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Gröbner geometry of Schubert polynomials through ice

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Abstract. The geometric naturality of Schubert polynomials and their combinatorial pipe dream representations was established by Knutson and Miller (2005) via antidiagonal Gröbner degeneration of matrix Schubert varieties. We consider instead diagonal Gröbner degenerations. In this dual setting, Knutson, Miller, and Yong (2009) obtained alternative combinatorics for the class of "vexillary" matrix Schubert varieties. We initiate a study of general diagonal degenerations, relating them to a neglected formula of Lascoux (2002) in terms of the 6-vertex ice model (recently rediscovered by Lam, Lee, and Shimozono (2018) in the guise of "bumpless pipe dreams").

Keywords: Schubert polynomial, bumpless pipe dream, matrix Schubert variety

1 Introduction

Let $\mathcal{F}_n$ be the complex flag variety, the parameter space for complete flags of nested vector subspaces of $\mathbb{C}^n$. The Schubert cell decomposition of $\mathcal{F}_n$ yields a distinguished $\mathbb{Z}$-linear basis for the cohomology ring $H^\star(\mathcal{F}_n)$. On the other hand, Borel [3] presented this ring as $H^\star(\mathcal{F}_n) \cong \mathbb{Z}[x_1, \ldots, x_n]/I$, where I is the ideal generated by the nonconstant elementary symmetric polynomials.

It is natural to desire polynomial representatives for the Schubert basis with respect to this presentation. Building on work of Bernstein, Gelfand, and Gelfand [2], Lascoux and Schützenberger [19] introduced *Schubert polynomials*. These are combinatorially well-adapted coset representatives for images of Schubert cohomology classes under the Borel isomorphism. In fact, Lascoux and Schützenberger introduced more general *double Schubert polynomials* that represent Schubert classes in the T-equivariant cohomology of $\mathcal{F}_n$ (where $T \subset GL_n(\mathbb{C})$ is the group of invertible diagonal matrices).

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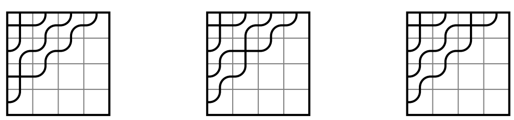
Since their introduction, (double) Schubert polynomials have become central objects in algebraic combinatorics. Knutson and Miller [13] gave a geometric justification for the naturality of Schubert polynomials by Gröbner degeneration of certain affine varieties. Moreover, they recovered aspects of the combinatorics of Schubert polynomials through this geometry, including identifying irreducible components of the degeneration with the *pipe dreams* of earlier combinatorial formulas [1, 6]. This explicit degeneration demonstrates the geometric naturality of pipe dream combinatorics.

Lascoux [18] introduced an alternate combinatorial model for (double) Schubert polynomials using states of the square-ice ("6-vertex") model from statistical physics. Recently, Lam, Lee, and Shimozono [17] rediscovered this Schubert polynomial model and gave a cleaner description in terms of *bumpless pipe dreams*. The connection between [17, 18] is detailed in [23].

Although both ordinary pipe dreams and bumpless pipe dreams compute the same double Schubert polynomials and appear superficially similar, they compute these polynomials in fundamentally different ways. In particular, (except in trivial cases) no weight-preserving bijection exists between these two sets. In light of this fact, the geometric content of bumpless pipe dreams and Lascoux's ice formula remains unclear.

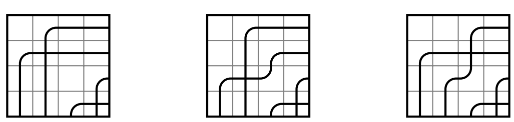
Example 1.1. Let w be the permutation $2143 \in S_4$. The three ordinary pipe dreams



for this permutation present the corresponding double Schubert polynomial as

$$\mathfrak{S}_w = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3).$$

There are also three bumpless pipe dreams



for w. These give a presentation of the same double Schubert polynomial as

$$\mathfrak{S}_w = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2).$$

Note that although these expressions are necessarily equal, this equality is only apparent after significant factoring and reorganizing. In particular, there is no weight-preserving way to match up the terms of the two summations. $\diamondsuit$

In Lie-theoretic terms, one may identify $\mathcal{F}_n$ with the homogeneous space $GL_n(\mathbb{C})/B$, where B denotes the Borel subgroup of invertible upper triangular matrices. Pulling back a Schubert cell in $\mathcal{F}_n$ to $GL_n(\mathbb{C})$, we may then consider its closure in the affine space of all $n \times n$ complex matrices. Fulton [7] showed that these *matrix Schubert varieties* are irreducible, gave set-theoretic defining equations for them, and showed that these equations define reduced schemes. The key observation of Knutson and Miller is that these *Fulton generators* form a Gröbner basis under any *antidiagonal* term order (that is, any term order under which the initial term of each minor of a generic matrix is the product of the entries along its main antidiagonal).

