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A combinatorial Chevalley formula for semi-infinite flag manifolds and its applications

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Abstract. We give a combinatorial Chevalley formula for an arbitrary weight, in the torus-equivariant K-group of semi-infinite flag manifolds, which is expressed in terms of the quantum alcove model. As an application, we prove the Chevalley formula for anti-dominant fundamental weights in the (small) torus-equivariant quantum K-theory $QK_T(G/B)$ of the flag manifold G/B; this has been a longstanding conjecture. We also discuss the Chevalley formula for partial flag manifolds G/P. Moreover, in type A_{n-1}, we prove that the so-called quantum Grothendieck polynomials indeed represent Schubert classes in the (non-equivariant) quantum K-theory $QK(SL_n/B)$.

Résumé. Nous donnons une formule combinatoire de Chevalley pour un poids arbitraire, dans la K-théorie équivariante des variétés de drapeau semi-infinies, exprimée en termes du modèle des alcôves quantique. En tant qu'application, nous prouvons la formule de Chevalley pour les poids fondamentaux anti-dominantes dans la (petite) K-théorie quantique équivariante $QK_T(G/B)$ des variétés de drapeau G/B; c'était une conjecture depuis longtemps. Nous discutons également de la formule plus générale de Chevalley pour les variétés de drapeau partielles G/P. De plus, dans le type A_{n-1}, nous montrons que les polynômes de Grothendieck quantiques représentent bien les classes de Schubert dans la K-théorie quantique (non-équivariante) $QK(SL_n/B)$.

Keywords: semi-infinite flag manifold, Chevalley formula, quantum Bruhat graph, quantum LS paths, quantum alcove model.

1 Introduction

This paper is concerned with a geometric application of the combinatorial model known as the *quantum alcove model*, introduced in [9]. Its precursor, the alcove model of the

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first author and Postnikov, was used to uniformly describe the *Chevalley formula* in the equivariant K-theory of flag manifolds G/B [13]. Also, the quantum alcove model was used to uniformly describe certain crystals of affine Lie algebras (single-column *Kirillov–Reshetikhin crystals*) and *Macdonald polynomials* specialized at $t = 0$ [12]. The objects of the quantum alcove model (indexing the crystal vertices and the terms of Macdonald polynomials) are paths in the *quantum Bruhat graph* on the Weyl group, introduced by Brenti-Fomin-Postnikov. In this paper we complete the above picture, by extending to the quantum alcove model the geometric application of the alcove model, namely the K-theory Chevalley formula.

To achieve our goal, we need to consider the so-called *semi-infinite flag manifold* $\mathbf{Q}_G$. We give a Chevalley formula for an arbitrary weight in the $T \times \mathbb{C}^*$-equivariant K-group $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ of $\mathbf{Q}_G$, which is described in terms of the quantum alcove model. In [6] and [14], the Chevalley formulas for $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ were originally given in terms of the *quantum LS path model* in the case of a dominant and an anti-dominant weight, respectively. For a general (not dominant nor anti-dominant) weight, there is no quantum LS path model, but there is a quantum alcove model. Hence, in order to obtain a Chevalley formula for an arbitrary weight, we first need to translate the formulas above to the quantum alcove model by using the weight-preserving bijection between the two models given by Proposition 7. Based on these translated formulas (Theorems 8 and 9), we obtain a Chevalley formula (Theorem 10) for an arbitrary weight.

