(T)) + (-1)^{|T|} S_{(n-1,1)}$ has Frobenius characteristic equal to an *integer* combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geq 2\}$. Theorem 1.4 establishes the truth of the following conjecture for all the rank-selected homology computed in this paper.

Conjecture 1.2. *Let A be an alphabet of size $n \geq 2$. Then the homology of any finite nonempty rank-selected subposet of subword order on A^*, plus or minus one copy of the reflection representation of S_n, is a permutation module. In fact its Frobenius characteristic is h-positive and supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.*

We give a simple criterion for when Conjecture 1.1 will imply Conjecture 1.2 in Lemma 7.3.

The main results of this paper are summarised below. Let A be an alphabet of size $n \geq 2$, and $T \subseteq [1,k]$ a subset of ranks. Let $A_{n,k}(T)$ denote the corresponding rank-selected subposet.

Theorem 1.3. *The action of S_n on the maximal chains of the rank-selected subposet $A_{n,k}(T)$ of A^* of words with lengths in T, is a nonnegative integer combination of tensor powers of the reflection representation $S_{(n-1,1)}$. If $|T| \geq 1$, this action has h-positive Frobenius characteristic supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\}$.*

Theorem 1.4. *The homology module $\tilde{H}(A_{n,k}(T))$ of words with lengths in T is an integer combination of positive tensor powers of the reflection representation $S_{(n-1,1)}$, with the property that $\tilde{H}(A_{n,k}(T)) + (-1)^{|T|} S_{(n-1,1)}$ has Frobenius characteristic equal to an integer combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geq 2\}$.*

Both integer combinations are nonnegative when T is one of the following rank sets:
(1) $[r,k], k \geq r \geq 1$; (2) $[1,k] \backslash \{r\}, k \geq r \geq 1$; (3) $\{1 \leq s_1 < s_2 \leq k\}$.

2 Subword order

The subword order poset A^* has a unique least element at rank 0, namely the empty word $\emptyset$ of length zero. In this section we collect the main facts on subword order from [4] that we will need. For general facts about posets, Möbius functions, etc. we refer the reader to [10].

Farmer computed the Möbius number of an arbitrary interval $(\hat{0}, \alpha)$ and also showed:

Theorem 2.1 ([6, Theorem 5 and preceding Remark])**.** *Let $|A| = n$ and let $A^*_{n,k}$ denote the subposet of A^* consisting of the first k nonzero ranks and the empty word, i.e. of words of length at most k, with an artificially appended top element $\hat{1}$. Then $\mu(A^*_{n,k}) = \mu(\hat{0}, \hat{1}) = (-1)^{k-1}(n-1)^k$ and $A^*_{n,k}$ has the homology of a wedge of $(n-1)^k$ spheres of dimension $(k-1)$.*

Björner generalised Farmer's computation of the Möbius function $\mu(\hat{0}, \alpha)$ to give a formula for the Möbius function of an arbitrary interval (β, α), and established the following generating functions. Recall that the zeta function [10] of a poset is defined by $\zeta(\beta, \alpha) = 1$ if $\beta \leq \alpha$, and equals zero otherwise.

Theorem 2.2 ([4])**.** *Let A be an alphabet of size n, and β a word in A^* of length k. The following generating functions hold:*

1. *[4, Theorem 2 (i)] For the Möbius function of subword order:*

$$\sum_{\alpha \in A^*} \mu(\beta, \alpha) t^{|\alpha|} = \frac{t^k (1-t)}{(1+(n-1)t)^{k+1}}.$$

2. *[4, 3. Remark (i)] For the zeta function of subword order:*

$$\sum_{\alpha \in A^*} \zeta(\beta, \alpha) t^{|\alpha|} = \frac{t^k}{(1-nt)(1-(n-1)t)^k}.$$

Farmer's result on the homology of $A^*_{n,k}$ was strengthened by Björner, who showed the following (see [2, 5]):

Theorem 2.3 (Björner [4, Theorem 3, Corollary 2])**.** *Every interval (β, α) in the subword order poset A^* is dual CL-shellable, and hence homotopy Cohen–Macaulay. In particular, for a finite alphabet A, the poset $A^*_{n,k}$ of nonempty words of length at most k, which may be viewed as the result of rank-selection from an appropriate interval of A^*, is also dual CL-shellable and hence also homotopy Cohen–Macaulay.*

It follows from the general theory of shellability [2, 5] that all rank-selected subposets of A^* are homotopy Cohen–Macaulay.

