Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #12, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Equidistributions of mesh patterns of length two

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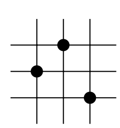
Abstract. A systematic study of *avoidance* of mesh patterns of length 2 was conducted by Hilmarsson *et al.* in 2015. In a recent paper Kitaev and Zhang examined the distribution of the aforementioned patterns. The aim of this paper is to prove more equidistributions of mesh pattern and confirm Kitaev and Zhang's four conjectures by constructing two involutions on permutations.

Keywords: permutation, mesh pattern, distribution, involution, continued fraction, antirecord

1 Introduction

Patterns in permutations and words have implicitly appeared in the mathematics literature for over a century, but interest in them has blown up in the past four decades (see [4, 6, 7, 10, 11, 14, 17] and references therein), and the research of this area continues to increase gradually.

A permutation $\sigma = \sigma(1)\cdots\sigma(n)$ of length n is an arrangement of $1\cdots n$. If π and σ are two permutations represented in this way, then π is said to contain σ as a pattern if some subsequence of the digits of π has the same relative order as all of the digits of σ. For example, the permutation 31542 contains two occurrences of the pattern 231 as the two subwords 352 and 342 all have the same ordering as 231. Let S_n be the set of all permutations of length n. We call $\sigma(i)$ the value of σ at position i $(1 \le i \le n)$ and draw a graphical presentation of σ by putting a dot at $(i, \sigma(i)$ for $i = 1, \ldots, n$ in the plan. For example, the permutation $231 \in S_3$ is presented as follows,



where the horizontal lines represent the values and the vertical lines denote the positions in the permutation. In [4] Brändén and Claesson introduced and studied *mesh patterns* as a common extension of several types of generalized permutation patterns.

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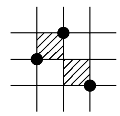
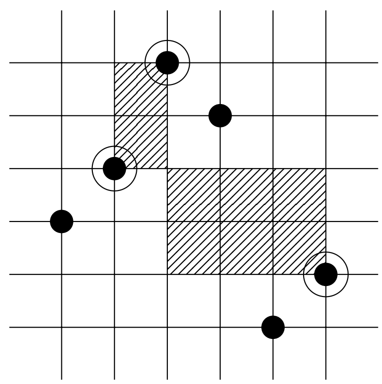
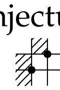
A *mesh pattern* of length k is a pair (τ, R), where τ is a permutation of length k and R is a subset of $[\![0,k]\!] \times [\![0,k]\!]$ with $[\![0,k]\!] = \{0,1,\dots,k\}$. Let (i,j) denote the box whose corners have coordinates $(i,j),(i,j+1),(i+1,j+1)$, and $(i+1,j)$. Mesh patterns can be depicted by shading the boxes in R. A mesh pattern with $\tau = 231$ and $R = \{(1,2),(2,1)\}$ is drawn as follows.



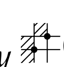
For example, the permutation 346512 depicted in the following picture contains the mesh pattern $(231, \{(1,2),(2,1)\})$ since the subsequence 462 forms the classical pattern 231 and there are no points in the shaded areas.

$346512 =$ 

Mesh patterns and their generalizations were studied in many papers; e.g. see [1, 2, 3, 8, 9, 10, 12, 13, 14, 18]. In the first systematic study of the mesh patterns avoidance, Hilmarsson et al. [10] solved 25 out of 65 non-equivalent *avoidance* cases of patterns of length 2. In a recent paper [14], Kitaev and Zhang further studied the distributions of mesh patterns considered in [10] by giving 27 distribution results, see [14, Table 1]. Moreover, for the unsolved case, they gave an equidistribution result and conjectured 6 more equidistributions (see Table 1). We prove 3 conjectured equidistributions and 2 more equidistribuions (see Table 2) by constructing two involutions. This extended abstract is a summary of the recent paper [9].

For a pattern p and a permutation π, we let $p(\pi)$ denote the number of occurrences of p in π. Kitaev and Zhang [14, Conjecture 6.1] conjectured a Stieltjes continued fraction formula for the distribution of pattern Nr. 3 =  (see [15, *A*200545]), which is equivalent to the following identity.

