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Geometric vertex decomposition and liaison

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Abstract. Geometric vertex decomposition and liaison are two frameworks that have been used to produce similar results about similar families of algebraic varieties. We establish an explicit connection between these approaches. In particular, we show that each geometrically vertex decomposable ideal is linked by a sequence of elementary G-biliaisons of height 1 to an ideal of indeterminates and, conversely, that every G-biliaison of a certain type gives rise to a geometric vertex decomposition.

As a consequence, we can immediately conclude that several well-known families of ideals are glicci, including Schubert determinantal ideals, defining ideals of varieties of complexes, and defining ideals of graded lower bound cluster algebras. We also use the structure of Knutson, Miller, and Yong's geometric vertex decomposition to provide a streamlined implementation of Gorla, Nagel, and Migliore's liaison-theoretic approach to establishing Gröbner bases.

Keywords: Vertex decomposition, liaison, Gröbner bases

1 Introduction

Determinantal ideals and their generalizations have been explored extensively both in the context of commutative algebra and also in the study of Schubert varieties in flag varieties. This overlap is to be expected because, for example, each ideal generated by the $k \times k$ minors of a generic matrix is the defining ideal of an open patch of a Schubert variety in a Grassmannian; each one-sided ladder determinantal ideal is a Schubert determinantal ideal for a vexillary (i.e., 2143-avoiding) permutation (see eg. [11]); each two sided mixed ladder determinantal ideal is a type A Kazhdan–Lusztig ideal; each ideal generated by the $k \times k$ minors of a generic symmetric matrix is the defining ideal of an open patch of a Schubert variety in a Lagrangian Grassmannian; and each defining ideal of a variety of complexes is a type A Kazhdan–Lusztig ideal, up to some extra indeterminate generators (see eg. [13, Ch. 17]).

While similar results on the above-mentioned families of ideals appear in the Schubert variety and commutative algebra literatures, it is often different techniques that are

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used to obtain them. For example, in [11], A. Knutson, E. Miller, and A. Yong introduced *geometric vertex decomposition*, a degeneration technique, and used this to study Gröbner geometry of Schubert determinantal ideals for vexillary permutations. Independently, *liaison-theoretic* methods were used by E. Gorla in [4] and E. Gorla, J. Migliore, and U. Nagel in [6] to obtain Gröbner bases for various classes of ladder determinantal ideals (including one sided ladder determinantal ideals, also known as Schubert determinantal ideals for vexillary permutations). In this extended abstract, we establish an explicit connection between *geometric vertex decomposition* and *liaison*, and we study implications of this connection. We have three main goals, which we now outline.

Our first goal is to show that it is no coincidence that geometric vertex decomposition and liaison can be used to obtain similar results. Indeed, we prove the following explicit connection between the two techniques (see Corollary 3.2 and Theorem 5.1):

Main Theorem. *Under mild hypotheses, every geometric vertex decomposition gives rise to an elementary G-biliason of height* 1*. Every sufficiently "nice" elementary G-biliaison of height* 1 *gives rise to a geometric vertex decomposition.*

The second motivation for our work comes from a long-standing open question in liaison theory, which asks whether subschemes of $\mathbb{P}^n$ are arithmetically Cohen–Macaulay if and only if they are in the *Gorenstein liaison class of a complete intersection* (or *glicci*). It is a standard homological argument that every glicci subscheme of $\mathbb{P}^n$ is arithmetically Cohen–Macaulay. Hence, the question may be phrased as follows:

Question 1.1 ([9, Question 1.6])**.** *Is every arithmetically Cohen–Macaulay subscheme of* $\mathbb{P}^n$ *glicci?*

By combining our main theorem with some straightforward consequences of geometric vertex decomposition, we give a corollary (stated precisely as Corollary 4.1) from which one can quickly deduce that certain well-known classes of varieties are glicci:

Corollary. *Let I be a homogenous ideal in a polynomial ring. If the Lex-initial ideal of I is the Stanley–Reisner ideal of a vertex decomposable simplicial complex and the vertex decomposition is compatible with the order of the variables, then I is glicci.*

We discuss three such classes in Section 4: matrix Schubert varieties, varieties of complexes, and varieties of graded lower bound cluster algebras. We present this as evidence in favor of Question 1.1, at least in combinatorially-natural settings. Using the first half of our main theorem, we recover a result of U. Nagel and T. Römer from [15], namely that the Stanley–Reisner ideal of a vertex decomposable simplicial complex is glicci. We show the following (appearing later as Theorem 3.3):

Theorem. *Geometrically vertex decomposable ideals are glicci.*

In [6, Lemma 1.12], it is shown that one can use liaison to compare Hilbert functions when the degrees of the isomorphisms of the G-biliaisons involved in an inductive argument are known. This approach is employed in many of the determinantal cases treated in the literature ([4, 5, 6]). It is worth noticing that the isomorphisms employed in these papers all have a similar form. We explain via geometric vertex decomposition in Theorem 3.1 why this similarity is not a coincidence but, rather, is to be expected. In that theorem, we associate an explicit isomorphism of degree 1 to a geometric vertex decomposition.

