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Eulerian representations for real reflection groups

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Abstract. The Eulerian idempotents of a real reflection group W generate a family of W-representations decomposing the regular representation, called the *Eulerian representations*. In Type A, the Eulerian representations are well-studied and have many elegant but mysterious connections to rings naturally associated with the braid arrangement. Here, we unify these results and show that they hold for any reflection group of *coincidental type*—that is, S_n, B_n, H_3 or the dihedral group $I_2(m)$—by giving six characterizations of the Eulerian representations, including as components of the associated graded Varchenko–Gelfand ring $\mathcal{V}$. As a consequence, we show that Solomon's descent algebra contains a commutative subalgebra generated by sums of elements with the same descent number if and only if W is coincidental. More generally, when W is any real, finite reflection group, we give a case-free construction of a family of Eulerian representations described by a flat-decomposition of the ring $\mathcal{V}$.

Keywords: Eulerian idempotents, hyperplane arrangements, Varchenko–Gelfand ring, configuration spaces, Solomon's descent algebra, reflection groups

1 Introduction

This abstract studies two related families of orthogonal idempotents within the group algebra $\mathbb{R}W$ of any *finite reflection group* W, that decompose the regular representation into W-representations recurring many times in the literature.

Recall that a reflection group is a finite subgroup W of the general linear group $GL(V)$ for $V = \mathbb{R}^r$, generated by orthogonal *reflections* through various *reflecting hyperplanes* H, each of which is a codimension one linear subspace of V. One then has an associated *reflection hyperplane arrangement*[1] $\mathcal{A} = \{H_i\}_{i \in I}$, and its *intersection lattice* $\mathcal{L}(\mathcal{A})$, which is simply the collection of all intersection subspaces (e.g. flats) $X = H_1 \cap \cdots \cap H_m$ of subsets of the hyperplanes. Work of Saliola [23, 24, 25] associates to each such intersection X an idempotent $\mathfrak{e}_X$ in the *face (Tits) algebra* of $\mathcal{A}$, and $\{\mathfrak{e}_X\}_{X \in \mathcal{L}(A)}$ turn out to give a complete family of orthogonal idempotents for this algebra; we call these *flat idempotents*[2] of $\mathcal{A}$.

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[1]We will write $\mathcal{A}(W)$ when we wish to specify W and $\mathcal{A}$ when W is clear from context.

[2]The family of idempotents depends on a choice of (homogeneous) *section map*. The technical definitions will not play a role in the statement of our results; for this reason, we omit them. The curious reader should consult [2] for an in-depth discussion.

We group the $\mathfrak{e}_X$ further into two coarser, complete families of orthogonal idempotents. Letting

$$[X] := \{Y = wX : w \in W\} \subset \mathcal{L}(\mathcal{A})$$

denote the W-orbit of the intersection space X, we will consider the idempotents

$$\mathfrak{e}_{[X]} := \sum_{Y \in [X]} \mathfrak{e}_Y \tag{1.1}$$

as $[X]$ runs through the W-orbits $\mathcal{L}(\mathcal{A})/W$ on $\mathcal{L}(A)$, which we call *flat-orbit idempotents*[3]. The $\mathfrak{e}_{[X]}$ can be realized as idempotents in $\mathbb{R}W$ via a result of Bidigare [9], and in this case they correspond to idempotents introduced by F. Bergeron, N. Bergeron, Howlett and Taylor in [6]. There are even coarser idempotents

$$\mathfrak{e}_k := \sum_{\substack{Y \in \mathcal{L}(\mathcal{A}) \\ \dim_V(Y) = k}} \mathfrak{e}_Y \tag{1.2}$$

for $k = 0, 1, \ldots, r$. This last family will be called the *Eulerian idempotents* for W and can also be realized in $\mathbb{R}W$.