Conjecture 1.1 ([14, Conjecture 6.1])**.** *We have*



$$\sum_{n\geq 0} t^n \sum_{\pi\in S_n} y^{p(\pi)} = \cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\cdots}}} \tag{1.1a}$$

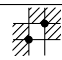
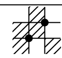
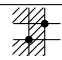
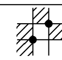
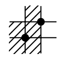
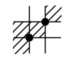
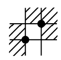
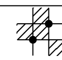
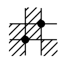
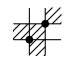
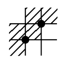
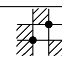
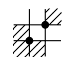
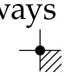
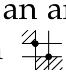
	Nr.	Repr. p	Ref.	Nr.	Repr. p	Ref.
proved equidistributions	48		[14, Theorem 5.1]			
	49					
conjectured equidistributions in [14]	23		Theorem 1.6	53		Corollary 1.1
	24			54		
	48		Corollary 1.1 and [14, Theorem 5.1]	57		N/A
	49			58		
	50			61		N/A
				62		

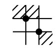
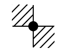
Table 1: Equidistributions for which enumeration is unknown. Pattern's numbers are adopted from [10, 14]

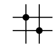
with coefficients

$$\alpha_{2k-1} = k, \quad \alpha_{2k} = y + k - 1. \tag{1.1b}$$

Presenting their conjecture in this way, we notice that the S-continued fraction (1.1a) appears in a recent paper of Sokal and Zeng [16]. Let us reformulate the relevant permutation statistics in [16] in terms of mesh patterns. Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called

- an *excedance* if $\pi(i) > i$;
- a *record* (rec) (or *left-to-right maximum*) if $\pi(j) < \pi(i)$ for all $j < i$ [note in particular that the index 1 is always a record and that the value n is always a record]; in other words, a record of π is one occurrence of pattern  of π;
- an *antirecord* (arec) (or *right-to-left minimum*) if $\pi(j) > \pi(i)$ for all $j > i$ [note in particular that the index n is always an antirecord and that the value 1 is always an antirecord]; in other words, an antirecord of π is one occurrence of pattern  of π;
- an *exclusive record* (erec) if it is a record and not also an antirecord; in other words, an exclusive record of π is one occurrence of pattern  of π, see (1.9);

- an *exclusive antirecord* (earec) if it is an antirecord and not also a record; in other words, an exclusive antirecord of π is one occurrence of pattern  of π, see (1.8);
- a *record-antirecord* (rar) (or *pivot*) if it is both a record and an antirecord; in other words, a record-antirecord of π is one occurrence of pattern  of π.

An *inversion* of a permutation $\pi \in S_n$ is a pair $(i,j) \in [n] \times [n]$ such that $i < j$ and $\pi(i) > \pi(j)$, in other words, an inversion of π is one occurrence of pattern  of π. We denote the number of excedances, records, antirecords, exclusive records, exclusive antirecords, record-antirecords and inversions in π by $\mathrm{exc}(\pi)$, $\mathrm{rec}(\pi)$, $\mathrm{arec}(\pi)$, $\mathrm{erec}(\pi)$, $\mathrm{earec}(\pi)$, $\mathrm{rar}(\pi)$ and $\mathrm{inv}(\pi)$ respectively.

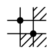
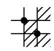
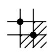
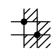
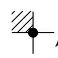
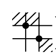
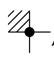
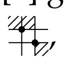
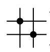
	Nr.	Repr. p	Ref.	Nr.	Repr. p	Ref.
proved	1^*		Theorem 1.6	3^*		Theorem 1.6
equidistributions	2^*			4^*		

Table 2: More proved equidistributions. Pattern's numbers are not considered in [10, 14]

Dumont and Kreweras [5] gave the joint distribution of (, ), Zeng [19] gave the joint distribution of (, , ). Recently Sokal and Zeng [16] proved much more general results. For example, define the generating function of the generalized Eulerian polynomials

$$F(x,y,z,v,q;t) = \sum_{n=0}^{\infty} t^n \sum_{\sigma \in S_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} z^{\mathrm{rar}(\sigma)} v^{\mathrm{exc}(\sigma)} q^{\mathrm{inv}(\sigma)}. \tag{1.2}$$

From [16, Theorems 2.7 and 2.8] we derive the following result.