In addition to the expository work of describing a unifying structure underlying examples already in the literature, Theorem 3.1 also provides a candidate isomorphism in the style of G-biliaison that, in good cases, allows one to use the framework of [6] to prove that a conjectured Gröbner basis is, indeed, a Gröbner basis. This corollary has been used in the study of diagonal degenerations of matrix Schubert varieties [8]. Some consequences of Theorem 3.1 on Gröbner bases and degenerations appear in Section 3.

The structure of this extended abstract. In Section 2, we review definitions and key lemmas from [11] on geometric vertex decomposition in the unmixed case and record some additional observations about the structure of a geometrically vertex decomposable ideal. We then briefly review background material on Gorenstein liaison that is necessary for this work. In Sections 3 and 5, we describe our main theorem (stated above), and related results and examples. In Section 4, we prove that certain well-known classes of combinatorially-defined ideals are glicci, via the material in Section 3.

Notational conventions. Throughout the extended abstract, we let κ be a field, which can be chosen arbitrarily except in Sections 3 and 4, where we require that κ be infinite.

2 Preliminaries

In this section we discuss geometric vertex decomposition, introduced by A. Knutson, E. Miller, and A. Yong in [11]. We first recall the basics of vertex decomposition of simplicial complexes and Stanley–Reisner ideals. Next, we move beyond the monomial ideal case and recall the basics of *geometric* vertex decomposition from [11]. Then we define and study *geometrically vertex decomposable ideals*. Finally, we recall the information of Gorenstein liaison essential for the purposes of this extended abstract.

Vertex decomposition and Stanley–Reisner ideals. Let Δ be a simplicial complex on vertex set $[n] = \{1, 2, \dots, n\}$ (without an insistence that every $v \in [n]$ necessarily be a face of Δ). Given a vertex $v \in \Delta$, define the following three subcomplexes: the **star** of v is the set $\operatorname{star}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta\}$, the **link** of v is the set $\operatorname{lk}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta, F \cap \{v\} = \varnothing\}$, and the **deletion** of v is the set $\operatorname{del}_\Delta(v) := \{F \in \Delta \mid F \cap \{v\} = \varnothing\}$. Recall that the **cone** from v on a simplicial complex Δ is the smallest simplicial complex that contains the set $\{F \cup \{v\} \mid F \in \Delta\}$. Then $\operatorname{star}_\Delta(v)$ is the cone from v on $\operatorname{lk}_\Delta(v)$ and

$\Delta = \text{star}_\Delta(v) \cup \text{del}_\Delta(v)$. The decomposition of Δ above is called a **vertex decomposition**.

A simplicial complex is called **pure** if all of its facets (i.e., maximal faces) are of the same dimension. A simplicial complex Δ is **vertex decomposable** if it is pure and if $\Delta = \emptyset$, or Δ is a simplex, or there is a vertex $v \in \Delta$ such that $\text{lk}_\Delta(v)$ and $\text{del}_\Delta(v)$ are vertex decomposable. Given a simplicial complex Δ on vertex set $[n]$, one defines the **Stanley–Reisner ideal** $I_\Delta \subseteq \kappa[x_1, \dots, x_n]$ associated to Δ as $I_\Delta := \langle \mathbf{x}_F \mid F \subseteq [n], F \notin \Delta \rangle$, where $\mathbf{x}_F := \prod_{i \in F} x_i$. The association $\Delta \mapsto I_\Delta$ determines a bijection between simplicial complexes on $[n]$ and squarefree monomial ideals in $\kappa[x_1, \dots, x_n]$. We write $\Delta(I)$ for the simplicial complex associated to a squarefree monomial ideal I.