Our goal in this abstract (based upon the paper [10]) is to analyze two families of representations. First, the *Eulerian representations* $\{\mathbb{R}W\mathfrak{e}_k\}_{0 \le k \le r}$ when W is a reflection group of *coincidental type*[4]; that is, a finite, irreducible real reflection group of rank r whose exponents (equivalently, degrees) can be expressed in terms of an *exponent gap* g:

$$1, 1+g, 1+2g, \ldots, 1+(r-1)g.$$

These are exactly reflection groups of Types A and B, H_3, and the dihedral group $I_2(m)$. Second, we study the family of flat-orbit idempotents $\{\mathbb{R}W\mathfrak{e}_{[X]}\}_{[X] \in \mathcal{L}(\mathcal{A})/W}$ for any real finite reflection group W. In both cases, we will give a description of these representations in terms of well-known topological spaces naturally associated with $\mathcal{A}$.

Outline

The remainder of the abstract proceeds as follows. Section 2 provides motivation for our results by reviewing previous work on the Type A and B Eulerian representations. Section 3 then discusses our methods and defines key constructions, including the Varchenko–Gelfand ring and its associated graded. In Section 4, we turn to the case that W is a coincidental reflection group. We show that $\mathbb{R}W$ contains a subalgebra generated by elements with the same number of descents if and only if W is coincidental (Theorem 4.3), and give six characterizations of the Eulerian representations in this setting (Theorem 4.5). Finally, in Section 5 we give a case-free construction of the family of representations $\{\mathbb{R}W\mathfrak{e}_{[X]}\}_{[X] \in \mathcal{L}(\mathcal{A})/W}$ for any real, finite reflection group (Theorem 5.3).

[3]The flat-orbit idempotents are sometimes also referred to as Eulerian idempotents in the literature.

[4]These groups are called *good reflection groups* by Aguiar–Mahajan in [2].

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2 Previous work on the Eulerian representations

2.1 Motivating story: Type A

The Eulerian idempotents $\mathfrak{e}_k$ described in (1.2) generalize the *Type A Eulerian idempotents*, which have been studied extensively beginning in the late 1980s, when they were discovered independently by both Reutenauer in [21] and Gerstenhaber–Schack in [15].

Reutenauer introduced a family of idempotents $\{\mathfrak{e}_k\}_{0\leq k\leq n-1}$ in $\mathbb{R}\,S_n$ as part of his work on the Campbell–Baker–Hausdorff formula. In [14], Garsia and Reutenauer showed that these idempotents can be defined via the generating function

$$\sum_{k=0}^{n-1} t^{k+1}\mathfrak{e}_k = \sum_{w\in S_n} \binom{t-1+n-\operatorname{des}(w)}{n} w, \tag{2.1}$$

where one defines the *Coxeter descent set* for any Coxeter system (W, S),

$$\operatorname{Des}(w) := \{s \in S : \ell(w) > \ell(ws)\}$$

and the *descent number*

$$\operatorname{des}(w) = |\operatorname{Des}(w)|.$$

By contrast, Gerstenhaber and Schack were interested in giving a Hodge-type decomposition of Hochschild homology, a homology theory for associative algebras. Earlier in [4], Barr had defined a "shuffle product" $\mathcal{S}(S_n)$ (*Barr's element*), which can be phrased in the language of descents as[5]

$$\mathcal{S}(S_n) := \sum_{s\in S} \sum_{\substack{w\in S_n \\ \operatorname{Des}(w)\subset\{s\}}} w.$$

Gerstenhaber and Schack built upon Barr's work, proving that $\mathcal{S}(S_n)$ 1) acts semisimply on $\mathbb{R}\,S_n$ with eigenvalues $\sigma_k = 2^{k+1} - 2$ for $0 \leq k \leq n-1$ and 2) commutes with

[5]Barr's element was originally defined by tensoring $\mathcal{S}(S_n)$ as defined above with the sign representation.

the Hochschild boundary operator. Using Lagrange interpolation[6], they constructed a family of idempotents that are polynomials in $\mathcal{S}(S_n)$ and for each k, project onto the σ_k-eigenspace of $\mathcal{S}(S_n)$. While it is not obvious that these viewpoints yield the same result, Loday showed in [18] that the σ_k-eigenspace projectors are precisely the $\mathfrak{e}_k$ in (2.1).