Theorem 1.2. *We have*

$$F(x,y,z,v,q;t) = \frac{F(x,y,1,v,q;t)}{1 + x(1-z)tF(x,y,1,v,q;t)}, \tag{1.3a}$$

where

$$F(x,y,1,v,q;t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \tag{1.3b}$$

with coefficients

$$\alpha_{2k-1} = q^{k-1}(x + q + q^2 + \cdots + q^{k-1}) \tag{1.3c}$$
$$\alpha_{2k} = q^k v(y + q + q^2 + \cdots + q^{k-1}). \tag{1.3d}$$

Remark 1.3.

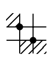
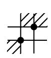
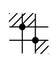
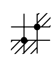
- *We can also prove* (1.3a) *by following the same steps in the special case as in [14] and then derive* (1.3b) *directly from [16, Theorem 2.8].*
- *The case* $x = y = v = q = 1$ *of Theorem* 1.2 *is Theorem 1.1 in [14].*
- *Since* $(\mathrm{arec}, \mathrm{inv})\pi = (\mathrm{rec}, \mathrm{inv})\pi^{-1}$ *we derive from [19] that*

$$F(x, 1, 1, 1, q; t) = \sum_{n=0}^{\infty} x(x+q)\ldots(x+q+\cdots q^{n-1})t^n. \tag{1.4}$$

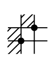
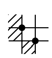
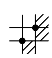
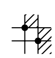
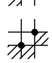
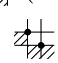
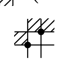
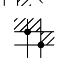
For $\pi = \pi(1)\ldots\pi(n) \in S_n$ we define the following three associated permutations:

$$\pi^{-1} := \pi^{-1}(1)\pi^{-1}(2)\cdots\pi^{-1}(n) \tag{1.5}$$
$$\pi^{\mathrm{r}} := \pi(n)\cdots\pi(2)\pi(1) \tag{1.6}$$
$$\pi^{c} := (n+1-\pi(1))(n+1-\pi(2))\cdots(n+1-\pi(n)) \tag{1.7}$$

Obviously we have

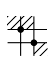
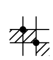
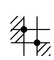
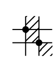
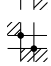
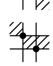
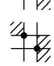
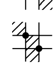
 $(\pi) =$  $(\pi^c) =$  $(\pi^{r\circ c}) =$  (π^r)

and

 $(\pi) =$  $(\pi^c) =$  $(\pi^{r\circ c}) =$  (π^r)
$=$  $(\pi^{-1}) =$  $(\tau^r) =$  $(\tau^{r\circ c}) =$  (τ^c)

with $\tau = \pi^{-1}$.

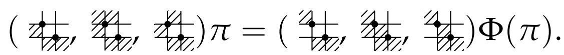
Lemma 1.1. *For* $\pi \in S_n$*, we have*

$\mathrm{earec}(\pi) =$  $(\pi) =$  $(\pi) =$  $(\pi) =$  $(\pi),$ (1.8)
$\mathrm{erec}(\pi) =$  $(\pi) =$  $(\pi) =$  $(\pi) =$  $(\pi).$ (1.9)

Proof. We just prove (1.8) as the proof of (1.9) is similar. In the entries placement representation of a permutation $\pi \in S_n$ the entry $y = (i, \pi(i))$ is an exclusive antirecord iff there is another entry $x = (j, \pi(j))$ at left of y, i.e., $j < i$ and higher than x, i.e., $\pi(j) > \pi(i)$. Hence there are four unique choices for such a entry x: the *highest*, *lowest*, *farthest* and *nearest*. This corresponds to the four mesh patterns in (1.8), respectively. □

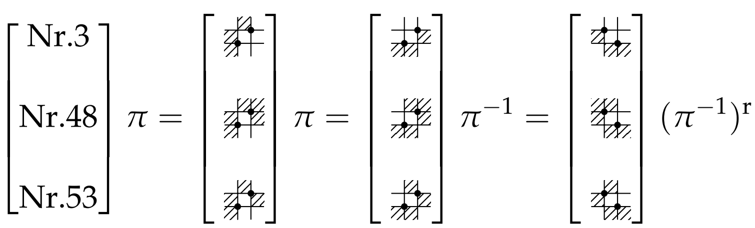
Remark 1.4. *As* $\mathrm{earec}(\pi) = \mathrm{erec}(\pi^{\mathrm{roc}})$ *for* $\pi \in S_n$*, we can also derive* (1.9) *from* (1.8).