It is at least as natural to consider the dual notion of *diagonal* term orders (that is, term orders where initial terms of minors are products along main diagonals). For example, much of the commutative algebra literature on determinantal ideals and generalizations focuses on this case (e.g., [22, 8, 4]). Indeed, Knutson and Miller first tried unsuccessfully to carry out their program in this context before they realized that the antidiagonal term orders were more amenable to their approach.

The geometry of diagonal degenerations, in fact, is more complicated than the antidiagonal case. In general, the Fulton generators are *not* a Gröbner basis with respect to diagonal term orders. In [14], it was shown that Fulton generators are diagonal Gröbner exactly for the class of matrix Schubert varieties called *vexillary*. For general matrix Schubert varieties, the diagonal Gröbner degenerations can even fail to be reduced. Moreover, in the nonreduced case, different diagonal term orders can yield distinct scheme structures on the limiting space of the degeneration.

In this paper, an extended abstract of [9], we return to the diagonal setting. Despite the additional geometric complication, we propose that diagonal Gröbner degenerations naturally give rise to bumpless pipe dreams in an exactly analogous fashion to how antidiagonal degenerations yield ordinary pipe dreams. Our main conjecture is:

Conjecture 1.2. *Let* $\mathtt{in}(X_w)$ *be the Gröbner degeneration of a matrix Schubert variety with respect to any diagonal term order. The irreducible components of* $\mathtt{in}(X_w)$*, counted with multiplicities, naturally correspond to the bumpless pipe dreams for the permutation* w*.*

In particular, Conjecture 1.2 implies that, although different choices of diagonal term orders may yield degenerations to distinct schemes, the reduced irreducible components of the degeneration and their multiplicities do not depend on such a choice. The vexillary case of Conjecture 1.2 follows from [14] and results in [23]. Our main result is to prove Conjecture 1.2 for a larger class of permutations, called *banner permutations*, extending the vexillary case. For these permutations, we are able to exhibit explicit diagonal Gröbner bases by modifying the Fulton generators in an appropriate fashion.

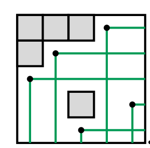
Theorem 1.3. *If* w *is a banner permutation, then the CDG generators for* X_w *are a diagonal Gröbner basis. The irreducible components of* $\mathtt{in}(X_w)$*, counted with multiplicities, naturally correspond to the bumpless pipe dreams for the permutation* w*. (Precise definitions of* banner permutations *and* CDG generators *appear in Section 3.)*

The recursive arguments in [13] rely on developing the combinatorics of a new *mitosis* recursion for ordinary pipe dreams. In contrast, bumpless pipe dreams appear well-adapted to the simpler and more classical *transition* formula of Lascoux and Schützenberger [20]. Our proof of Theorem 1.3 relies heavily on this latter recursion. Recently, Knutson [12] has developed a dual notion of *cotransition*, which can be used to simplify antidiagonal arguments of [13] in a similar fashion to the arguments here [21, Section 4].

We believe Theorem 1.3 holds in more generality than proved in this paper (see Conjecture 3.13) and we hope that Theorem 1.3 can be extended using similar techniques. However, we do not know a description of diagonal Gröbner bases in the most general case. Indeed, since different choices of diagonal term order can lead to different initial ideals, it is not guaranteed that there exists an explicit uniform description of Gröbner bases for all diagonal orders. Nonetheless, Conjecture 1.2 is supported by calculations in such cases. By computer, we have systematically verified Conjecture 1.2 through the symmetric group S_7 for one choice of diagonal term order, as well as in a variety of other experiments for larger permutations and for other diagonal term orders.