Notice that if $\Delta = \Delta_1 \cup \Delta_2$ is a union of simplicial complexes on $[n]$, then F is a non-face of Δ if and only if it is a non-face of both Δ_1 and Δ_2. Thus, $I_\Delta = I_{\Delta_1} \cap I_{\Delta_2}$. In particular, if v is a vertex of Δ, we may decompose Δ to get $I_\Delta = I_{\text{star}_\Delta(v)} \cap I_{\text{del}_\Delta(v)}$.

Lemma 2.1. *Let $v \in [n]$ be a vertex of Δ. Writing $I_\Delta = \langle x_v^{d_i} q_i \mid 1 \leq i \leq m \rangle$ where q_i is a squarefree monomial that is not divisible by x_v and $d_i = 0$ or 1. Then $I_{star_\Delta(v)} = \langle q_i \mid 1 \leq i \leq m \rangle$, $I_{lk_\Delta(v)} = I_{star_\Delta(v)} + \langle x_v \rangle$, and $I_{del_\Delta(v)} = \langle q_i \mid d_i = 0 \rangle + \langle x_v \rangle$.*

Geometric vertex decomposition We now discuss *geometric vertex decomposition*, introduced by A. Knutson, E. Miller, and A. Yong in [11]. Let $R = \kappa[x_1, \dots, x_n]$ be a polynomial ring in n indeterminates and let $y = x_j$ for some $1 \leq j \leq n$. Define the **initial y-form** $\text{in}_y f$ of a polynomial $f \in R$ to be the sum of all terms of f having the highest power of y. That is, if $f = \sum_{i=0}^n \alpha_i y^i$, where each $\alpha_i \in \kappa[x_1, \dots, \widehat{x_j}, \dots x_n]$ and $\alpha_n \neq 0$, define $\text{in}_y f := \alpha_n y^n$, which is usually not a monomial. Given an ideal $I \subseteq R$, define $\text{in}_y I := \langle \text{in}_y f \mid f \in I \rangle$. We say that a monomial order $<$ on R is **y-compatible** if $\text{in}_< f = \text{in}_<(\text{in}_y f)$ for every $f \in R$, in which case $\text{in}_<(\text{in}_y I) = \text{in}_< I$ for any ideal $I \subseteq R$.

Let $I \subseteq R$ be an ideal and $<$ a y-compatible monomial order. With respect to $<$, let $\mathcal{G} := \{y^{d_i} q_i + r_i \mid 1 \leq i \leq m\}$ be a Gröbner basis of I where y does not divide any q_i and $\text{in}_y(y^{d_i} q_i + r_i) = y^{d_i} q_i$. One easily checks that the ideal $\text{in}_y I$ is generated by $\text{in}_y \mathcal{G} := \{y^{d_i} q_i \mid 1 \leq i \leq m\}$ ([11, Theorem 2.1(a)]). That is, $\text{in}_y I = \langle y^{d_i} q_i \mid 1 \leq i \leq m \rangle$.

Definition 2.2 ([11, Section 2.1])**.** As above, let let $\mathcal{G} := \{y^{d_i} q_i + r_i \mid 1 \leq i \leq m\}$ be a Gröbner basis of the ideal I with respect to the y-compatible term order $<$. Define $C_{y,I} := \langle q_i \mid 1 \leq i \leq m \rangle$ and $N_{y,I} = \langle q_i \mid d_i = 0 \rangle$. When $\text{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, this decomposition is called a **geometric vertex decomposition of I with respect to y**.

The ideals $C_{y,I}$ and $N_{y,I}$ do not depend on the choice of Gröbner basis and, in particular, do not depend on the choice of y-compatible term order $<$. This follows from the facts that $C_{y,I} = (\text{in}_y I : y^\infty)$ by [11, Theorem 2.1(d)] and that $N_{y,I} + \langle y \rangle = \text{in}_y I + \langle y \rangle$ by [11, Theorem 2.1 (a)], together with the observation that y does not appear in the generators of $N_{y,I}$ given in its definition.