3 Rank-selection in A^*

In this section we will assume the alphabet A is finite of size n.

We follow the standard convention as in [9], [10]: By the homology of a poset P with greatest element $\hat{1}$ and least element $\hat{0}$, we mean the reduced homology $\tilde{H}(P)$ of the simplicial complex whose faces are the chains of $P \backslash \{\hat{0}, \hat{1}\}$. In order to determine the homology representation of S_n on rank-selected subposets of $A^*_{n,k}$, we will use the techniques developed in [11].

The Whitney homology of a poset was originally defined by Baclawski [1]. Björner showed [3] that the ith Whitney homology of a graded Cohen–Macaulay poset P with least element $\hat{0}$ is given by the isomorphism

$$WH_i(P) \simeq \bigoplus_{x:\text{rank}(x)=i} \tilde{H}_{i-2}(\hat{0}, x). \tag{3.1}$$

Note that if P has a top element $\hat{1}$, then the top Whitney homology coincides with the top homology of P. The present author observed that the isomorphism (3.1) is in fact group-equivariant, leading to a powerful technique, the equivariant acyclicity of Whitney homology, to determine group actions on the homology of posets [11].

It is also computationally useful to consider the *dual* Whitney homology of the Cohen–Macaulay poset P when P has a top element $\hat{1}$, that is, the Whitney homology of the dual poset P^*, which we denote by $WH^*(P)$. Note that we now have an equivariant isomorphism

$$WH^*_i(P) \simeq \bigoplus_{x:\text{rank}(x)=r-i} \tilde{H}_{i-2}(x, \hat{1}), 0 \le i \le r. \tag{3.2}$$

Here r is the length of the longest chain from $\hat{0}$ to $\hat{1}$.

Theorem 3.1 ([11, Lemma 1.1, Theorem 1.2, Proposition 1.9])**.** *Let P be a graded Cohen–Macaulay poset of rank r carrying an action of a group G. Then the unique nonvanishing top homology of P coincides with the top Whitney homology module $WH_r(P)$, and as a G-module, can be computed as an alternating sum of Whitney homology modules:*

$$\tilde{H}_{r-2}(P) \simeq \bigoplus_{i=0}^{r-1} (-1)^i WH_{r-1-i}(P). \tag{3.3}$$

In particular, if $P(\underline{k})$ denotes the subposet consisting of the first k nonzero ranks, with a bottom and top element attached, then one has the G-module decomposition

$$\tilde{H}_{k-2}(P(\underline{k-1})) \oplus \tilde{H}_{k-1}(P(\underline{k})) \simeq WH_k(P), r \geq k \geq 1. \tag{3.4}$$

Note that $WH_0(P)$ is the trivial G-module, while $WH_r(P)$ gives the reduced top homology of the poset P.

Denote by S_λ the irreducible representation of the symmetric group S_n indexed by the partition λ of n, and write $S_\lambda^{\otimes i}$ for the ith tensor power of the module S_λ.

The theorem below computes all but the top Whitney homology S_n-modules for subword order. The proof requires a key formula, which we derive from the generating function for the Möbius function of A^* given in Theorem 2.2.