2 Background

The **Rothe diagram** of a permutation $w \in S_n$ is $D_w = \{(i,j) \in [n] \times [n] : w(i) > j, w^{-1}(j) > i\}$. We visualize D_w by placing $\bullet$ in $(i, w(i))$ for each $i \in [n]$, then drawing lines below and to the right of each $\bullet$. Then D_w is the complement of the marked boxes. For example, D_{42153} is $\{(1,1),(1,2),(1,3),(2,1),(4,3)\}$, which can be visualized as shown.



The **essential set** $\mathrm{Ess}(w)$ of w is the maximally southeast cells in each connected component of D_w. The ith **row** of D_w is $\{j \in [n] : (i,j) \in D_w\}$. The **Lehmer code** of w is $c(w) = (c_1, \ldots, c_n)$ where c_i is the cardinality of the ith row of D_w. We say w is **dominant** if $c(w)$ is nonincreasing; w is **vexillary** if its rows are totally ordered by inclusion. To each w, we associate a **rank function** $r_w : [n] \times [n] \to \mathbb{Z}$, where $r_w(i,j) = \#\{k \leq i : w(k) \leq j\}$. For $v, w \in S_n$, we say $v \leq w$ in **Bruhat order** if $r_v(i,j) \geq r_w(i,j)$ for all $i, j \in [n]$. We write $\lessdot$ for the covering relation in Bruhat order.

A **partial permutation** is a 0–1 matrix with at most one 1 in each row and column. The definitions of the above notions naturally extend to partial permutations. Let $M_{m,n}$ denote the set of $m \times n$ matrices over $\mathbb{C}$ and define $M_n := M_{n,n}$. An $m \times n$ partial permutation $w \in M_{m,n}$ can be (uniquely) completed to a permutation matrix $\widetilde{w} \in M_{\max\{m,n\}}$. This completion respects diagrams and essential sets.

Let $Z = (z_{ij})_{i \in [m], j \in [n]}$ be a matrix of distinct indeterminates and let $R = \mathbb{C}[Z]$. We identify $M_{m,n}$ with the mn-dimensional affine space $\operatorname{Spec} R$. For $A \in M_n$ and $I, J \subset [n]$, let $A_{I,J} = (a_{ij})_{i \in I, j \in J}$. Then the **matrix Schubert variety** for $w \in S_n$ is the affine variety $X_w = \left\{ A \in M_n : \mathtt{rank}(A_{[i],[j]}) \leq r_w(i,j) \text{ for all } i, j \in [n] \right\}$, and the *Schubert determinantal* *ideal* I_w is the prime ideal associated to X_w, so $X_w \cong \operatorname{Spec} R/I_w$ as reduced schemes. Fulton [7, Proposition 3.3] showed that I_w is generated by $r_w(i,j)+1$-size minors in $Z_{[i],[j]}$. In fact, he described a smaller generating set for I_w:

$$I_w = \left\langle (r_w(i,j)+1)\text{-size minors in } Z_{[i],[j]} : (i,j) \in \operatorname{Ess}(w) \right\rangle. \tag{2.1}$$

The minors in Equation (2.1) are called the **Fulton generators** of I_w.

For example, suppose $w = 42153$. Then the Fulton generators of I_w are

$$z_{11}, z_{12}, z_{13}, z_{21}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{41} & z_{42} & z_{43} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{31} & z_{32} & z_{33} \\ z_{41} & z_{42} & z_{43} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \\ z_{41} & z_{42} & z_{43} \end{vmatrix}. \tag{2.2}$$

Following Equation (2.1), we also define matrix Schubert varieties in $M_{m,n}$, indexed by partial permutations.

Following [17], a **bumpless pipe dream** is a tiling of the $n \times n$ grid with the six tiles

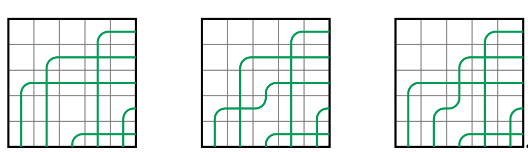
(2.3)

so that there are n pipes which start at the right edge of the grid, end at the bottom of the grid, and pairwise cross at most once.

If P is a bumpless pipe dream, we define a permutation w by setting $w(i)$ to be the column where pipe i exits (labeling rows top to bottom). Write $\mathsf{BPD}(w)$ for the set of bumpless pipe dreams for w. The **diagram** of P is $D(P) := \{(i,j) : (i,j) \text{ is a blank tile in } P\}$. Each bumpless pipe dream has weight $\operatorname{wt}(P) = \prod_{(i,j)\in D(P)} (x_i - y_j)$. The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ can be expressed as a sum over bumpless pipe dreams:

Theorem 2.1 ([17, Theorem 5.13])**.** $\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{P \in \mathsf{BPD}(w)} \operatorname{wt}(P)$.