We say that a geometric vertex decomposition is **degenerate** if $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$ or if $C_{y,I} = \langle 1 \rangle$ and **nondegenerate** otherwise. As we will see through Lemma 2.4, if $C_{y,I} =$

$\langle 1\rangle$, then some polynomial whose initial y-form is a unit multiple of y is an element of I, in which case $R/I \cong R/(N_{y,I}+\langle y\rangle)$. If $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$, then $\sqrt{\text{in}_y I} = \sqrt{C_{y,I}} \cap \sqrt{N_{y,I}+\langle y\rangle} = \sqrt{C_{y,I}}$, in which case $\text{in}_y I$, $C_{y,I}$, and $N_{y,I}$ all determine the same variety. In both of these cases, we may often prefer to study $N_{y,I}$ in the smaller polynomial ring that omits y. This is especially true when I is radical for the following reason:

Proposition 2.3. *If I is radical and has a degenerate geometric vertex decomposition $in_y I = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ with $\sqrt{N_{y,I}} = \sqrt{C_{y,I}}$, then the reduced Gröbner basis of I does not involve y and $I = in_y I = C_{y,I} = N_{y,I}$.*

If an ideal $I \subseteq R$ has a generating set $\mathcal{G}$ in which y^2 does not divide any term of g for any $g \in \mathcal{G}$, then we say that I is **squarefree in** y. It is easy to see (for example, by considering S-pair reductions) that every ideal that is squarefree in y has a Gröbner basis, with respect to any y-compatible term order, such that y^2 does not divide any term of any element of the Gröbner basis.

Lemma 2.4. *If $I \subseteq R$ possesses a geometric vertex decomposition with respect to a variable $y = x_j$ of R, then I is squarefree in y, and the reduced Gröbner basis of I with respect to any y-compatible term order has the form $\{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$ where y does not divide any term of any q_i or r_i for any $1 \leq i \leq k$ nor any h_j for any $1 \leq j \leq \ell$.*

Geometrically vertex decomposable ideals. A geometric vertex decomposition of an ideal is analogous to a vertex decomposition of a simplicial complex into a deletion and star. In this subsection, we extend this analogy by considering *geometrically vertex decomposable ideals*, which are analogous to vertex decomposable simplicial complexes. We again let $R = \kappa[x_1, \ldots, x_n]$ throughout this subsection. Recall that an ideal P of R is an **associated prime** of the ideal I if R/P is isomorphism to a submodule of R/I and that an ideal $I \subseteq R$ is **unmixed** if $\dim(R/P) = \dim(R/I)$ for all associated primes P of I .

Definition 2.5. An ideal $I \subseteq R$ is **geometrically vertex decomposable** if I is unmixed and if (1) $I = \langle 1\rangle$ or I is generated by indeterminates in R, or (2) for some variable $y = x_j$ of R, $\text{in}_y I = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is a geometric vertex decomposition and the contractions of $N_{y,I}$ and $C_{y,I}$ to $\kappa[x_1, \ldots, \widehat{y}, \ldots, x_n]$ are geometrically vertex decomposable.

We take case (1) to include the zero ideal, whose (empty) generating set vacuously consists only of indeterminates. We will soon need observations about the relative heights of the ideals I, $C_{y,I}$, and $N_{y,I}$ in the circumstances of condition (2). The degenerate cases are clear: if $C_{y,I} = \langle 1\rangle$, then $\text{ht}(I) = \text{ht}(N_{y,I}) + 1$ and, if $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$, then $\text{ht}(I) = \text{ht}(\text{in}_y I) = \text{ht}(C_{y,I}) = \text{ht}(N_{y,I})$. The nondegenerate case is handled by the lemma below.

Lemma 2.6. *If $I \subseteq R$ is an ideal so that R/I is equidimensional and $in_y I = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is a nondegenerate geometric vertex decomposition with respect to some variable $y = x_j$ of R, then $ht(C_{y,I}) = ht(I) = ht(N_{y,I}) + 1$.*

Note that if Δ be a simplicial complex on vertex set $[n]$, then its Stanley–Reisner ideal $I_\Delta \subseteq R$ is geometrically vertex decomposable if and only if Δ is vertex decomposable. We next discuss some properties of geometrically vertex decomposable ideals and further connections to vertex decomposable simplicial complexes.

Proposition 2.7. *A geometrically vertex decomposable ideal is radical.*

We next consider geometrically vertex decomposable ideals that have a certain compatibility with a given lexicographic monomial order. The main result in our discussion of these ideals is Theorem 2.10, which we will need in Section 4 on applications.

Definition 2.8. Fix a lexicographic monomial order $<$ on R. We say that an ideal $I \subseteq R$ is **$<$-compatibly geometrically vertex decomposable** if I satisfies Definition 2.5 upon replacing item (2) with (2*) for the $<$-largest variable y in R, $\text{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition and the contractions of $N_{y,I}$ and $C_{y,I}$ to $\kappa[x_1, \ldots, \widehat{y}, \ldots, x_n]$ are $<$-compatibly geometrically vertex decomposable for the naturally induced monomial order on $\kappa[x_1, \ldots, \widehat{y}, \ldots, x_n]$ (which we also call $<$).