Theorem 3.2. *Consider the subword order poset $A^*_{n,k}$, with $|A| = n$. As S_n-modules, the Whitney homology $WH(A^*_{n,k})$ and the dual Whitney homology $WH^*(A^*_{n,k})$, for $1 \leq i \leq k$, are as follows. Note that $WH_0(A^*_{n,k}) = S_{(n)} = WH^*_{k+1}(A^*_{n,k})$ (the trivial S_n-module).*

$$WH_i(A^*_{n,k}) = S^{\otimes i}_{(n-1,1)} \oplus S^{\otimes (i-1)}_{(n-1,1)}; \tag{3.5}$$

$$WH^*_{k+1-i}(A^*_{n,k}) = \binom{k}{i} S^{\otimes (k-i)}_{(n-1,1)} \otimes (S_{(n-1,1)} \oplus S_{(n)})^{\otimes i} = \bigoplus_{j=0}^{i} \binom{k}{i}\binom{i}{j} S^{\otimes j+(k-i)}_{(n-1,1)}. \tag{3.6}$$

Since the sum from (3.5) telescopes, acyclicity of Whitney homology, Theorem 3.1, immediately allows us to deduce the top homology:

Corollary 3.3 (Björner–Stanley [4, Theorem 4])**.** *The top homology of $A^*_{n,k}$ as an S_n-module is given by $S^{\otimes k}_{(n-1,1)}$.*

We can now state the main result of this section, which generalises the preceding corollary to the rank-set $[r,k]$ consisting of the interval of consecutive ranks $r, r+1, \ldots, k$. To do this, we must rewrite the partial alternating sums of terms appearing in the dual Whitney homology (3.6) as a nonnegative linear combination rather than a signed sum. The poset of words in an alphabet of size n, with lengths bounded above by k and below by r, has homology as follows.

Theorem 3.4. *Fix $k \geq 1$ and let T be the interval of consecutive ranks $[r,k]$ for $1 \leq r \leq k$. Then the rank-selected subposet $A^*_{n,k}(T)$ has unique nonvanishing homology in degree $k-r$, and the S_n-homology representation on $\tilde{H}_{k-r}(A^*_{n,k}(T))$ is given by the decomposition*

$$\bigoplus_{i=1+k-r}^{k} b_i\, S^{\otimes i}_{(n-1,1)}, \text{ where } b_i = \binom{k}{i}\binom{i-1}{k-r}, i = 1+k-r, \ldots, k. \tag{3.7}$$

The technically difficult proof of the preceding result, which we have omitted, also establishes the following combinatorial identity, which is instrumental in the proof of Theorem 1.4.

Corollary 3.5. *The alternating sum $\sum_{i=0}^{k+1-r}(-1)^i \dim WH^*_{k+1-(r+i)}(A^*_{n,k})$ equals*

$$\sum_{i=0}^{k-r}(-1)^i\binom{k}{r+i} n^{r+i}(n-1)^{k-(r+i)} + (-1)^{k+1-r} = \sum_{i=1+k-r}^{k}\binom{k}{i}\binom{i-1}{k-r}(n-1)^i.$$

4 A curious isomorphism of homology

In this section we will determine the homology representation of the rank-selected subposet $A^*_{n,k}(S)$ of $A^*_{n,k}$ when S is obtained by deleting one rank from the interval $[1,k]$. In this special case the computation will reveal a curious duality in homology.

We use a method developed in [11, Theorem 1.10] which is particularly useful for computation of the Lefschetz homology, i.e. the alternating sum of homology modules, when the deleted set is an antichain. Applying this technique to the poset $A^*_{n,k}$ and the rank-set $T = [1,k]\backslash\{r\}$, removing all words of length r, for a fixed r in $[1,k]$, we obtain:

Theorem 4.1. *As an S_n-module, we have*

$$\tilde{H}_{k-2}(A^*_{n,k}(T)) \simeq \left[\binom{k}{r} - 1\right] S^{\otimes k}_{(n-1,1)} \oplus \binom{k}{r} S^{\otimes k-1}_{(n-1,1)}.$$

In particular, Conjecture 1.1 is true for this case of rank-selection.

An immediate and intriguing corollary is the following.