Example 2.2. When $n = 3$, one has $\mathcal{S}(S_3) = 2 + (12) + (23) + (123) + (132)$ and

$$\begin{aligned}
\mathfrak{e}_0 &= \frac{1}{6}\big(2 - (12) - (23) - (123) - (132) + 2(13)\big),\\
\mathfrak{e}_1 &= \frac{1}{2}\big(1 - (13)\big),\\
\mathfrak{e}_2 &= \frac{1}{6}\big(1 + (12) + (23) + (13) + (123) + (132)\big).
\end{aligned}$$

From our perspective, perhaps the most interesting aspect of the Type A Eulerian idempotents are the properties of the S_n-representations $\mathbb{R}\,S_n\mathfrak{e}_k$ they generate. In the $k = 0$ case, $\mathbb{R}\,S_n\mathfrak{e}_0 \otimes \mathrm{sgn}_{S_n}$ is isomorphic to the top homology of the partition lattice Π_n (see Barcelo [3], Joyal [17], Wachs [29]), and $\mathbb{R}\,S_n\mathfrak{e}_0$ is isomorphic to the multilinear component of the free Lie algebra on n generators (see Garsia [13], Reutenauer [22]).

Even more surprising is the following "folklore" fact:

$$\mathbb{R}\,S_n\mathfrak{e}_k \cong_{S_n} H^{(n-1-k)d}(\mathrm{Conf}_n(\mathbb{R}^d);\mathbb{R})$$

when $d \geq 3$ and odd, where $\mathrm{Conf}_n(\mathbb{R}^d)$ is the space of n distinct labeled points in $\mathbb{R}^d$. This can be deduced by comparing a result of Sundaram and Welker for subspace arrangements [27, Thm 4.4(iii)] with descriptions of the characters of $\mathbb{R}\,S_n\mathfrak{e}_k$ by Hanlon [16]; see Sundaram [26, Sec. 2: Thm. 2.2, Eq. 23] for history, or Early–Reiner [12, Eq. 1.1]. The space $H^*\,\mathrm{Conf}_n(\mathbb{R}^d)$ is well-studied and connects the $\mathfrak{e}_k$ to other rings associated with the braid arrangement (to be discussed in Section 3). These Type A properties are the inspiration for our results.

2.3 Hint at a more general phenomenon: Type B

As in Type A, the Eulerian idempotents in (1.2) generalize earlier work by F. Bergeron and N. Bergeron in [5] for Type B. Like Garsia and Reutenauer, Bergeron and Bergeron define the *Type B Eulerian idempotents* as elements in $\mathbb{R}\,B_n$ using the generating function[7]

$$\sum_{k=0}^{n} t^k \mathfrak{e}_k = \sum_{w \in B_n} \binom{\frac{t-1}{2} + n - \mathrm{des}(w)}{n} w. \tag{2.2}$$

[6] Anytime an element x of an algebra A acts semisimply on A, one can produce a family of idempotents that project onto each eigenspace of x via the Lagrange interpolation formula. This will be important in Sections 4 and 5; idempotents constructed in this way will be called the *eigenspace projectors of x*.

[7] The idempotents that Bergeron and Bergeron define are actually obtained by tensoring the $\mathfrak{e}_k$ in (2.2) with the sign representation.

Like Gerstenhaber and Schack, they show that the $\mathfrak{e}_k$ give a Hodge-type decomposition of Hochschild homology for a commutative hyperoctahedral algebra[8], although they do not use a Barr-like element to do so.

In [7], N. Bergeron gave a description of the B_n representation $\mathbb{R}\, B_n \mathfrak{e}_0 \otimes \mathrm{sgn}_{B_n}$ as the top homology of the intersection lattice $\mathcal{L}(\mathcal{A})$ for the Type B hyperplane arrangement—thus hinting that the features of the Eulerian representations in Type A might hold more generally. We will show this to be true.