We take this theorem to be the definition of the double Schubert polynomial; the single Schubert polynomial is obtained from this by setting all y variables to 0. For example, the bumpless pipe dreams for $w = 42153$ are



. (2.4)

Hence, $\mathfrak{S}_{42153}(\mathbf{x};\mathbf{y}) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)\Big((x_4-y_3)+(x_3-y_1)+(x_2-y_2)\Big)$.

We also need to consider bumpless pipe dreams for partial permutations. Let $w \in M_{m,n}$ be a partial permutation and $\widetilde{w}$ its completion to a permutation. We define

$\mathsf{BPD}(w) = \{P\ |_{m\times n} : P \in \mathsf{BPD}(\widetilde{w})\}$, where $P\ |_{m\times n}$ denotes the restriction of P to its first m rows and n columns.

(Double) Schubert polynomials satisfy a recurrence called **transition**. Let t_{ij} be the transposition $(i\ j) \in S_n$. For $v \in S_n$ and $r \in [n]$, we define $I(v,r) = \{i < r : v \lessdot vt_{ir}\}$ and $\Phi(v,r) = \{vt_{ir} : i \in I(v,r)\}$. An **inversion** in $w \in S_n$ is a pair (i,j) such that $i < j$ and $w(i) > w(j)$. **Lexicographic order** on inversions of w is given by $(i_1, j_1) > (i_2, j_2)$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_1 > j_2$.

Theorem 2.2 (Equivariant Transition, [16, Proposition 4.1])**.** *Let $w \in S_n$ with lexicographically largest inversion $(r, w^{-1}(s))$ and let $v := wt_{rw^{-1}(s)}$. Then $v \lessdot w$ and*

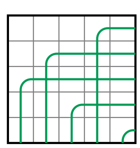
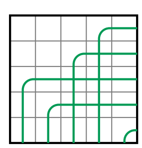
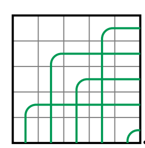
$$\mathfrak{S}_w = (x_r - y_s)\mathfrak{S}_v + \sum_{u\in\Phi(v,r)} \mathfrak{S}_u.$$

The combinatorics of bumpless pipe dreams is compatible with transition.

Lemma 2.3. *There is a bijection $\Psi : \mathsf{BPD}(v) \cup \bigcup_{u\in\Phi(v,r)} \mathsf{BPD}(u) \to \mathsf{BPD}(w)$ so that*

$$D(\Psi(P)) = \begin{cases} D(P) \cup \{(r,s)\} & \text{if } P \in \mathsf{BPD}(v) \text{ and} \\ D(P) & \text{otherwise.} \end{cases}$$

Continuing our running example $w = 42153$, the lexicographically largest inversion is $(r, w^{-1}(s)) = (4, 5)$, so we have $v = wt_{45} = 42135$. Since $\Phi(v, 4) = \{u^{(1)} = 43125, u^{(2)} = 42315\}$, Lemma 2.3 claims a bijection between $\mathsf{BPD}(w)$ and the unions of $\mathsf{BPD}(u^{(1)})$, $\mathsf{BPD}(u^{(2)})$, and $\mathsf{BPD}(v)$. Indeed, in this case, each of these three permutations $u^{(1)}, u^{(2)}, v$ is dominant and has a unique bumpless pipe dream:

v:  $u^{(1)}$:  $u^{(2)}$: . (2.5)

The diagram of the first bumpless pipe dream of (2.4) consists of the diagram of the bumpless pipe dream for v together with $(r,s) = (4,3)$. The diagram of the second bumpless pipe dream of (2.4) is that of $u^{(2)}$; the diagram of the third is that of $u^{(1)}$.

We use a diagrammatic interpretation of transition, described in [15, Section 2]. The **maximal corner** of w is the lexicographically maximal cell (r,s) in D_w. Amongst the •'s in D_w that are northwest of the maximal corner, we call the maximally southeast ones **pivots**. For (i,j) a pivot of w, the **marching operation** is a two-step procedure on D_w. First remove the lines emanating from the • at (i,j). Next, for every cell in D_w in the rectangle with corners (i,j) and (r,s), move that cell strictly to the northwest in the unique way such that each cell fills a position vacated either by the removed lines or by another cell. The resulting diagram is D_u for some $u \in S_n$, and we say $w \xrightarrow{i} u$.