Proposition 4.2. *Let $|A| = n$. Fix a rank $r \in [1, k-1]$. Then the homology modules of the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$ are S_n-isomorphic.*

It would be interesting to explain this isomorphism topologically. More precisely:

Question 4.3. Is there an S_n-homotopy equivalence between the simplicial complexes associated to the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$?

5 Chains and arbitrary rank-selected homology

Assume $|A| = n$. For a subset $S \subseteq [1,k]$, denote by $\alpha_n(S)$ the permutation module of S_n afforded by the maximal chains of the rank-selected subposet $A^*_{n,k}(S)$. In this section we derive a recurrence for the action, and hence an explicit formula. The proof uses Part (2) of Theorem 2.2.

Theorem 5.1. *For any subset $T \subseteq [1,k]$, the action of S_n on the maximal chains of the rank-selected subposet $A^*_{n,k}(T)$ is a nonnegative integer combination of tensor powers of the irreducible indexed by $(n-1,1)$. Hence the S_n-action on the homology of the rank-selected subposet $A^*_{n,k}(T), T \neq \emptyset$, is an integer combination of positive tensor powers of the irreducible indexed by $(n-1,1)$. The highest tensor power that can occur is the mth, where $m = \max(T)$.*

Thus Theorem 5.1 supports Conjecture 1.1.

It is worth pointing out the special case for the full poset $A^*_{n,k}$.

Theorem 5.2. *The action of S_n on the maximal chains of $A^*_{n,k}$ decomposes into the sum*

$$S_{(n)} \oplus \bigoplus_{j=1}^{k} c(k+1,j) S_{(n-1,1)}^{\otimes k+1-j},$$

where $c(k+1,j)$ is the number of permutations in S_{k+1} with exactly j cycles in its disjoint cycle decomposition.

6 The Whitney homology and the tensor powers $S_{(n-1,1)}^{\otimes k}$

In this section we explore the tensor powers $S_{(n-1,1)}^{\otimes k}$. We use symmetric functions to describe some of the results that follow. See [7] and [8, Chapter 7]. The homogeneous symmetric function h_n is the Frobenius characteristic, denoted ch, of the trivial representation of S_n. Also let $*$ denote the internal product on the ring of symmetric functions, so that the Frobenius characteristic of the inner tensor product, or Kronecker product, of two S_n-modules is the internal product of the two characteristics.

The following lemma is an easy exercise in permutation actions.

Lemma 6.1. *Let $V_{j,n}$ denote the permutation module obtained from the S_n-action on the cosets of the Young subgroup $S_1^j \times S_{n-j}$. Then the kth tensor power of the natural representation $V_{1,n}$ of S_n decomposes into a sum of $S(k,j)$ copies of $V_{j,n}$, where $S(k,j)$ is the Stirling number of the second kind:*

$$V_{1,n}^{\otimes k} = \sum_{j=1}^{\min(n,k)} S(k,j)\, V_{j,n}, \quad \text{and thus} \quad (h_1 h_{n-1})^{*k} = \sum_{j=1}^{\min(n,k)} S(k,j)\, h_1^j h_{n-j}. \tag{6.1}$$

Using the observation that the Frobenius characteristic of $(S_{(n-1,1)})^{\otimes k}$ is the k-fold internal product of $(h_1 h_{n-1} - h_n)$, and standard manipulations in the ring of symmetric functions, we obtain the following description of the top homology.

Theorem 6.2. *The top homology of $A^*_{n,k}$ has Frobenius characteristic*

$$\sum_{i=0}^{\min(n,k)} h_1^i h_{n-i} \left(\sum_{r=0}^{k-i} (-1)^r \binom{k}{r} S(k-r,i) \right).$$

Theorem 7.2 gives a different description of this module, from which it will be evident that the coefficients of $h_1^i h_{n-i}$ are positive for $i \geq 2$.

This gives the multiplicity of the trivial representation in the top homology of $A^*_{n,k}$:

Corollary 6.3. *Let $n \geq 2$. The multiplicity of the trivial representation in $S_{(n-1,1)}^{\otimes k}$ equals*

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{i=0}^{\min(n,k)} S(k-r,i).$$

When $n \geq k$, this is the number of set partitions of $\{1,\ldots,k\}$ with no singleton blocks.