3 Methods and Key Constructions

We aim to describe the Eulerian representations in terms of three closely related spaces:

1. The *associated graded Varchenko–Gelfand ring* $\mathcal{V}(\mathcal{A})$, to be defined and discussed in Section 3.1 below. Intuitively, $\mathcal{V}(\mathcal{A})$ can be thought of as a commutative version of the (better studied) Orlik–Solomon algebra;

2. The cohomology[9] of a "d-dimensionally thickened" hyperplane complement

$$\mathcal{M}^d_{\mathcal{A}} := V \otimes \mathbb{R}^d - \Big(\bigcup_{H_i \in \mathcal{A}} H_i \otimes \mathbb{R}^d \Big); \text{ and}$$

3. The homology of open intervals (V, X) in $\mathcal{L}(\mathcal{A})$: for each X in $\mathcal{L}(\mathcal{A})$, the set-wise W-stabilizer subgroup N_X acts on the order complex $\Delta(V, X)$ and on its homology $H_*(V, X)$, which is nonvanishing only in degree $\mathrm{codim}(X) - 2$. We will abbreviate the name of this N_X-representation[10] as

$$\mathrm{WH}_X := H_{\mathrm{codim}(X)-2}(V, X)$$

and define from it an induced W-representation

$$\mathrm{WH}_{[X]} := \mathrm{Ind}^W_{N_X} \mathrm{WH}_X \otimes \det_{V/X},$$

where $\det_{V/X}(w)$ is the determinant of $w \in N_X$ acting on V/X.

The relationship between the associated graded Varchenko–Gelfand ring and the Orlik–Solomon algebra is best understood through $\mathcal{M}^d_{\mathcal{A}}$. In the $d = 2$ case, $\mathcal{M}^2_{\mathcal{A}}$ is the complexification of the hyperplane complement $\mathcal{M}^1_{\mathcal{A}}$, and $H^*(\mathcal{M}^2_{\mathcal{A}})$ is (W-equivariantly) isomorphic to the Orlik–Solomon algebra. A recent result of Moseley in [19] shows that when $d \geq 3$ and odd, $H^*(\mathcal{M}^d_{A})$ (W-equivariantly) describes $\mathcal{V}(\mathcal{A})$ as a graded ring.

[8] A hyperoctahedral algebra is an algebra with an involutive automorphism.

[9] Henceforth, all cohomology and homology groups are assumed to have coefficients in $\mathbb{R}$.

[10] The notation here refers to the fact that WH_X is a summand of *Whitney Homology*.

In the case of the Braid arrangement, $\mathcal{M}^d_{\mathcal{A}(S_n)} = \mathrm{Conf}_n(\mathbb{R}^d)$ and there is a description of the cohomology due to F. Cohen [11] for d of any parity. Similarly, in Type B, $\mathcal{M}^d_{\mathcal{A}(B_n)}$ is $\mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$, the $\mathbb{Z}_2$ *orbit configuration space* with cohomology presentation given by Xicotencatl [30] for any d.

Our contribution will be to connect all of these spaces—which already have well-known relationships to each other—to the Eulerian idempotents (in all of their guises). In doing so, we will avoid any character computations and rather tie together various equivariant versions of results in the literature, such as work by Aguiar–Mahajan [2], Reiner–Saliola–Welker [20], and Sundaram–Welker [27]. The main novelties in our methods are 1) to define generalizations and extensions of Barr's element and study their action on $\mathbb{R}W$ and 2) to further analyze the associated graded Varchenko–Gelfand ring in order to use it as a stepping-stone between other spaces.

3.1 The Varchenko–Gelfand ring

Every real hyperplane H in a vector space V partitions $V \setminus H$ into two disjoint half-spaces, denoted H^+ and H^-. For a arrangement $\mathcal{A} = \{H_i\}_{i \in I}$, the ring of locally constant functions on $\mathcal{M}^1_{\mathcal{A}}$ is precisely $H^0(\mathcal{M}^1_{\mathcal{A}})$, and has a filtration by *Heaviside functions*, where for each $i \in I$, the Heaviside function $x_i \in H^0(\mathcal{M}^1_{\mathcal{A}})$ is given by