Lemma 2.4 (Implicit in [15])**.** *If $w \in S_n$ with maximal corner (r,s) and $v = wt_{rw^{-1}(s)}$, then the pivots of w are $\{(i, w(i)) : i \in I(v,r)\}$ and $\Phi(v,r) = \{u^{(i)} : w \xrightarrow{i} u^{(i)}$ for $i \in I(v,r)\}$.*

Recall $R = \mathbb{C}[Z]$. A **monomial order** is a linear ordering on monomials in R such that, for any monomials m, n, and p, we have $\mathsf{m} < \mathsf{n}$ if and only if $\mathsf{mp} < \mathsf{np}$; and $\mathsf{m} \leq \mathsf{mp}$. An **(anti)diagonal** term order on R is a monomial order so that the initial term of any minor of Z is the product of the entries on its main (anti)diagonal. Fix a monomial order on R. Given $f \in R$ its **initial term** $\mathtt{in}(f)$ is the term whose monomial is largest with respect to the order. For a set of polynomials F, we define $\mathtt{in}(F) = \{\mathtt{in}(f) : f \in F\}$. If F is an ideal, then $\mathtt{in}(F)$ is called the **initial ideal** of F. If $X = \mathrm{Spec}(R/I)$, the **initial scheme** $\mathtt{in}(X)$ is $\mathrm{Spec}(R/\mathtt{in}(I))$. A **Gröbner basis** of an ideal I is a subset G such that $\langle \mathtt{in}(G) \rangle = \mathtt{in}(I)$. Every ideal $I \subseteq R$ admits a finite Gröbner basis; if G is a Gröbner basis for I, then $I = \langle G \rangle$.

We need some basic notions of equivariant cohomology. Consider the torus $T \subset GL_n(\mathbb{C})$ of invertible diagonal matrices and its Lie algebra $\mathfrak{t}$ of diagonal matrices. There is a natural left action of $T \times T$ on $\mathrm{Spec}\, R$ given by scaling rows and columns separately: $(t, \tau) \cdot M = tM\tau^{-1}$. Now, $\mathrm{Spec}\, R$ has a $(T \times T)$-equivariant cohomology ring $H_{T\times T}(\mathrm{Spec}\, R)$. Since $\mathrm{Spec}\, R$ is contractible, we have by definition that

$$H_{T\times T}(\mathrm{Spec}\, R) \cong H_{T\times T}(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{t} \otimes \mathfrak{t}) \cong \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Every setwise-stable subscheme $X \subseteq \mathrm{Spec}\, R$ has an equivariant class $[X]_{T\times T}$, which we may thereby identify with an integral polynomial in $2n$ variables. For $\mathcal{B} \subset [n] \times [n]$, let $C_{\mathcal{B}}$ be the coordinate subspace $\mathrm{Spec}(R/\langle z_{ij} : (i,j) \notin \mathcal{B} \rangle)$.

We need only the following three properties of equivariant classes in $H_{T\times T}(\mathrm{Spec}\, R)$:
Normalization: For any coordinate subspace $C_{\mathcal{B}}$, we have $[C_{\mathcal{B}}]_{T\times T} = \prod_{(i,j)\in\mathcal{B}}(x_i - y_j)$.
Additivity: For any $X \subseteq \mathrm{Spec}\, R$, $[X]_{T\times T} = \sum_j \mathrm{mult}_{X_j}(X)\, [X_j]_{T\times T}$, where the sum is over the top-dimensional components of X and $\mathrm{mult}_Y(X)$ denotes the multiplicity of X along the reduced irreducible variety Y. In particular, if $X = \bigcup_{i=1}^m X_i$, is a reduced scheme, then $[X]_{T\times T} = \sum_j [X_j]_{T\times T}$, summing over components X_j with $\dim X_j = \dim X$.
Degeneration: If $\mathtt{in}(X)$ is a Gröbner degeneration of X, then $[X]_{T\times T} = [\mathtt{in}(X)]_{T\times T}$.