Theorem 1.5. *There exists an involution* Φ *on* S_n *such that for* $\pi \in S_n$*,*



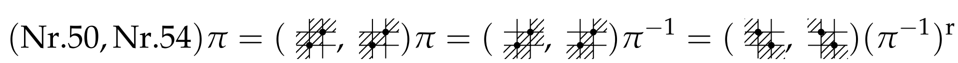
Corollary 1.1. *The triple pattern* (Nr.3, Nr.48, Nr.53) *is equidistributed with the triple pattern* (erec, Nr.50, Nr.54) *on* S_n*.*

Proof. For any $\pi \in S_n$ we have



(1.10)

and



(1.11)

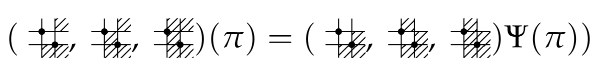
By Theorem 1.5 the result follows from (1.9), (1.10) and (1.11). □

Corollary 1.2. *Conjecture 1.1 holds true.*

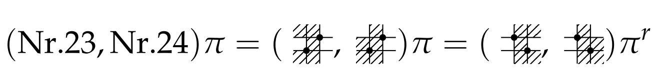
Proof. By Corollary 1.1 this follows from (1.3b) with $x = v = q = 1$. □

As the equidistribution of Nr.48 and Nr.49 is known [14, Theorem 5.1], Corollary 1.1 confirms two conjectured equidistributions in Table 1.

Theorem 1.6. *There exist an involution* Ψ *on* S_n *such that for* $\pi \in S_n$*,*



For the patterns Nr.23 and Nr.24, we have



By Theorem 1.6, we confirm another conjecture in Table 1, i.e., the patterns Nr.23 and Nr.24 are equidistributed.

We shall prove Theorem 1.5 and Theorem 1.6 in Section 2 and Section 3, respectively.

2 Proof outlines of Theorem 1.5

For $\pi \in S_n$ let $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$ be the sequence of antirecord positions of π from left to right. So $\pi(i_1) = 1$, $i_1 < \cdots < i_l$ and $i_l = n$. For each antirecord position i_k define two mappings

$$\varphi_1^{(i_k)} : \pi \mapsto \pi' \tag{2.1a}$$
$$\varphi_2^{(i_k)} : \pi \mapsto \pi'' \tag{2.1b}$$

as follows:

- let $w = w_1 \ldots w_r$ be the subword of π consisting of letters greater than $\pi(i_k)$ on the left of $\pi(i_k)$ (resp. $\pi(i_{k-1})$);
- let $w' = w'_1 \ldots w'_r$ be the word obtained by substituting the jth largest letter with the jth smallest letter in w for $j = 1, \ldots, r$;
- let π' (resp. π'') be the word obtained by replacing w_j with w'_j for $j = 1, \ldots, r$ in π.

Remark 2.1. *By convention, we define $\varphi_2^{(i_1)}$ to be the identity mapping. Clearly the two operations keep the sequence of antirecords for both values and positions, that is,*

$$\mathrm{AREC}(\pi) = \mathrm{AREC}(\pi') = \mathrm{AREC}(\pi'') \tag{2.2a}$$
$$\pi'(i_k) = \pi''(i_k) = \pi(i_k) \quad \textit{for} \quad k = 1, \ldots l. \tag{2.2b}$$

Let $P = \{p_1 < \cdots < p_r\}$ and $Q = \{q_1 < \cdots < q_r\}$ be two ordered sets and $\pi = p_1 \ldots p_r$ and $\tau = q_1 \ldots q_r$ are permutations of P and Q, respectively. We say that π and τ are *order isomorphic* and write $\pi \sim \tau$ if for any two indices r and s we have the equivalence $p_r < p_s$ if and only if $q_r < q_s$. In other words, τ is the permutation obtained from π by substituting p_i by q_i for $i = 1, \ldots, r$.

Let $w = w_1 \ldots w_n$ be a permutation of $a_1 < a_2 < \cdots < a_n$. We define the *complement* of w by w^c[1], which is the word obtained by substituting a_i by a_{n+1-i} in w for $i = 1, \ldots, n$. If x is a subset of the letters in w, we write $[w]_x$ as the subword of w consisting of the letters $a \in x$.