$$x_i(v) := \begin{cases} 1 & v \in H_i^+ \\ 0 & v \in H_i^-. \end{cases}$$

In [28], Varchenko and Gelfand use these Heaviside functions to describe $H^0(\mathcal{M}^1_{\mathcal{A}})$, which we now review[11]. Define $E(\mathcal{A}) := \mathbb{R}[e_i]_{H_i \in \mathcal{A}}$, and for a k-tuple of hyperplanes $M = (H_1, \ldots, H_k)$, write $e_M = e_1 e_2 \ldots e_k$. The set M is *dependent* if $\mathrm{codim}_V \left(\bigcap_{H_i \in M} H_i \right) < |M|$ and *independent* otherwise. If M is minimally dependent—that is, for any $H_j \in M$, one has $M \setminus H_j$ independent—M is called a *circuit*. Any circuit C of $\mathcal{A}$ can be uniquely partitioned into two sets, C^+ and C^- such that

$$\bigcap_{H_i \in C^+} H_i^+ \cap \bigcap_{H_j \in C^-} H_j^- = \emptyset.$$

Theorem 3.2 (Varchenko–Gelfand [28, Thm. 4.5])**.** *The ring morphism defined by*

$$\begin{array}{rcl} \Psi : E(\mathcal{A}) & \longrightarrow & H^0(\mathcal{M}^1_{\mathcal{A}}) \\ e_i & \longmapsto & x_i \end{array}$$

induces a ring isomorphism $H^0(\mathcal{M}^1_{\mathcal{A}}) \cong E(\mathcal{A})/\mathcal{J}$, with $\mathcal{J} = \ker(\Psi)$ generated by:

[11] For simplicity, we will assume that $\mathcal{A}$ is a central arrangement, although Varchenko and Gelfand also give a presentation of $H^0(\mathcal{M}^1_{\mathcal{A}})$ for non-central arrangements in [28].

1. *$e_i^2 - e_i$ for $H_i \in \mathcal{A}$,*

2. *For every circuit C in $\mathcal{A}$,*

$$\prod_{H_i \in C^+} e_i \prod_{H_j \in C^-} (e_j - 1) - \prod_{H_i \in C^+} (e_i - 1) \prod_{H_j \in C^-} e_j.$$

The ring $E(\mathcal{A})/\mathcal{J}$ is the *Varchenko–Gelfand ring*. The map Ψ imposes an ascending filtration on $H^0(\mathcal{M}^1_{\mathcal{A}})$ obtained from the natural degree grading on $E(\mathcal{A})$: the m^{th} layer in the filtration is the span of monomials in the variables x_i having degree at most m. We will call the associated graded ring with respect to this filtration the *associated graded Varchenko–Gelfand ring*, which has the following presentation:

Definition 3.3 (Associated graded Varchenko–Gelfand ring)**.** For a central hyperplane arrangement $\mathcal{A}$, let $\mathcal{V}(\mathcal{A}) := E(\mathcal{A})/\mathcal{I}$ be the *associated graded Varchenko–Gelfand ring*, where $\mathcal{I}$ is generated by:

1. e_i^2 for each $H_i \in \mathcal{A}$;

2. For every circuit C in $\mathcal{A}$

$$\tilde{\partial}(e_C) := \sum_{H_i \in C} c(i) e_{C \setminus H_i},$$

   where

$$c(i) = \begin{cases} 1 & \text{if } H_i \in C^-, \\ -1 & \text{if } H_i \in C^+. \end{cases}$$

Let $\mathcal{V}^k(\mathcal{A})$ be the k-th graded piece of $\mathcal{V}(\mathcal{A})$ spanned by degree k polynomials in the e_i.

Remark 3.4. There is a finer decomposition of $\mathcal{V}(\mathcal{A})$ by flats $X \in \mathcal{L}(\mathcal{A})$. Define for each $X \in \mathcal{L}(\mathcal{A})$ the $\mathbb{R}$-subspace $E_X(\mathcal{A}) := \mathbb{R}\{e_M : \left(\bigcap_{H_i \in M} H_i\right) = X\}$. Note that $E(\mathcal{A})$ has a vector space decomposition $E(\mathcal{A}) = \bigoplus_{X \in \mathcal{L}(\mathcal{A})} E_X(\mathcal{A})$. We show in [10, Thm. 5.5] that this decomposition holds for $\mathcal{V}(\mathcal{A})$ as well:

$$\mathcal{V}(\mathcal{A}) = \bigoplus_{X \in \mathcal{L}(\mathcal{A})} \mathcal{V}_X(\mathcal{A}),$$

where $\mathcal{V}_X(\mathcal{A}) := E_X(\mathcal{A})/\mathcal{I} \cap E_X(\mathcal{A})$.