The study of the equivariant K-group of semi-infinite flag manifolds was started in [6]. A breakthrough in this study is [4] (see also [5]), in which Kato established a $\mathbb{Z}[P]$-module isomorphism from the (small) T-equivariant quantum K-theory $QK_T(G/B)$ of the finite-dimensional flag manifold G/B onto (a version of) the T-equivariant K-group $K'_T(\mathbf{Q}_G)$ of $\mathbf{Q}_G$; here P is the weight lattice generated by the fundamental weights ϖ_i, $i \in I$. Here we should mention that in [4], he also established a $\mathbb{Z}[P]$-module embedding of (a certain localization of) the T-equivariant K-group of the affine Grassmannian into the T-equivariant K-group of the full semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$, which is a certain inductive limit of copies of $K_T(\mathbf{Q}_G)$, thus verifying a conjectural K-theoretic generalization of Peterson's isomorphism proposed by Lam-Li-Mihalcea-Shimozono ([7]). The isomorphism above sends each (opposite) Schubert class in $QK_T(G/B)$ to the corresponding semi-infinite Schubert class in $K'_T(\mathbf{Q}_G)$; moreover, it respects the quantum multiplication in $QK_T(G/B)$ with the class of the line bundle associated to an anti-dominant fundamental weight and the tensor product in $K'_T(\mathbf{Q}_G)$ with the class of the line bundle associated to the corresponding anti-dominant fundamental weight. Based on this result, a longstanding conjecture on the multiplicative structure of $QK_T(G/B)$, i.e., the Chevalley formula (Theorem 12) for anti-dominant fundamental weights $-\varpi_k$, $k \in I$, for $QK_T(G/B)$ is proved by our Chevalley formula for $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ specialized to $q = 1$. In Section 5, we also discuss the quantum K-theory Chevalley formula for partial flag manifolds G/P.

As an application of our quantum K-theory Chevalley formula, we prove an important conjecture for the non-equivariant quantum K-theory $QK(SL_n/B)$ of the type A_{n-1} flag manifold (Theorem 13): the *quantum Grothendieck polynomials*, introduced in [10], represent Schubert classes in $QK(SL_n/B)$. Thus, we generalize the results of [3], where the *quantum Schubert polynomials* are constructed as representatives for Schubert classes in the quantum cohomology of SL_n/B. Therefore, we can use quantum Grothendieck polynomials to compute any structure constant in $QK(SL_n/B)$ (with respect to the Schubert basis); indeed, we just need to expand their products in the basis they form, which is done by [10, Algorithm 3.28], see [10, Example 7.4]. This is important, since computing even simple products in quantum K-theory is notoriously difficult.

2 Background on the combinatorial models

2.1 Root systems

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}}$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Given $\alpha \in \Phi$, we let $\mathrm{sgn}(\alpha)$ be 1 or -1 depending on α being positive or negative, and $|\alpha| := \mathrm{sgn}(\alpha)\alpha$. Let $\rho := \frac{1}{2}(\sum_{\alpha\in\Phi^+}\alpha)$. Let θ be the highest root, and $\alpha_i \in \Phi^+$ the *simple roots*, for i in an indexing set I. We denote $\langle\cdot,\cdot\rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given $\alpha \in \Phi$, we consider the *coroot* $\alpha^\vee$ and reflection s_α. The root and coroot lattices are denoted by Q and $Q^\vee$, as usual, while the positive part of the coroot lattice is denoted by $Q^{\vee,+}$. The *weight lattice P* is generated by the *fundamental weights ϖ_i*, for $i \in I$. Let P^+ be the set of *dominant weights*.

Let W be the *Weyl group*, with length function $\ell(\cdot)$ and longest element $w_\circ$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\ell(ws_\alpha) = \ell(w)+1$, where $\alpha \in \Phi^+$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle\lambda,\alpha^\vee\rangle = k\}$. These reflections generate the *affine Weyl group* $W_{\mathrm{af}} = W \ltimes Q^\vee$ for the *dual root system* $\Phi^\vee$. The hyperplanes $H_{\alpha,k}$ divide the vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves*. The *fundamental alcove* is denoted by $A_\circ$.

The *quantum Bruhat graph* $\mathrm{QB}(W)$ on W is defined by adding downward (quantum) edges, denoted $w \lhd ws_\alpha$, to the covers of the Bruhat order, i.e., the edges of $\mathrm{QB}(W)$ are:

$$w \xrightarrow{\alpha} ws_\alpha \;\text{ if }\; w \lessdot ws_\alpha \;\text{ or }\; \ell(ws_\alpha) = \ell(w) - 2\langle\rho,\alpha^\vee\rangle + 1\,, \quad \text{where } \alpha \in \Phi^+\,.$$

We define the *weight* of an edge $w \xrightarrow{\alpha} ws_\alpha$ to be either $\alpha^\vee$ or 0, depending on whether it is a quantum edge or not, respectively. Then the weight of a directed path is the sum of the weights of its edges. It turns out that the weight of a shortest directed path from v to w is independent of the mentioned path, so we will denote it by $\mathrm{wt}(w \Rightarrow v)$; see [12].