7 Proofs of main results

In this section we indicate the main ideas leading to the proofs of Theorems 1.3 and 1.4 announced in the Introduction.

Theorem 1.3 is proved by an inductive argument using Theorem 5.1 and Lemma 6.1.

The proof of Theorem 1.4 relies on the precise homology computations of the preceding sections, as well as Theorem 7.2 and Lemma 7.3 below, which are crucial. We first derive a different expression for the Whitney homology modules of $A^*_{n,k}$, thereby obtaining a new expression for the top homology module as well.

Recall that in Theorem 3.2, the jth Whitney homology of $A^*_{n,k}$ $j \geq 2$, was determined as a sum of two consecutive tensor powers of $S_{(n-1,1)}$. We now have the following surprising result.

Proposition 7.1. *Each Whitney homology module of subword order, and hence the sum of two consecutive tensor powers of the reflection representation, has h-positive Frobenius characteristic, and in particular it is a permutation module. We have* $\mathrm{ch}\, WH_0 = h_n, \mathrm{ch}\, WH_1 = h_1 h_{n-1}$, *and for* $k \geq j \geq 2$, *the jth Whitney homology of* $A^*_{n,k}$ *has Frobenius characteristic*

$$(h_1 h_{n-1}) * s_{(n-1,1)}^{*(j-1)} = \sum_{d=2}^{j} S(j-1, d-1)\, h_1^d h_{n-d} = \sum_{d=2}^{j} S^*_{j,d}\, h_1^d h_{n-d} = h_1 (h_1 h_{n-2})^{*j-1}, \quad (7.1)$$

a permutation module with orbits whose stabilisers are Young subgroups indexed by partitions of the form $(n-d, 1^d), d \geq 0$.

Theorem 7.2. *Fix* $k \geq 1$. *The kth tensor power of the reflection representation* $S_{(n-1,1)}^{\otimes k}$, *i.e. the homology module* $\tilde{H}_{k-1}(A^*_{n,k})$, *has the following property:* $S_{(n-1,1)}^{\otimes k} \oplus (-1)^k S_{(n-1,1)}$ *is a permutation module* $U_{n,k}$ *whose Frobenius characteristic is h-positive, and is supported on the set* $\{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. *If* $k = 1$, *then* $U_{n,1} = 0$. *More precisely, the k-fold internal product* $s_{(n-1,1)}^{*k}$ *has the following expansion in the basis of symmetric functions* h_λ :

$$\sum_{d=0}^{n} g_n(k,d) h_1^d h_{n-d}, \text{ where} \quad (7.2)$$

$$g_n(k,0) = (-1)^k = -g_n(k,1), \text{ and } g_n(k,d) = \sum_{i=d}^{k} (-1)^{k-i} S(i-1, d-1) \text{ for } 2 \leq d \leq n.$$

Hence $s_{(n-1,1)}^{*k} = (-1)^{k-1} s_{(n-1,1)} + \mathrm{ch}\,(U_{n,k})$, *where* $\mathrm{ch}\,(U_{n,k}) = \sum_{d=2}^{n} g_n(k,d) h_1^d h_{n-d}$.

The integers $g_n(k,d)$ *are independent of n for* $k \leq n$, *nonnegative for* $2 \leq d \leq k$, *and* $g_n(k,d) = 0$ *if* $d > k$.

There follows a key lemma connecting Conjectures 1 and 2, namely:

Lemma 7.3. *Suppose V is an* S_n*-module which can be written as an integer combination* $V = \oplus_{k=1}^{m} c_k S_{(n-1,1)}^{\otimes k}$ *of positive tensor powers of* $S_{(n-1,1)}$. *If* $\sum_{k=1}^{m} (-1)^{k-1} c_k = 0$, *then the Frobenius characteristic of V is supported on the set* $\{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. *If in addition* $c_k \geq 0$ *for all* $k \geq 2$, *the Frobenius characteristic is h-positive and hence V is a permutation module.*

As an example of its application, we can deduce from this lemma and Theorem 3.2 that

Corollary 7.4. *The dual Whitney homology modules* $WH^*_{k+1-i}(A^*_{n,k}), 1 < i \leq k$, *are permutation modules whose Frobenius characteristic is a nonnegative integer combination of the set* $T_2 = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.