Lemma 2.1. *1. If $w = w_1 w_2$ and $w^c = w'_1 w'_2$, then $(w'_1)^c \sim w_1$.*[2]

2. Let $w = w_1 w_2 w_3$ and $v = v_1 v_2 v_3$ with $|w_1| = |v_1|$. If $w_1 w_2 \sim v_1 v_2$ with $(w_1 w_2)^c = w'_1 w'_2$ and $(v_1 v_2)^c = v'_1 v'_2$, then $w_1 \sim v_1$, $w_2 \sim v_2$, $w'_1 \sim v'_1$ and $w'_2 \sim v'_2$. Moreover, we have $(w'_1)^c \sim (v'_1)^c$ and $(w'_2)^c \sim (v'_2)^c$.

[1]When $a_i = i$, w^c reduces to π^c, see (1.7).

[2]The word w'_1 is the complement of w_1 in the alphabet of w, while $(w'_1)^c$ is the complement of w'_1 in the alphabet of w'_1.

3. *If $w \sim v$ and $[w]_x = [v]_x$ for some set x of some common letters in w and v, then*
   - *$w^c \sim v^c$ and $[w^c]_x = [v^c]_x$.*
   - *$[w]_y \sim [v]_z$, where y (resp. z) is the complementary of x in the alphabet of w (resp. v).*

For example, if $w = 359147286$, then $w^c = 751963824$. Let $w = w_1 w_2$ with $w_1 = 359147$ and $w_2 = 286$, then $w_1' = 751963$ and $(w_1')^c = 369157$. Clearly $(w_1')^c \sim w_1$ and $[(w_1')^c]_x = [w_1]_x$ with $x = \{1, 3, 9\}$. We see that $w_1^c = 741953$ and $[w_1']_x = [w_1^c]_x = 193$.

Lemma 2.2. *For any antirecord position i of $\pi \in S_n$ the mappings $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ are involutions and commute, namely,*

$$\varphi_1^{(i)} \circ \varphi_1^{(i)}(\pi) = \varphi_2^{(i)} \circ \varphi_2^{(i)}(\pi) = \pi \tag{2.3}$$

and

$$\varphi_2^{(i)} \circ \varphi_1^{(i)}(\pi) = \varphi_1^{(i)} \circ \varphi_2^{(i)}(\pi). \tag{2.4}$$

Proof. From the definitions of $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ in Eq. (2.1), it is easy to check Eq. (2.3) holds and

$$\varphi_2^{(i)} \circ \varphi_1^{(i)}(\pi) \sim \varphi_1^{(i)} \circ \varphi_2^{(i)}(\pi).$$

Since the set of letters greater than $\pi(i)$ on the left of $\pi(i)$ are invariant under the operation $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ on π, we obtain Eq.(2.4) immediately. □

Let $\pi \in S_n$ with sequence of antirecord positions $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$. We define the operation Φ on π by

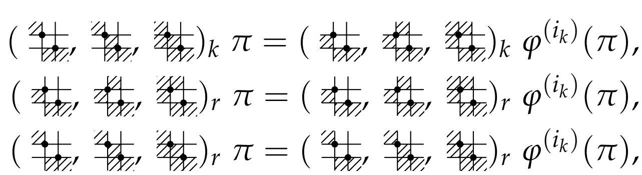
$$\Phi(\pi) = \varphi^{(i_1)} \circ \varphi^{(i_2)} \circ \cdots \circ \varphi^{(i_l)}(\pi) \tag{2.5}$$

with $\varphi^{(i_k)} = \varphi_2^{(i_k)} \circ \varphi_1^{(i_k)}$ for $k = 1, \ldots, l$.

Lemma 2.3. *For $\pi \in S_n$ with $\mathrm{AREC}(\pi) = \{i_1, \ldots, i_l\}$. The mappings $g := \varphi^{(i_{k-1})}$ and $f := \varphi^{(i_k)}$ commute, i.e.,*

$$g \circ f(\pi) = f \circ g(\pi).$$

Lemma 2.4. *The mapping $\varphi^{(i_k)}$ is an involution such that for $\pi \in S_n$ and $r \neq k$,*



(2.6a)
(2.6b)
(2.6c)

where $(pattern)_k$ means the number of the patterns between $\pi(i_{k-1})$ and $\pi(i_k)$.