For the remainder of this section, we fix $\lambda \in P^+$. Let W_J be the stabilizer of λ, as a parabolic subgroup with $J \subset I$ and root system Φ_J. We denote the set of minimum-length coset representatives for W/W_J by W^J, and the minimum-length coset representative of wW_J by $\lfloor w \rfloor$. We consider the *parabolic quantum Bruhat graph* on W^J, denoted by $\mathrm{QB}(W^J)$; this generalizes $\mathrm{QB}(W)$, see [11]. Its directed edges are labeled by $\alpha \in \Phi^+ \setminus \Phi_J^+$. The upward edges are the covers of the Bruhat order on W^J, while the downward (quantum) edges $w \xrightarrow{\alpha} \lfloor ws_\alpha \rfloor$ are given by the condition $\ell(\lfloor ws_\alpha \rfloor) = \ell(w) - 2\langle \rho - \rho_J, \alpha^\vee \rangle + 1$. Given a rational number b, we define $\mathrm{QB}_{b\lambda}(W^J)$ to be the subgraph of $\mathrm{QB}(W^J)$ with the same vertex set but having only the edges with labels α satisfying $b\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$.

We now recall the quantum Bruhat graph analogue of a certain lift from W/W_J to W which was previously defined by Deodhar. Let $\ell(w \Rightarrow x)$ denote the length of the shortest path from w to x in $\mathrm{QB}(W)$. It was shown in [11] that, given $v, w \in W$, there exists a unique element $x \in vW_J$ such that $\ell(w \Rightarrow x)$ attains its minimum value as a function of $x \in vW_J$. For reasons explained in [11], we denote the unique element by $\min(vW_J, \preceq_w)$, and call it a *quantum Deodhar lift*.

2.2 Quantum LS paths

Definition 1 ([12]). *A* quantum LS path $\eta \in \mathrm{QLS}(\lambda)$, *for* $\lambda \in P^+$, *is given by two sequences*

$$(0 = b_1 < b_2 < b_3 < \cdots < b_t < b_{t+1} = 1)\,; \qquad (\phi(\eta) = \sigma_1, \sigma_2, \ldots, \sigma_t = \iota(\eta))\,, \tag{2.1}$$

where $b_k \in \mathbb{Q}$, $\sigma_k \in W^J$, *and there is a directed path in* $\mathrm{QB}_{b_k\lambda}(W^J)$ *from* σ_{k-1} *to* σ_k, *for each* $k = 2, \ldots, t$. *The elements* σ_k *are called the* directions *of* η, *while* $\iota(\eta)$ *and* $\phi(\eta)$ *are the* initial *and* final directions, *respectively.*

This data encodes the sequence of vectors $v_t := (b_{t+1} - b_t)\sigma_t\lambda, \ldots, v_2 := (b_3 - b_2)\sigma_2\lambda$, $v_1 := (b_2 - b_1)\sigma_1\lambda$. We can view $\eta \in \mathrm{QLS}(\lambda)$ as a piecewise-linear path given by the sequence of points $0, v_t, v_{t-1} + v_t, \ldots, v_1 + \cdots + v_t$. The endpoint of the path, also called its weight, is $\mathrm{wt}(\eta) := \eta(1) = v_1 + \cdots + v_t$. Given $w \in W$, we define the *initial direction of* η *with respect to* w as $\iota(\eta, w) := w_t \in W$, where the sequence (w_k) is calculated by the following recursive formula: $w_0 := w$, $w_k := \min(\sigma_k W_J, \preceq_{w_{k-1}})$ for $k = 1, \ldots, t$. Also, we set $\xi(\eta, w) := \sum_{k=1}^{t} \mathrm{wt}(w_{k-1} \Rightarrow w_k)$ and $\deg_w(\eta) := -\sum_{k=1}^{t}(1 - b_k)\langle \lambda, \mathrm{wt}(w_{k-1} \Rightarrow w_k)\rangle$.