Applying Stanley's theory of rank-selection [9] in conjunction with Theorem 1.3 now establishes the result below.

Theorem 7.5. *Let $T \subseteq [1,k]$ be any nonempty subset of ranks in $A^*_{n,k}$. The following statements hold for the Frobenius characteristic $F_n(T)$ of the homology representation $\tilde{H}(A^*_{n,k}(T))$:*

1. *its expansion in the basis of homogeneous symmetric functions is an integer combination supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\}$.*
2. *$F_n(T) + (-1)^{|T|} s_{(n-1,1)}$ is supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.*

Theorem 1.4 describes the cases for which we are able to establish that the symmetric function $F_n(T) + (-1)^{|T|} s_{(n-1,1)}$ is in fact a *nonnegative* integer combination of $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.

8 Enumerative Consequences

We conclude by pointing out a representation-theoretic consequence, and some enumerative implications, of the expansion (7.2). Fix $n \geq 3$ and consider the n by $n-1$ matrix D_n whose kth column consists of the coefficients $g_n(n-k, n-d), d = 1, \ldots, n-1$. Thus the kth column contains the coefficients in the expansion of $S^{\otimes n-k}_{(n-1,1)}$ in the h-basis: we have $\mathrm{ch}\, S^{\otimes k}_{(n-1,1)} = \sum_{d=1}^{n} g_n(k, n-d) h_1^{n-d} h_d, 1 \leq k \leq n-1$. From Theorem 7.2 it is easy to see that the matrix D_n has rank $(n-1)$; the last two rows, consisting of alternating ± 1s, differ by a factor of (-1), and the matrix is lower triangular with 1's on the diagonal, hence it has rank $(n-1)$. Similarly the $(n+1)$ by $(n-1)$ matrix obtained by appending to D_n a first column consisting of the h-expansion of the nth tensor power of $S^{\otimes n}_{(n-1,1)}$ also has rank $(n-1)$.

Theorem 8.1. *In the representation ring of the symmetric group S_n, the first $(n-1)$ tensor powers of $S_{(n-1,1)}$ are an integral basis for the vector space spanned by the positive tensor powers. The nth tensor power of $S_{(n-1,1)}$ is an integer linear combination of the first $(n-1)$ tensor powers:*

$$S^{\otimes n}_{(n-1,1)} = \bigoplus_{k=1}^{n-1} a_k(n) S^{\otimes k}_{(n-1,1)}, \text{ with } a_{n-1}(n) = \binom{n-1}{2}.$$

Let $c(n,j)$ be the number of permutations in S_n with exactly j disjoint cycles. The coefficients $a_k(n)$ are determined by the polynomial $P(t) = t^n - \sum_{k=1}^{n-1} a_k(n) t^k$, defined by

$$P(t) = \tfrac{t+1}{t-(n-2)} \sum_{j=1}^{n} c(n,j) t^j (-1)^{n-j}. \tag{8.1}$$

The preceding result gives a recurrence for the coefficients $a_k(n)$; we have

$$\begin{aligned} a_{n-1}(n) &= \tbinom{n-1}{2}; \\ (n-2)a_j(n) - a_{j-1}(n) &= (-1)^{n-j}[c(n,j) - c(n,j-1)], 2 \le j \le n-1; \\ (n-2)a_1(n) &= c(n,1)(-1)^{n-1} \\ \implies a_1(n) &= \tfrac{(n-1)!}{n-2}(-1)^{n-1} = (-1)^{n-1}[(n-2)! + (n-3)!] \end{aligned}$$

Theorem 7.2 has the following interesting corollary.