Example 3.5 (Braid Arrangement)**.** Let $W = S_n$. The braid arrangement has hyperplanes of the form $H_{ij} := \{x_i - x_j = 0\}$. One can show that the circuits needed to generate $\mathcal{I}$ are $C^+ = \{H_{ij}, H_{jk}\}$, $C^- = \{H_{ik}\}$ for $1 \leq i < j < k \leq n$. Thus $\mathcal{I}$ is generated by

1. e_{ij}^2 for every $H_{ij} \in \mathcal{A}$

2. $e_{ij}e_{jk} - e_{ij}e_{ik} - e_{jk}e_{ik}$ for every $H_{ij}, H_{jk}, H_{ik} \in \mathcal{A}$.

This recovers F. Cohen's presentation of $H^* \operatorname{Conf}_n(\mathbb{R}^d)$ when $d \geq 3$ is odd [11].

The ring $\mathcal{V}(\mathcal{A})$ will play an essential role in Sections 4 and 5.

4 Results for coincidental reflection groups

We first turn to the case where W is a coincidental reflection group of rank r with Coxeter generators S. Recall that because W is coincidental, the exponents (equivalently, degrees) of W can be expressed in terms of the *exponent gap* g as $1, 1+g, 1+2g, \ldots, 1+(r-1)g$.

Below are the ranks r, exponent gaps g, and exponents for the coincidental groups:

W	r	g	**exponents**
S_n	$n-1$	1	$1, 2, \cdots, n-1$
B_n	n	2	$1, 3, \cdots, 2n-1$
H_3	3	4	$1, 5, 9$
$I_2(m)$	2	$m-2$	$1, m-1$

One might wonder why we focus on coincidental reflection groups. For our purposes, it is because this class of group is well-behaved with respect to the *restriction arrangement* $\mathcal{A}^X := \{H \cap X : H \in \mathcal{A}, X \not\subset H\}$ (see Figure 1 for an example). In particular, using a result of Abramenko [1, Prop 5.], Aguiar–Mahajan [2, Thm. 5.28] show the following.

Theorem 4.1 (Abramenko, Aguiar–Mahajan)**.** *Let $\mathcal{A}$ be an irreducible reflection arrangement. Then for any $X \in \mathcal{L}(\mathcal{A})$, the restriction arrangement $\mathcal{A}^X$ is also a reflection arrangement (not necessarily irreducible) if and only if W is of coincidental type. In this case, the combinatorics of $\mathcal{A}^X$ depends only on the dimension of X.*

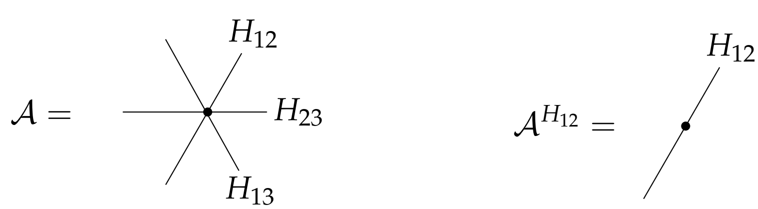


Figure 1: The rank-two braid arrangement (left) and its restriction at H_{12} (right).