2.3 The quantum alcove model

We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ for $\beta \in \Phi$ if the common wall is orthogonal to β and β points in the direction from A to B.

Definition 2 ([13]). *An* alcove path *is a sequence of alcoves* $(A_0, A_1, \dots, A_m)$ *such that* A_{j-1} *and* A_j *are adjacent, for* $j = 1, \dots, m$. *We say that* $(A_0, A_1, \dots, A_m)$ *is* reduced *if it has minimal length among all alcove paths from* A_0 *to* A_m.

Let $\lambda \in P$ be any weight, although dominant and anti-dominant λ will play a special role. Let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by λ.

Definition 3 ([13]). *The sequence of roots* $\Gamma(\lambda) = (\beta_1, \beta_2, \dots, \beta_m)$ *is called a* λ-chain *if*

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

A reduced alcove path $(A_0 = A_\circ, A_1, \dots, A_m = A_{-\lambda})$ can be identified with the corresponding total order on the hyperplanes, to be called *λ-hyperplanes*, which separate $A_\circ$ from $A_{-\lambda}$. This total order is given by the sequence $H_{\beta_i, -l_i}$ for $i = 1, \dots, m$, where $H_{\beta_i, -l_i}$ contains the common wall of A_{i-1} and A_i. Note that $\langle \lambda, \beta_i^\vee \rangle \geq 0$, and that the integers l_i, called *heights*, have the following ranges:

$$0 \leq l_i \leq \langle \lambda, \beta_i^\vee \rangle - 1 \ \text{ if } \ \beta_i \in \Phi^+ , \quad \text{and} \quad 1 \leq l_i \leq \langle \lambda, \beta_i^\vee \rangle \ \text{ if } \ \beta_i \in \Phi^- . \tag{2.2}$$

Note also that a λ-chain $(\beta_1, \dots, \beta_m)$ determines the corresponding reduced alcove path, so we can identify them as well.

Remark 4. A reduced alcove path corresponds to the choice of a reduced word for the affine Weyl group element sending $A_\circ$ to $A_{-\lambda}$ [13, Lemma 5.3].

For dominant λ, we have a particular choice of a λ-chain, denoted by $\Gamma_{\text{lex}}(\lambda)$, which we call the *lexicographic (lex) λ-chain* (see [13, Prposition 6.7]). For a λ-hyperplane $H_{\beta,-l}$, the rational number $l/\langle \lambda, \beta^\vee \rangle$ is called the *relative height*; by definition, the sequence of relative heights in the lex λ-chain is weakly increasing.

The objects of the quantum alcove model are defined next; for examples, we refer to [9, 12]. Compared with the original construction in [9], here we consider a generalization of this model, by letting λ be any weight, as opposed to only a dominant weight; another aspect of the generalization is making the model depend on a fixed element $w \in W$, such that the initial model corresponds to w being the identity. In addition to w, we fix an arbitrary λ-chain $\Gamma(\lambda) = (\beta_1, \dots, \beta_m)$, and let $r_i := s_{\beta_i}$, $\widehat{r}_i := s_{\beta_i, -l_i}$.

Definition 5 ([9]). *A subset* $A = \{j_1 < \cdots < j_s\}$ *of* $[m] := \{1, \dots, m\}$ *(possibly empty) is a* w-admissible subset *if we have the following directed path in* $\mathrm{QB}(W)$*:*

$$\Pi(w, A): \quad w \xrightarrow{|\beta_{j_1}|} wr_{j_1} \xrightarrow{|\beta_{j_2}|} wr_{j_1}r_{j_2} \xrightarrow{|\beta_{j_3}|} \cdots \xrightarrow{|\beta_{j_s}|} wr_{j_1}r_{j_2} \cdots r_{j_s} =: \mathrm{end}(w, A) . \tag{2.3}$$

We let $\mathcal{A}(w, \Gamma(\lambda))$ *be the collection of all w-admissible subsets of* $[m]$.