Corollary 8.2. *Let $k \ge 2$.*

1. *For $\min(n,k) \ge d \ge 2$, the coefficient of $h_1^d h_{n-d}$ in $s_{(n-1,1)}^{*k} = \operatorname{ch} S_{(n-1,1)}^{\otimes k}$ is the nonnegative integer $g_n(k,d)$ given by the two equal expressions:*

$$\sum_{j=d}^{k} (-1)^{k-j} S(j-1, d-1) = \sum_{r=0}^{k-d} (-1)^r \binom{k}{k-r} S(k-r, d). \tag{8.2}$$

   In particular, when $n \ge k$, this multiplicity is independent of n.

2. *The positive integer $\beta_n(k) = \sum_{d=2}^{\min(n,k)} g_n(k,d)$ is the multiplicity of the trivial representation in $S_{(n-1,1)}^{\otimes k}$. When $n \ge k$, it equals the number of set partitions $B_k^{\ge 2}$ of the set $\{1, \ldots, k\}$ with no singleton blocks. We have $\beta_n(n+1) = B_{n+1}^{\ge 2} - 1$ and $\beta_n(n+2) = B_{n+2}^{\ge 2} - \binom{n+1}{2}$.*

Question 8.3. The identity (8.2) holds for all $d = 2, \ldots, k$. Is there a combinatorial explanation?

Question 8.4. For fixed k and n, what do the positive integers $g_n(k,d)$ count? Is there a combinatorial interpretation for $\beta_n(k) = \sum_{j=d}^{\min(n,k)} g_n(k,d)$, the multiplicity of the trivial representation in the top homology of $A_{n,k}^*$, in the nonstable case $k > n$? Recall that for $k \le n$ this is the number $B_k^{\ge 2}$ of set partitions of $[k]$ with no singleton blocks, and is sequence OEIS A000296.

Question 8.5. Recall that $a_{n-1}(n) = \binom{n-1}{2}$. Is there a combinatorial interpretation for the signed integers $a_i(n)$? There are many interpretations for $(-1)^{n-1}a_1(n) = (n-2)! + (n-3)!$, see OEIS A001048. For $n \ge 4$ it is the size of the largest conjugacy class in S_{n-1}. We were unable to find the other sequences $\{a_i(n)\}_{n \ge 3}$ in OEIS.

References

[1] K. BacLawski. "Whitney numbers of geometric lattices". *Advances in Math.* **16** (1975), pp. 125–138. DOI.

[2] A. Björner. "Shellable and Cohen–Macaulay partially ordered sets". *Trans. Amer. Math. Soc.* **260**.1 (1980), pp. 159–183. DOI.

[3] A. Björner. "On the homology of geometric lattices". *Algebra Universalis* **14**.1 (1982), pp. 107–128. DOI.

[4] A. Björner. "The Möbius function of subword order". *Invariant theory and tableaux (Minneapolis, MN, 1988)*. Vol. 19. IMA Vol. Math. Appl. Springer, New York, 1990, pp. 118–124.

[5] A. Björner and M. Wachs. "On lexicographically shellable posets". *Trans. Amer. Math. Soc.* **277**.1 (1983), pp. 323–341. DOI.

[6] F. D. Farmer. "Cellular homology for posets". *Math. Japon.* **23**.6 (1978/79), pp. 607–613.

[7] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second edition. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[8] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI.

[9] R. P. Stanley. "Some aspects of groups acting on finite posets". *J. Combin. Theory Ser. A* **32**.2 (1982), pp. 132–161. DOI.

[10] R. P. Stanley. *Enumerative combinatorics. Vol. 1*. Vol. 49. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. Cambridge University Press, Cambridge, 1997, pp. xii+325. DOI.

[11] S. Sundaram. "The homology representations of the symmetric group on Cohen–Macaulay subposets of the partition lattice". *Adv. Math.* **104**.2 (1994), pp. 225–296. DOI.