Let Δ be a simplicial complex on a vertex set $[n]$, and let $<$ be a total order on $[n]$. We say that a simplicial complex Δ is **$<$-compatibly vertex decomposable** if either $\Delta = \emptyset$ or Δ is a simplex or, for the $<$-largest vertex $v \in \Delta$, $\text{del}_\Delta(v)$ and $\text{lk}_\Delta(v)$ are $<$-compatibly vertex decomposable. The following is an easy consequence of [11, Theorem 2.1].

Lemma 2.9. *Suppose that $I \subseteq R$ is squarefree in $y = x_j$, and suppose that $<$ is a y-compatible monomial order on R. Then $in_< I = in_< C_{y,I} \cap (in_< N_{y,I} + \langle y \rangle)$.*

We now state the main result of this section:

Theorem 2.10. *An ideal $I \subseteq R$ is $<$-compatibly geometrically vertex decomposable for the lexicographic monomial order $x_1 > x_2 > \cdots > x_n$ if and only if $in_< I$ is the Stanley–Reisner ideal of a $<$-compatibly vertex decomposable simplicial complex on $[n]$ for the vertex order $1 > 2 > \cdots > n$.*

The next example shows that there exist geometrically vertex decomposable ideals that are *not* geometrically vertex decomposable compatible with any lexicographic monomial order.

Example 2.11. Let $I = \langle y(zs - x^2), ywr, wr(z^2 + zx + wr + s^2) \rangle \subseteq \kappa[x, y, z, w, r, s]$. Observe that I is squarefree in y, and we have a geometric vertex decomposition with $C_{y,I} = \langle zs - x^2, wr \rangle$ and $N_{y,I} = \langle (wr)(zx + s^2 + z^2 + wr) \rangle$. Furthermore, the contractions of $C_{y,I}$ and $N_{y,I}$ to $\kappa[x, z, w, r, s]$ are geometrically vertex decomposable. (To see this, let C^c and N^c denote these contracted ideals. Then C^c and N^c are squarefree in s and x, respectively, and $\text{in}_s C^c = \langle zs, wr \rangle$ and $\text{in}_x N^c = \langle wrzx \rangle$.) Hence I is geometrically vertex decomposable. Yet one readily checks that I has no squarefree initial ideals, so cannot be $<$-compatibly geometrically vertex decomposable for any order $<$ by Theorem 2.10.

Gorenstein Liaison. Here we review standard definitions and lemmas in Gorenstein liaison that we will need in this extended abstract. We restrict to an understanding of Gorenstein liaison as it arises in the context of Gröbner bases and as it is required to understand Question 1.1 for our study in combinatorial settings. For a more thorough introduction, see [12]. We follow definitions and some notation from [6], which provides a careful discussion of how liaison theory can be used to make inferences about Gröbner bases. Throughout this subsection, we let $R = \kappa[x_0, x_1, \ldots, x_n]$ with the standard grading.

Definition 2.12. Let $V_1, V_2, X \subseteq \mathbb{P}^n$ be subschemes defined by saturated ideals I_{V_1}, I_{V_2}, and I_X of R, respectively, and assume that X is arithmetically Gorenstein. If $I_X \subseteq I_{V_1} \cap I_{V_2}$ and if $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$, then V_1 and V_2 are **directly algebraically** G**-linked** by X, and we write $I_{V_1} \sim I_{V_2}$.

One may generate an equivalence relation using these direct links.

Definition 2.13. If there is a sequence of links $V_1 \sim \cdots \sim V_k$ for some $k \geq 2$, then we say that V_1 and V_k are in the same G**-liaison class (or Gorenstein liaison class)** and that they are G**-linked** in $k-1$ steps. Of particular interest is the case in which V_k is a complete intersection, in which case we say that V_1 is **in the Gorenstein liaison class of a complete intersection (abbreviated glicci)**.

We will say that a homogeneous, saturated, unmixed ideal of R is glicci if it defines a glicci subscheme of $\mathbb{P}^n$. It is because liaison was developed to study subschemes of projective space that the restriction to homogeneous, saturated ideals is natural. Throughout this extended abstract, we will be interested in G-links coming from *elementary G-biliaisons*. Indeed, it is through elementary G-biliaisons that we connect geometric vertex decomposition to liaison theory. Let S be a ring. If S_P is Gorenstein for all prime ideals P of height 0, then we say that S is $\mathbf{G_0}$.