For any X and any term order, $\mathtt{in}(X)$ is cut out of $\mathrm{Spec}\, R$ by a monomial ideal. Hence, $\mathtt{in}(X)$ is a (schemy) union of coordinate subspaces, and its equivariant class may be computed by Additivity and Normalization. Thus, the equivariant class of any $X \subseteq \mathrm{Spec}\, R$ may be computed from these properties, given enough information about $\mathtt{in}(X)$. One of the key results of [13] is the following.

Theorem 2.5 ([13, Theorem A])**.** *The matrix Schubert variety X_w satisfies*

$$[X_w]_{T\times T} = \mathfrak{S}_w(\mathbf{x}; \mathbf{y}).$$

3 CDG generators, skew sums, and monomial ideals

For λ an integer partition, let Z^λ be the matrix obtained from $Z = (z_{ij})$ by specializing z_{ij} to 0 for all (i,j) in the Young diagram of λ. The **dominant part** of D_w is $\mathrm{Dom}(w) = \{(i,j) \in D_w : r_w(i,j) = 0\}$. The cells of $\mathrm{Dom}(w)$ make up the Young diagram of a partition λ. Define $\mathrm{Ess}'(w) := \mathrm{Ess}(w) - \mathrm{Dom}(w)$. For example, with $w = 42153$ we have $\mathrm{Dom}(w) = \{(1,1),(1,2),(1,3),(2,1)\}$ = the partition $(3,1)$. Furthermore, $\mathrm{Ess}'(w) = \{(4,3)\}$ and

$$Z^{(3,1)}_{[4],[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \\ z_{41} & z_{42} & z_{43} \end{bmatrix}.$$

Let $G'_w = \bigcup_{(i,j)\in \mathrm{Ess}'(w)} \left\{ \text{minors of size } r_w(i,j)+1 \text{ in } Z^{\mathrm{Dom}(w)}_{[i],[j]} \right\}$. Then I_w is generated by $G_w = G'_w \cup \{z_{ij} : (i,j) \in \mathrm{Dom}(w)\}$. We call this set G_w of polynomials the **CDG generators** of I_w (after the authors of [5] who studied something similar). We are interested in when G_w is a diagonal Gröbner basis for I_w; in this case, we say that w and I_w are **CDG**. Note that if w is CDG, then $\mathtt{in}(I_w)$ is reduced, since the initial terms of the polynomials in G_w are all squarefree.

Example 3.1. Let $w = 42153$. Then the CDG generators of I_w are

$$z_{11}, z_{12}, z_{13}, z_{21}, z_{22}z_{33}z_{41} + z_{23}z_{31}z_{42} - z_{22}z_{31}z_{43} - z_{23}z_{32}z_{41}. \tag{3.1}$$

Note that this is a smaller set than the Fulton generators from (2.2). ♢

In S_4, all permutations are CDG; in S_5, 13254 and 21543 are the only permutations which are not CDG. Note that $\mathrm{Dom}(13254) = \emptyset$, so the CDG generators are simply the Fulton generators in this case.

Given two diagrams $D_1 \subseteq [a] \times [b]$ and $D_2 \subseteq [c] \times [d]$, let $\square = [c] \times [b]$ and define

$$D_1 \boxplus D_2 = \begin{array}{|c|c|} \hline \square & D_2 \\ \hline D_1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Lemma 3.2. *For u and v partial permutations, there exists $w \in S_\infty$ with $D(w) = D(u) \boxplus D(v)$.*

Given partial permutations u, v with n, m rows respectively, the **skew sum** $u \ominus v$ of u and v is the unique partial permutation with $n + m$ rows constructed as in Lemma 3.2. When u and v are permutations, $u \ominus v$ is the usual skew sum.

Lemma 3.3. *For u, v partial permutations and $w = u \ominus v$, there is a bijection from* $\mathsf{BPD}(u) \times \mathsf{BPD}(v)$ *to* $\mathsf{BPD}(w)$ *mapping the pair (B_u, B_v) to B_w satisfying $D(B_w) = D(B_u) \boxplus D(B_v)$.*

For z_{ij} an indeterminate, let

$$\downarrow_a(z_{ij}) := \begin{cases} z_{i+a\ j} & \text{if } i+a \leq m \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \rightarrow_b(z_{ij}) := \begin{cases} z_{i\ j+b} & \text{if } j+b \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Extend these operators to act indeterminate-by-indeterminate on monomials, linearly on polynomials and pointwise on sets of polynomials. From now on, all term orders are assumed to be diagonal, unless otherwise specified.