Proof of Theorem 1.5. By (2.5) the reverse of the mapping Φ is given by

$$\Phi^{-1}(\pi) = \varphi^{(i_l)} \circ \cdots \circ \varphi^{(i_2)} \circ \varphi^{(i_1)}(\pi). \tag{2.7}$$

Theorem 1.5 follows from Lemma 2.2, Lemma 2.3 and Lemma 2.4. □

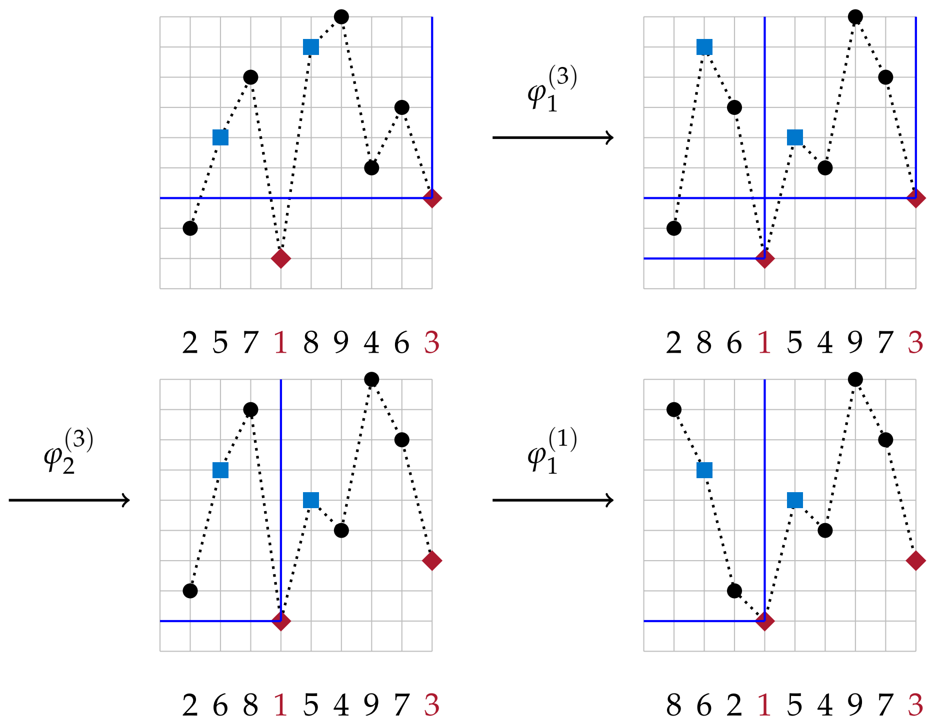


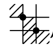
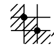
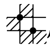
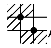
Figure 1: The involution Φ on the permutation 257189463

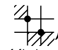
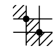
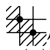
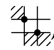
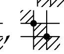
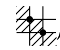
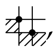
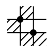
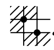
Example 2.2. *We show the process of the involution Φ in Figure 1, For $\pi = 257189463$, we have $\mathrm{AREC}(\pi) = (4, 9)$. We proceed from right to left.*

1. *For position 9 with value 3, w have $w = 578946$ and $w' = 865497$. Thus $\varphi_1^{(9)} : \pi \mapsto \pi' = 286154973$. Next, we have $w = 86$ and $w' = 68$. Thus $\varphi_2^{(9)} : \pi' \mapsto \pi'' = 268154973$.*

2. *For position 4 with value 1 we have $w = 268$ and $w' = 862$. Finally we obtain $\Phi(\pi) = 862154973$.*

Now, we check the mesh patterns.

- *First,* $\varphi_1^{(9)} : \pi = 257189463 \mapsto \pi' = 286154973$, *the pair* $(8,3)$ *of* π *contributes the pattern*  *without the patterns* , *), the pair* $(5,3)$ *of* π' *contributes the pattern*  *without the patterns* , *, the operations* $\varphi_2^{(9)}, \varphi_1^{(1)}$ *do not change the corresponding mesh pattens at position* 9 *of* π'.