Our primary tool will be a generalization of Barr's element in $\mathbb{R}W$. Define

$$\mathcal{S}(W) := \sum_{s \in S} \sum_{\substack{w \in W \\ \mathrm{Des}(w) \subset \{s\}}} w.$$

Proposition 4.2. *The element $\mathcal{S}(W)$ acts semisimply on $\mathbb{R}W$. When W is coincidental, $\mathcal{S}(W)$ has eigenvalues $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ in $\mathbb{Z}_{\geq 0}$, where σ_k counts the number of rays (i.e. halfspaces for lines) lying in $\mathcal{A}^X$ for any k-dimensional flat $X \in \mathcal{L}(\mathcal{A})$. Furthermore, in the coincidental case, the eigenspace projectors of $\mathcal{S}(W)$ are precisely the Eulerian idempotents in (1.2).*

As a consequence, we are able to determine when the *Eulerian subspace*

$$\mathcal{E}(W) := \Big\{ \sum_{w \in W} c_w w : c_w = c_{w'} \in \mathbb{R} \text{ if } \mathrm{des}(w) = \mathrm{des}(w') \Big\} \subset \mathbb{R}W$$

is a commutative subalgebra, as it is known to be in Types A and B (see [14] and [5].)

Theorem 4.3. *The Eulerian subspace $\mathcal{E}(W)$ is a subalgebra if and only if W is coincidental. Moreover, when the Eulerian subalgebra exists, it is always commutative.*

Proof idea. Write $\mathcal{E}(W) = \mathcal{E}$ and $\mathcal{S}(W) = \mathcal{S}$. The Eulerian subspace $\mathcal{E}$ always has dimension $r+1$ and contains $\mathcal{S}$. If $\mathcal{E}$ is a subalgebra, it will contain the subalgebra $\mathbb{R}\,\mathcal{S}$ generated by $\mathcal{S}$. Since $\mathcal{S}$ acts semisimply on $\mathbb{R}\,W$, the algebra $\mathbb{R}\,\mathcal{S}$ will be commutative and have dimension equal to the number of distinct eigenvalues of $\mathcal{S}$. In the coincidental case, $\mathcal{S}$ has $r+1$ eigenvalues by Proposition 4.2, so $\mathbb{R}\,\mathcal{S} = \mathcal{E}$. In the non-coincidental case, one can show that $\mathcal{S}$ always has more than $r+1$ eigenvalues and so $\mathbb{R}\,\mathcal{S} \not\subseteq \mathcal{E}$. $\square$

4.4 Main Results

We will now develop a unified theory of Eulerian representations for coincidental reflection groups. Recall that $(t)_k := (t)(t+1)\ldots(t+k-1)$ is the rising factorial. Let

$$\beta_{W,k}(t) := \frac{\left(\frac{t+g-1}{g} - k\right)_k \left(\frac{t+1}{g}\right)_{r-k}}{\left(\frac{2}{g}\right)_r},$$

where g is the exponent gap of W. Our main theorem is a description of the Eulerian representations, which follows from Theorem 4.1 and Proposition 4.2 in conjunction with results from Aguiar–Mahajan [2], Bidigare [9], Bidigare–Hanlon–Rockmore [8], Moseley [19], Reiner–Saliola–Welker [20] and Sundaram–Welker [27].

Theorem 4.5. *When W is a coincidental reflection group of rank r, for each $0 \leq k \leq r$, the following are equivalent as W-representations:*

1. *The k-th graded piece of the associated graded Varchenko–Gelfand ring, $\mathcal{V}^k(\mathcal{A})$;*
2. *$H^{k(d-1)}(\mathcal{M}^d_{\mathcal{A}})$ for $d \geq 3$ and odd;*
3. *$\bigoplus_{[X]} \mathrm{WH}_{[X]}$, where the direct sum is over all $[X] \in \mathcal{L}(\mathcal{A})/W$ with $\mathrm{codim}_V(X) = k$;*
4. *The σ_{r-k} eigenspace of $\mathcal{S}(W)$ in $\mathbb{R}\,W$;*
5. *The left $\mathbb{R}\,W$-module $\mathbb{R}\,W\mathfrak{e}_{r-k}$, where $\mathfrak{e}_{r-k}$ is an Eulerian idempotent as in (1.2);*
6. *The left $\mathbb{R}\,W$-module $\mathbb{R}\,WE_{r-k}$, where $\{E_k\} \subset \mathcal{E}(W)$ are idempotents defined by*

$$\sum_{k=0}^{r} t^k E_k := \sum_{w \in W} \beta_{W,\mathrm{des}(w)}(t) \cdot w.$$

Theorem 4.5 recovers all known descriptions of the Type A and B Eulerian representations and also implies that the Type B Eulerian representations are isomorphic to the non-trivial pieces of the $\mathbb{Z}_2$-orbit configuration space $H^* \mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$ when $d \geq 3$ and odd. See Table 1 for an example of the Eulerian representations in Type A.