Lemma 3.4. *Let u and v be partial permutations such that u has b columns and v has a rows. If F_u and F_v are Gröbner bases of the Schubert determinantal ideals I_u and I_v, then $\downarrow_a(F_u) \cup \rightarrow_b(F_v) \cup \{z_{ij} : 1 \leq i \leq a,\ 1 \leq j \leq b\}$ is a Gröbner basis for $I_{u \ominus v}$.*

Corollary 3.5. *Let u and v be CDG partial permutations. Then $u \ominus v$ is CDG.*

We say that a partial permutation w is **predominant** if there is a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ so that $c(w) = (\lambda_1, \dots, \lambda_k, 0, \dots, 0, \ell, 0, \dots) = \lambda 0^h \ell$, for some $h, k, \ell \in \mathbb{Z}_{\geq 0}$. Note that we allow h to be zero, so ℓ can immediately follow λ_k, even if $\lambda_k < \ell$. We say a partial permutation is **copredominant** if it is the transpose of a predominant permutation. A partial permutation w is **block predominant** if it is a finite skew sum of predominant partial permutations. A predominant permutation is **indecomposable** if it cannot be written as the permutation associated to a skew sum of predominant partial permutations. A predominant partial permutation is **indecomposable** if its associated permutation is indecomposable. Note that only identity permutations are simultaneously dominant and indecomposable. The class of block predominant permutations is closed under transition:

Corollary 3.6. *Let π be block predominant with maximal corner (r,s) and $\Phi(\pi t_{r\pi^{-1}(s)}, r) = \{\tau^{(1)}, \dots, \tau^{(m)}\}$. Then each $\tau^{(i)}$ is block predominant.*

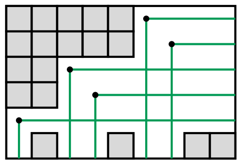
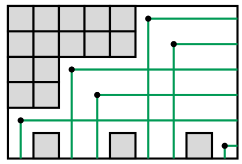
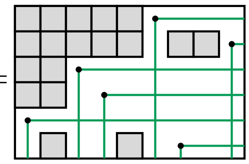
We define the monomial ideal $J_w = \langle \mathtt{in}(g) : g \in G_w \rangle$. Note that, by definition, G_w is a Gröbner basis for I_w if and only if $J_w = \mathtt{in}(I_w)$. Similarly, let J_{ij}^{λ} be the ideal generated by initial terms of non-zero maximal minors in the matrix $Z_{[i],[j]}^{\lambda}$. By construction,

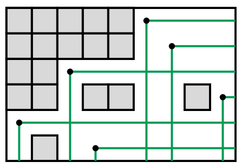
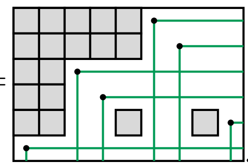
$$J_w = \langle z_{ij} : (i,j) \in \mathrm{Dom}(w) \rangle + \sum_{(i,j) \in \mathrm{Ess}'(w)} J_{ij}^{\mathrm{Dom}(w)}. \tag{3.2}$$

We will show that transition gives a recurrence on the monomial ideals J_w for block predominant permutations. Consulting Example 3.8 may help clarify this result.

Theorem 3.7. *If w is a block predominant permutation, then $J_w = \left(J_v + \langle z_{rs} \rangle\right) \cap \left(\bigcap J_{u^{(i)}}\right)$.*

Example 3.8. Let $w = 67341\ 10\ 2589$. Transition gives $v = 673419258$ and $\Phi(w, 6) = \{u^{(1)} = 693417258, u^{(2)} = 673914258, u^{(3)} = 673491258\}$ with diagrams

$D_w =$  $D_v =$  $D_{u^{(1)}} =$ 

$D_{u^{(2)}} =$  $D_{u^{(3)}} =$ .

The dominant component in each is $\lambda = (5^2, 2^2, 0^1)$ except that $\mathrm{Dom}(u^{(3)}) = \lambda' = (5^2, 2^3)$. The monomials coming from minors corresponding to the essential boxes $(6,2)$ and $(6,5)$ are in J_v, as do those corresponding to $(6,9)$ that do not involve z_{69}. A lemma guarantees that $J_v \subseteq J_w, J_{u^{(1)}}, J_{u^{(2)}}, J_{u^{(3)}}$. Therefore, Theorem 3.7 follows in this case by showing $J_w/J_v = \left(\langle z_{69}\rangle \cap J_{u^{(1)}} \cap J_{u^{(2)}} \cap J_{u^{(3)}}\right)/J_v$. Recall J^λ_{ij} is the ideal generated by initial terms of non-zero maximal minors in the matrix $Z^\lambda_{[i],[j]}$.