- *Second,* $\varphi_2^{(9)} \circ \varphi_1^{(9)} : \pi = 257189463 \mapsto \pi'' = 268154973$ *it is easy to see that* $257 \sim 268$*. The pair* $(5,1)$ *of* π *contributes the patterns* ,  *without the pattern* *, the pair* $(6,1)$ *of* π'' *also contributes the pattern* ,  *without the pattern* , $\varphi_1^{(1)} : \pi'' = 268154973 \mapsto \pi''' = 862154973$*, the pair* $(6,1)$ *of* π''' *contributes the pattern* ,  *without the pattern* .

3 Proof outlines of Theorem 1.6

First we introduce two mappings different from Section 2. For $\pi \in S_n$, recall that $\mathrm{AREC}(\pi) = (i_1, i_2, \dots, i_l)$ be the sequence of antirecord positions of π from left to right. For any antirecord position i_k we define two mappings

$$\psi_1^{(i_k)} : \pi \mapsto \pi' \tag{3.1a}$$

$$\psi_2^{(i_k)} : \pi \mapsto \pi'' \tag{3.1b}$$

as follows:

- let $w = w_1 \dots w_r$ is the subword of π consisting of letters greater than $\pi(i_k)$ on the right side of $\pi(i_{k-1})$ (resp. $\pi(i_k)$) with $\pi(i_0) = 0$;
- let $w' = w'_1 \dots w'_r$ be the word obtained by substituting the jth largest letter with the jth smallest letter in w for $j = 1, \dots, r$;
- let π' (resp. π'') is defined to be the word obtained by replacing w_j with w'_j in π.

Note that $\pi'(i_k) = \pi(i_k)$.

Lemma 3.1. *For any antirecord positions* i_{k-1} *and* i_k *of* $\pi \in S_n$ *the mappings* $\psi_1^{(i_k)}$ *and* $\psi_2^{(i_k)}$ *are involutions and commutate, namely,*

$$\psi_1^{(i_k)} \circ \psi_1^{(i_k)}(\pi) = \psi_2^{(i_k)} \circ \psi_2^{(i_k)}(\pi) = \pi \tag{3.2}$$

and

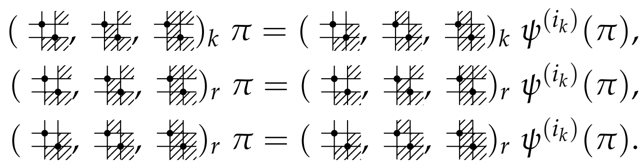
$$\psi_2^{(i_k)} \circ \psi_1^{(i_k)}(\pi) = \psi_1^{(i_k)} \circ \psi_2^{(i_k)}(\pi). \tag{3.3}$$

Let $\psi^{(i_k)} = \psi_2^{(i_k)} \circ \psi_1^{(i_k)}$*. Then* $\psi^{(i_k)}(\pi)$ *and* π *have the same sequence of antirecord positions.*

Lemma 3.2. *For $\pi \in S_n$ with $\mathrm{AREC}(\pi) = \{i_1, \ldots, i_l\}$. For $k = 2, \ldots, l$ the mappings $\psi^{(i_{k-1})}$ and $\psi^{(i_k)}$ commute, i.e.,*

$$\psi^{(i_k)} \circ \psi^{(i_{k-1})}(\pi) = \psi^{(i_{k-1})} \circ \psi^{(i_k)}(\pi).$$

Lemma 3.3. *The mapping $\psi^{(i)}$ is an involution such that for $\pi \in S_n$ and $r \neq k$*



(3.4a)
(3.4b)
(3.4c)

where $(pattern)_k$ *means the number of the patterns between* $\pi(i_{k-1})$ *and* $\pi(i_k)$.

Proof of Theorem 1.6. For $\pi \in S_n$ and $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$, we define the operation Ψ on π by

$$\Psi(\pi) = \psi^{(i_l)} \circ \cdots \circ \psi^{(i_2)} \circ \psi^{(i_1)}(\pi). \tag{3.5}$$

By (3.5) the mapping Ψ is reversible with reverse

$$\Psi^{-1}(\pi) = \psi^{(i_1)} \circ \psi^{(i_2)} \circ \ldots \circ \psi^{(i_l)}(\pi).$$

Theorem 1.6 follows from Lemma 3.1, Lemma 3.2 and Lemma 3.3. □

An example of the involution Ψ could be found in [9, Example 3.4].

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