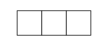
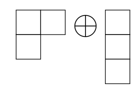
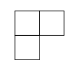
Eulerian representation	**Configuration space cohomology**	**Irreducible decomposition**
$\mathbb{R}\, S_3\mathfrak{e}_2 = \sigma_2$-eigenspace	$\mathcal{V}^0(\mathcal{A}) = H^0\,\mathrm{Conf}_3(\mathbb{R}^d)$	
$\mathbb{R}\, S_3\mathfrak{e}_1 = \sigma_1$-eigenspace	$\mathcal{V}^1(\mathcal{A}) = H^{1(d-1)}\,\mathrm{Conf}_3(\mathbb{R}^d)$	
$\mathbb{R}\, S_3\mathfrak{e}_0 = \sigma_0$-eigenspace	$\mathcal{V}^2(\mathcal{A}) = H^{2(d-1)}\,\mathrm{Conf}_3(\mathbb{R}^d)$	

Table 1: The Eulerian representations for S_3 when $d \geq 3$ and odd.

5 Results for arbitrary reflection groups

We now turn to the case that W is an arbitrary finite Coxeter group with reflection arrangement $\mathcal{A}$. For our purposes, the key differences in this setting are 1) in general $\mathcal{A}^X$ is not necessarily a reflection arrangement, and 2) if $\dim(X) = \dim(Y)$ for $X, Y \in \mathcal{L}(\mathcal{A})$, it is not necessarily true that $\mathcal{L}(\mathcal{A}^X) \cong \mathcal{L}(\mathcal{A}^Y)$. Hence the methods used in the coincidental case are not applicable. To combat this problem, we do two new things. First, we utilize the flat grading $\mathcal{V}(\mathcal{A}) = \oplus_{X\in\mathcal{L}(\mathcal{A})} \mathcal{V}_X(\mathcal{A})$ discussed in Remark 3.4. Second, we introduce an element $\mathcal{T} \in \mathbb{R}\, W$ whose eigenspaces will be indexed by flat orbits $[X] \in \mathcal{L}(\mathcal{A})/W$.

Definition 5.1. Let

$$\mathcal{T} := \sum_{T\subset S} \sum_{\substack{w\in W \\ \mathrm{Des}(w)\subset T}} c_T w$$

where the coefficients $\{c_T\}_{T\subset S} \subset \mathbb{R}$ are positive and algebraically independent over $\mathbb{Q}$.

Proposition 5.2. *The element $\mathcal{T}$ acts semisimply on $\mathbb{R}\, W$ with eigenvalues $\tau_{[X]}$ for $[X] \in \mathcal{L}(\mathcal{A})/W$, such that $\tau_{[X]} = \tau_{[Y]}$ if and only if $[X] = [Y]$. Furthermore, the eigenspace projectors of τ recover a family of flat-orbit idempotents as in* (1.1).

We can thus use properties of $\mathcal{T}$ to give a case-free proof of the following result.

Theorem 5.3. *For each $[X] \in \mathcal{L}(\mathcal{A})/W$, the following are isomorphic as W-representations:*

1. *The direct summand $\mathrm{Ind}_{N_X}^W \mathcal{V}_X(\mathcal{A})$, where N_X is the set-wise stabilizer of X;*
2. *The representation $\mathrm{WH}_{[X]} = \mathrm{Ind}_{N_X}^W WH_X \otimes \det_{V/X}$;*
3. *The $\tau_{[X]}$-eigenspace of $\mathcal{T}$ in $\mathbb{R}\, W$ and*
4. *The representation $\mathbb{R}\, W\mathfrak{e}_{[X]}$ generated by a flat-orbit idempotent $\mathfrak{e}_{[X]}$ as in* (1.1).

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