We now associate several parameters with the pair (w, A). The weight of (w, A) is

$$\mathrm{wt}(w, A) := -w\widehat{r}_{j_1} \cdots \widehat{r}_{j_s}(-\lambda)\,. \tag{2.4}$$

Given the height sequence $(l_1, \ldots, l_m)$ above, we define the complementary height sequence $(\widetilde{l}_1, \ldots, \widetilde{l}_m)$ by $\widetilde{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i$. Given $A = \{j_1 < \cdots < j_s\} \in \mathcal{A}(w, \Gamma(\lambda))$, let

$$A^- := \left\{ j_i \in A \mid w r_{j_1} \cdots r_{j_{i-1}} > w r_{j_1} \cdots r_{j_{i-1}} r_{j_i} \right\};$$

in other words, we record the quantum steps in the path $\Pi(w, A)$ defined in (2.3). Let

$$\mathrm{down}(w, A) := \sum_{j \in A^-} |\beta_j|^\vee \in Q^{\vee,+}\,, \quad \mathrm{height}(w, A) := \sum_{j \in A^-} \mathrm{sgn}(\beta_j) \widetilde{l}_j\,. \tag{2.5}$$

3 Chevalley formulas for the semi-infinite flag manifold

Consider a simply-connected simple algebraic group G over $\mathbb{C}$, with Borel subgroup $B = TN$, maximal torus T, and unipotent radical N. The full *semi-infinite flag manifold* $\mathbf{Q}_G^{\mathrm{rat}}$ is the reduced (ind-)scheme associated to $G(\mathbb{C}((z)))/(T \cdot N(\mathbb{C}((z))))$; in this paper, we concentrate on its semi-infinite Schubert subvariety $\mathbf{Q}_G := \mathbf{Q}_G(e) \subset \mathbf{Q}_G^{\mathrm{rat}}$ corresponding to the identity element $e \in W_{\mathrm{af}}$, which we also call the semi-infinite flag manifold. The $T \times \mathbb{C}^*$-equivariant K-group $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ of $\mathbf{Q}_G$ has a (topological) $\mathbb{Z}[q, q^{-1}][P]$-basis of *semi-infinite Schubert classes*, and its multiplicative structure is determined by a *Chevalley formula*, which expresses the tensor product of a semi-infinite Schubert class with the class of a line bundle. In [6] and [14], the Chevalley formulas were given in the cases of a dominant and an anti-dominant weight λ, respectively. These formulas were expressed in terms of the quantum LS path model. We will express them in terms of the quantum alcove model based on the lexicographic λ-chain. The goal is to generalize these formulas for an arbitrary weight λ, and we will also see that an arbitrary λ-chain can be used. Throughout this section, W_J is the stabilizer of λ, and we use freely the notation in Section 2.

The $T \times \mathbb{C}^*$-equivariant K-group $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ is the $\mathbb{Z}[q, q^{-1}][P]$-submodule of the (Iwahori-) equivariant K-group $K_{\mathbf{I} \rtimes \mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ of $\mathbf{Q}_G^{\mathrm{rat}}$, introduced in [6], consisting of all (possibly infinite) linear combinations of the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ of the structure sheaves of the semi-infinite Schubert varieties $\mathbf{Q}_G(x) (\subset \mathbf{Q}_G)$ with coefficients $a_x \in \mathbb{Z}[q, q^{-1}][P]$ for $x \in W_{\mathrm{af}}^{\geq 0} = W \times Q^{\vee,+}$ such that the sum $\sum_{x \in W_{\mathrm{af}}^{\geq 0}} |a_x|$ of the absolute values $|a_x|$ lies in $\mathbb{Z}_{\geq 0}((q^{-1}))[P]$. Here $\mathbb{C}^*$ acts on $\mathbf{Q}_G$ by loop rotation, and $\mathbb{Z}[P]$ is the group algebra of P, spanned by formal exponentials $\mathbf{e}^\mu$, for $\mu \in P$, with $\mathbf{e}^\mu \mathbf{e}^\nu = \mathbf{e}^{\mu+\nu}$; note that $\mathbb{Z}[P]$ is identified with the representation ring of T. We also consider the $\mathbb{Z}[q, q^{-1}][P]$-submodule $K'_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ of $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ consisting of all finite linear combinations of the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ with coefficients in $\mathbb{Z}[q, q^{-1}][P]$ for $x \in W_{\mathrm{af}}^{\geq 0}$. The T-equivariant K-groups of $\mathbf{Q}_G$, denoted by $K_T(\mathbf{Q}_G)$ and $K'_T(\mathbf{Q}_G)$, are obtained from the $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ and