A direct argument shows $J_w/J_v = z_{69}J^\lambda_{58}/J_v$ and $J_{u^{(1)}}/J_v = J^\lambda_{28}/J_v$. We claim $(J_{u^{(1)}} \cap J_{u^{(2)}})/J_v = J^\lambda_{48}$. Observe that $J^\lambda_{48} \subseteq J_{u^{(2)}}$ by Equation (3.2). The opposite containment follows since the minors coming from $(4,5) \in \mathrm{Ess}(u^{(2)})$ correspond to J^λ_{45}, while $(J^\lambda_{45} \cap J^\lambda_{28}) \subseteq J^\lambda_{48}$. Next, we show $(J_{u^{(1)}} \cap J_{u^{(2)}} \cap J_{u^{(3)}})/J_v = J^\lambda_{58}/J_v$. To show the forward containment, we study $(5,2), (5,5), (5,8) \in \mathrm{Ess}(u^{(3)})$ individually. We have $J^\lambda_{52} \cap J^\lambda_{48}, J^{\lambda'}_{55} \cap J^\lambda_{28}, J^{\lambda'}_{58} \subseteq J^\lambda_{58}$. Moreover, the only monomials generating J^λ_{58} not found in $J^{\lambda'}_{58}$ are those involving z_{51} and z_{52}, so we see $J^\lambda_{58} \subseteq J^{\lambda'}_{58} + J^\lambda_{52} \cap J^\lambda_{48}$. ♢

Given a bumpless pipe dream P, write $\mathcal{L}_P = \langle z_{ij} : (i,j) \in D(P)\rangle$.

Proposition 3.9. *Suppose w is a block predominant permutation. Then the CDG generators are a diagonal Gröbner basis for I_w, and* $\mathtt{in}(I_w) = \bigcap_{P \in \mathsf{BPD}(w)} \mathcal{L}_P$.

A similar fact is true for vexillary permutations.

Proposition 3.10. *If w is vexillary, then w is CDG and* $\mathtt{in}(I_w) = \bigcap_{P \in \mathsf{BPD}(w)} \mathcal{L}_P$.

We say a permutation is **banner** if it is a skew sum of predominant, copredominant, and vexillary partial permutations. The following theorem is our main result.

Theorem 3.11. *Let w be a banner permutation. Then w is CDG, and* $\mathtt{in}(I_w) = \bigcap_{P \in \mathsf{BPD}(w)} \mathcal{L}_P$. *In particular,* $\mathtt{in}(I_w)$ *is radical.*

Theorem 3.11 follows from Propositions 3.9 and 3.10 together with the following.

Proposition 3.12. *If $w = u^{(1)} \ominus \cdots \ominus u^{(k)}$ is a skew sum of partial permutations and $\mathtt{in}(I_{u^{(i)}}) = \bigcap_{P \in \mathsf{BPD}(u^{(i)})} \mathcal{L}_P$ for all $i \in [k]$, then $\mathtt{in}(I_w) = \bigcap_{P \in \mathsf{BPD}(w)} \mathcal{L}_P$, and so $\mathtt{in}(I_w)$ is radical.*

Theorem 3.11 is a special case of Conjecture 1.2, and provides further evidence for the general statement of the conjecture. Theorem 3.11 does not exhaust the set of all CDG permutations. For example, $w = 25143$ is not banner, but one can compute that its CDG generators are a diagonal Gröbner basis for I_w. We conjecture the following characterization of CDG permutations.

Conjecture 3.13. *Let w be a permutation. The CDG generators are a diagonal Gröbner basis for I_w if and only if w avoids all eight of the patterns* 13254, 21543, 214635, 215364, 241635, 315264, 215634, 4261735*. In particular, this holds if $\mathfrak{S}_w(\mathbf{x})$ is a multiplicity-free sum of monomials.*

Conjecture 3.13 has now been proven by Klein [11]. If Conjecture 1.2 holds, then as a consequence, Spec $R/\mathtt{in}(I_w)$ is reduced if and only if each $P \in \mathsf{BPD}(w)$ has a distinct diagram. Data suggests that this condition is also governed by pattern containment [10].

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