Definition 2.14. Let I and C be homogeneous, saturated, unmixed ideals of R with $\text{ht}(I) = \text{ht}(C)$. Suppose there exist $\ell \in \mathbb{Z}$, a homogeneous Cohen–Macaulay ideal $N \subseteq I \cap C$ of height $\text{ht}(I) - 1$, and an isomorphism $I/N \cong [C/N](-\ell)$ as graded R/N-modules. If N is G_0, then we say that I is obtained from C by an **elementary** G**-biliaison of height** ℓ.

Theorem 2.15 ([7, Theorem 3.5])**.** *Let I and C be homogeneous, saturated, unmixed ideals defining subschemes V_I and V_C, respectively, of $\mathbb{P}^n$. If I is obtained from C by an elementary G-biliaison, then V_I is G-linked to V_C in two steps.*

3 Geometrically vertex decomposable ideals are glicci

Throughout this section, we assume that the field κ is infinite, and we let R denote the standard graded polynomial ring $\kappa[x_1, \ldots, x_n]$.

An elementary G-biliaison arising from a geometric vertex decomposition. Our first result uses a geometric vertex decomposition to construct an isomorphism that will constitute an elementary G-biliaison when the setting is appropriate.

Theorem 3.1. *Suppose that $I \subseteq R$ is an unmixed ideal possessing a nondegenerate geometric vertex decomposition with respect to some variable $y = x_j$ of R. If $N_{y,I}$ is unmixed, then there is an isomorphism $I/N_{y,I} \cong C_{y,I}/N_{y,I}$ as $R/N_{y,I}$-modules. If $N_{y,I}$, $C_{y,I}$, and I are homogeneous, then the same map is an isomorphism $I/N_{y,I} \cong [C_{y,I}/N_{y,I}](-1)$ in the category of graded $R/N_{y,I}$-modules.*

Corollary 3.2. *Let I be a homogeneous, saturated, unmixed ideal of R and $in_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ a nondegenerate geometric vertex decomposition with respect to some variable $y = x_j$ of R. Assume that $N_{y,I}$ is Cohen–Macaulay and G_0 and that $C_{y,I}$ is also unmixed. Then I is obtained from $C_{y,I}$ by an elementary G-biliaison of height 1.*

Theorem 3.3. *If $I = I_0 \subseteq R$ is a homogeneous, geometrically vertex decomposable proper ideal, then there is a finite sequence of homogeneous, saturated, unmixed ideals $I_1, \ldots, I_t$ so that I_{j-1} is obtained from I_j by an elementary G-biliaison of height 1 for every $1 \leq j \leq t$ and I_t is a complete intersection. In particular, I is glicci and, hence, Cohen–Macaulay.*

Applications to Gröbner bases and degenerations One cannot in general transfer the Cohen–Macaulay property from an ideal to its initial ideal or from one component of a variety to the whole variety. However, in the context of geometric vertex decomposition, we can use the combination of Cohen–Macaulayness of a homogeneous ideal I and of the component $N_{y,I} + \langle y \rangle$ (equivalently, of $N_{y,I}$) to infer the same about $\text{in}_y I$.

Corollary 3.4. *Suppose that $in_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a nondegenerate geometric vertex decomposition of the homogeneous ideal $I \subseteq R$ and that both $N_{y,I}$ and I are Cohen–Macaulay. Then, $C_{y,I}$ and $in_y I$ are Cohen–Macaulay as well.*

We will now describe conditions that allow one to use the map constructed in Theorem 3.1 in order to conclude that a known set of generators for I forms a Gröbner basis when Gröbner bases for $C_{y,I}$ and $N_{y,I}$ are known. The result complements the framework of [11], in which one begins with a Gröbner basis of I and concludes that the resultant generating sets of $C_{y,I}$ and $N_{y,I}$ are also Gröbner bases. The corollary below gives a way to implement the approach of [6, Lemma 1.12].