$K'_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ above, respectively, by the specialization $q = 1$. Hence the Chevalley formulas in $K_T(\mathbf{Q}_G)$ (for arbitrary weights) and $K'_T(\mathbf{Q}_G)$ (for anti-dominant weights) are obtained from the corresponding one in $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ by setting $q = 1$. Note that $K_T(\mathbf{Q}_G)$ turns out to be the $\mathbb{Z}[P]$-module consisting of all (possibly infinite) linear combinations of the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients in $\mathbb{Z}[P]$; also $K'_T(\mathbf{Q}_G)$ is the $\mathbb{Z}[P]$-submodule of $K_T(\mathbf{Q}_G)$ consisting of all finite linear combinations of the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients in $\mathbb{Z}[P]$.

3.1 Chevalley formulas for dominant and anti-dominant weights

We start with the Chevalley formula for dominant weights, which was derived in terms of semi-infinite LS paths in [6], and then restated in [14, Corollary C.3] in terms of quantum LS paths.

Let $\lambda = \sum_{i\in I} \lambda_i \varpi_i$ be a dominant weight. We denote by $\overline{\mathrm{Par}(\lambda)}$ the set of I-tuples of partitions $\boldsymbol{\chi} = (\chi^{(i)})_{i\in I}$ such that $\chi^{(i)}$ is a partition of length at most λ_i for all $i \in I$. For $\boldsymbol{\chi} = (\chi^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}$, we set $|\boldsymbol{\chi}| := \sum_{i\in I} |\chi^{(i)}|$, with $|\chi^{(i)}|$ the size of the partition $\chi^{(i)}$. Also set $\iota(\boldsymbol{\chi}) := \sum_{i\in I} \chi_1^{(i)} \alpha_i^{\vee} \in Q^{\vee,+}$, with $\chi_1^{(i)}$ the first part of the partition $\chi^{(i)}$.

Theorem 6 ([6, 14]). *Let $x = wt_{\xi} \in W_{\mathrm{af}}^{\geq 0} = W \times Q^{\vee,+}$. Then, in $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G}(-w_{\circ}\lambda)] \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] =$$

$$= \sum_{\eta\in\mathrm{QLS}(\lambda)} \sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}(\lambda)}} q^{\deg_w(\eta)-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|} \mathbf{e}^{\mathrm{wt}(\eta)} [\mathcal{O}_{\mathbf{Q}_G(\iota(\eta,w)t_{\xi+\xi(\eta,w)+\iota(\boldsymbol{\chi})})}] \,.$$

We now translate this formula in terms of the quantum alcove model for the lex λ-chain $\Gamma_{\mathrm{lex}}(\lambda)$. To this end, given $w \in W$, we construct a bijection $A \mapsto \eta$ between $\mathcal{A}(w, \Gamma_{\mathrm{lex}}(\lambda))$ and $\mathrm{QLS}(\lambda)$, for which several properties are then proved.

In order to construct the forward map, let $A = \{j_1 < \cdots < j_s\}$ be in $\mathcal{A}(w, \Gamma_{\mathrm{lex}}(\lambda))$. The corresponding heights are within the first range in (2.2). Consider the weakly increasing sequence of relative heights $h_i := l_{j_i}/\langle\lambda, \beta_{j_i}^{\vee}\rangle \in [0, 1) \cap \mathbb{Q}$ for $i = 1, \ldots, s$. Let $0 < b_2 < \cdots < b_t < 1$ be the distinct nonzero values in the set $\{h_1, \ldots, h_s\}$, and let $b_1 := 0$, $b_{t+1} := 1$. For $k = 1, \ldots, t$, let $I_k := \{1 \leq i \leq s \mid h_i = b_k\}$.