Corollary 3.5. *Let $I = \langle yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell \rangle$ be a homogenous ideal of R with $y = x_j$ some variable of R and y not dividing any term of any q_i for $1 \leq i \leq k$ nor of any h_j for $1 \leq j \leq \ell$. Fix a term order $<$, and suppose that $\mathcal{G}_C = \{q_1, \ldots, q_k, h_1, \ldots, h_\ell\}$ and $\mathcal{G}_N = \{h_1, \ldots, h_\ell\}$ are Gröbner bases for the ideals they generate, which we call C and N, respectively. Assume also that the leading term of each $yq_i + r_i$ is yq_i for all $1 \leq i \leq k$, that $ht(I), ht(C) > ht(N)$, and that N is unmixed. Let $M = \begin{pmatrix} q_1 & \cdots & q_k \\ r_1 & \cdots & r_k \end{pmatrix}$. If the ideal of 2-minors of M is contained in N, then the given generators of I are a Gröbner basis with respect to $<$.*

Example 3.6 (The Veronese Embedding)**.** Using Corollary 3.5, one can give a concise inductive proof that the usual set of homogeneous equations defining the image of the d^{th} Veronese $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ forms a Gröbner basis for any $d \geq 1$. With homogeneous coordinates $[s : t]$ on $\mathbb{P}^1$ and $[x_0 : \cdots : x_d]$ on $\mathbb{P}^d$, recall that the d^{th} Veronese is the map $[s : t] \mapsto [s^d : s^{d-1}t : \cdots : st^{d-1} : t^d]$. Let $M_d = \begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}$, let $\mathcal{G}_d$ denote the set of 2×2 minors of M_d, and let $I = \langle \mathcal{G}_d \rangle$ be the ideal generated by $\mathcal{G}_d$. The image of the ν_d is defined by I, which is to say that there is a ring isomorphism $\frac{\kappa[x_0, \ldots, x_d]}{I} \to \kappa[s^d, s^{d-1}t, \ldots, st^{d-1}, t^d] \subseteq \kappa[s,t]$ given by $x_i \mapsto s^{d-i}t^i$ for $0 \leq i \leq d$. One may use Corollary 3.5 in an inductive argument to see that $\mathcal{G}_d$ is a Gröbner basis of I with respect to the lexicographic monomial order with $x_d > x_{d-1} > . \cdots > x_1 > x_0$ by observing that the ideal generated by the 2×2 minors of $\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} \\ x_1x_{d-1} & x_2x_{d-1} & \cdots & x_{d-1}^2 \end{pmatrix}$ is equal to $x_{d-1} \cdot N$ and so is contained in N.

4 Some well-known families of ideals are glicci

Many well-known classes of ideals Gröbner degenerate to Stanley–Reisner ideals of vertex decomposable complexes. In this section, we recall a few of these classes and deduce that they are glicci, providing evidence for an affirmative answer to the question of whether every homogeneous Cohen–Macaulay ideal is glicci [9, Question 1.6], at least in combintorially-natural settings. As in Section 3, we will assume throughout this section that the field κ is infinite.

The main result we need for our applications is as follows. It is immediately obtained by combining Theorem 2.10 with Theorem 3.3.

Corollary 4.1. *Let $I \subseteq \kappa[x_1, \ldots, x_n]$ be a homogeneous ideal, and let $<$ denote the lexicographic order with $x_1 > x_2 > \cdots > x_n$. If $\text{in}_< I$ is the Stanley–Reisner ideal of a $<$-compatibly vertex decomposable simplicial complex on $[n]$ for the vertex order $1 > 2 > \cdots > n$, then I is glicci.*

We now discuss three classes of ideals which satisfy the hypotheses of Corollary 4.1. We omit many definitions of the ideals in question and instead provide references.

Schubert determinantal ideals. Let $X = (x_{ij})$ be an $n \times n$ matrix of variables and let $R = \kappa[x_{ij}]$ be the polynomial ring in the matrix entries of X. Given a permutation $w \in S_n$, there is an associated generalized determinantal ideal $I_w \subseteq R$, called a *Schubert determinantal ideal*. Schubert determinantal ideals and their corresponding *matrix Schubert varieties* were introduced by Fulton in [3].

Fix the lexicographical monomial order $<$ on R defined by $x_{ij} > x_{kl}$ if $i < k$ or $i = k$ and $j > l$. This monomial order is *antidiagonal*, that is, the initial term of the determinant of a submatrix Y of X is the product of the entries along the antidiagonal of Y. For this monomial order, $\text{in}_< I_w$ is the Stanley–Reisner ideal of a simplicial complex, called a *subword complex*, which is $<$-compatibly vertex decomposable (see [10] or [13, Ch. 16.5]). Corollary 4.1 thus immediately implies:

Proposition 4.2. *Schubert determinantal ideals are glicci.*

Graded lower bound cluster algebras. Cluster algebras are a class of combinatorially-defined commutative algebras that were introduced by S. Fomin and A. Zelevinsky at the turn of the century [2]. *Lower bound algebras*, introduced in [1] are related objects: each lower bound algebra is contained in an associated cluster algebra, and this containment is equality in certain cases (i.e. in the *acyclic* setting, see [1, Theorem 1.20]).

Each (skew-symmetric) lower bound algebra is defined from a quiver. Indeed, given a quiver Q, there is an associated polynomial ring $R_Q = \kappa[x_1, \dots, x_n, y_1, \dots, y_n]$ and ideal $K_Q \subseteq R_Q$ such that the lower bound algebra $\mathcal{L}_Q$ associated to Q can be expressed as $\mathcal{L}_Q = R_Q/K_Q$. Fix the lexicographical monomial order with $y_1 > \cdots > y_n > x_1 > \cdots > x_n$. By [14, Theorem 1.7] and the proof of [14, Theorem 3.3], $\text{in}_< K_Q$ is the Stanley–Reisner ideal of a simplicial complex Δ on vertex set $\{y_1, \dots, y_n, x_1, \dots, x_n\}$, which has vertex decomposition compatible with $<$. Consequently, by Theorem 2.10, we have the following:

Proposition 4.3. *The ideal K_Q is geometrically vertex decomposable. When K_Q is homogeneous, it is glicci.*

Remark 4.4. It follows from [14, Theorem 1.7] that K_Q is homogeneous if and only if Q has no *frozen vertices* and Q has exactly two arrows entering each vertex and two arrows exiting each vertex.

Ideals defining equioriented type A quiver loci. Let $d_0, d_1, \dots, d_n$ be a sequence of positive integers and consider the product of matrix spaces $\texttt{Mat}$, and product of general linear group $\texttt{GL}$ defined as follows:

$$\texttt{Hom} := \oplus_{i=1}^n \text{Mat}_{d_{i-1}\times d_i}(\kappa), \quad \texttt{GL} := \oplus_{i=0}^n \text{GL}_{d_i}(\kappa).$$

The group $\texttt{GL}$ acts on $\texttt{Hom}$ on the right by conjugation: $(M_i)_{i=1}^n \bullet (g_i)_{i=0}^n = (g_{i-1}^{-1} M_i g_i)_{i=1}^n$. Closures of $\texttt{GL}$-orbits are called **equioriented type A quiver loci**. *Buchsbaum-Eisenbud varieties of complexes* are special cases of these quiver loci. An introduction to equioriented type A quiver loci and related combinatorics can be found in [13, Ch. 17].

Proposition 4.5. *Equioriented type A quiver loci are glicci. In particular, varieties of complexes are glicci.*

This follows because, up to adding some additional indeterminate generators, the ideal of such a quiver locus is a type A Kazhdan–Lusztig ideal. As shown in [16], each Kazhdan–Lusztig ideal Gröbner degenerates to the Stanley–Reisner ideal of a subword complex, and this degeneration is compatible with the vertex decomposition of the complex.

5 From G-biliaisons to geometric vertex decompositions

We now state something of a converse to Theorem 3.1.

Theorem 5.1. *Let I, C, and $N \subseteq I \cap C$ be ideals of R, and let $<$ be a y-compatible term order. Suppose that I is squarefree in y and that no term of any element of the reduced Gröbner basis of N is divisible by y. Suppose further that there exists an isomorphism $\phi : C/N \xrightarrow{f/g} I/N$ of R/N-modules for some $f, g \in R$, and $in_y(f)/g = y$. Then $in_y I = C \cap (N + \langle y \rangle)$ is a geometric vertex decomposition of I.*

Example 5.2. To illustrate this correspondence between elementary G-biliaisons and geometric vertex decomposition, we consider a classical example. If I is the ideal of 2-minors of the matrix $M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}$, $C = \langle x_{11}, x_{12} \rangle$, $N = \langle x_{22}x_{11} - x_{21}x_{12} \rangle$, $f = x_{23}x_{12} - x_{22}x_{13}$, and $g = x_{12}$ in $\kappa[x_{11}, \ldots, x_{23}]$, then the multiplication by f/g map $[C/N](-1) \xrightarrow{f/g} I/N$ gives an elementary G-biliaison. Using any lexicographic order with x_{23} largest, we take $C = C_{x_{23},I}$ and $N = N_{x_{23},I}$, and then $\text{in}_{x_{23}}(I) = C \cap (N + \langle x_{23} \rangle)$ is a geometric vertex decomposition.

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