Recall the path $\Pi(w, A)$ in $\mathrm{QB}(W)$ defined in (2.3). We divide this path into subpaths corresponding to the sets I_k, and record the last element in each subpath; more precisely, for $k = 0, \ldots, t$, we define the sequence of Weyl group elements

$$w_k := w \overrightarrow{\prod_{i\in I_1\cup\cdots\cup I_k}} r_{j_i} \,,$$

where the non-commutative product is taken in the increasing order of the indices i, and $w_0 := w$. For $k = 1, \ldots, t$, let $\sigma_k := \lfloor w_k \rfloor \in W^J$. We can now define the forward map as

$$(w, A) \mapsto \eta := ((b_1, b_2, \ldots, b_t, b_{t+1}); (\sigma_1, \ldots, \sigma_t)) \,.$$

We will verify below that the image is in $\mathrm{QLS}(\lambda)$.

The inverse map is constructed using the quantum Deodhar lift and the related *shellability property* of the quantum Bruhat graph, due to Brenti-Fomin-Postnikov.

Proposition 7. *The map $A \mapsto \eta$ constructed above is a bijection between $\mathcal{A}(w, \Gamma_{\mathrm{lex}}(\lambda))$ and $\mathrm{QLS}(\lambda)$. It maps the corresponding parameters in the following way:*

$$\mathrm{wt}(w, A) = \mathrm{wt}(\eta),\ \mathrm{end}(w, A) = \iota(\eta, w),$$
$$\mathrm{down}(w, A) = \xi(\eta, w),\ -\mathrm{height}(w, A) = \deg_w(\eta).$$

We translate the formula in Theorem 6 to the quantum alcove model via Proposition 7.

Theorem 8. *Let λ be a dominant weight, $\Gamma_{\mathrm{lex}}(\lambda)$ the lex λ-chain, and let $x = wt_\xi \in W_{\mathrm{af}}^{\geq 0}$. Then, in $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ\lambda)] \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] =$$
$$\sum_{A \in \mathcal{A}(w, \Gamma_{\mathrm{lex}}(\lambda))} \sum_{\chi \in \overline{\mathrm{Par}(\lambda)}} q^{-\mathrm{height}(w,A) - \langle \lambda, \xi \rangle - |\chi|} \mathbf{e}^{\mathrm{wt}(w,A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(w,A) t_{\xi + \mathrm{down}(w,A) + \iota(\chi)})}] \,.$$

A similar Chevalley formula for an anti-dominant weight λ was derived in [14, Theorem 1], also in terms of quantum LS paths. Using a similar procedure to the one above, we translate it to the quantum alcove model, as stated in Theorem 9. We work with the lex λ-chain $\Gamma_{\mathrm{lex}}(\lambda)$, defined as the reverse of the lex $(-\lambda)$-chain; note that the alcove path corresponding to the former (ending at $A_\circ - \lambda$) is the translation by $-\lambda$ of the alcove path corresponding to the latter (ending at $A_\circ + \lambda$).

Theorem 9. *Let λ be an anti-dominant weight, $\Gamma_{\mathrm{lex}}(\lambda)$ the lex λ-chain, and let $x = wt_\xi \in W_{\mathrm{af}}^{\geq 0}$. Then, in $K'_{T\times\mathbb{C}^*}(\mathbf{Q}_G) \subset K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ\lambda)] \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] =$$
$$\sum_{A \in \mathcal{A}(w, \Gamma_{\mathrm{lex}}(\lambda))} (-1)^{|A|} q^{-\mathrm{height}(w,A) - \langle \lambda, \xi \rangle} \mathbf{e}^{\mathrm{wt}(w,A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(w,A) t_{\xi + \mathrm{down}(w,A)})}] \,.$$

3.2 The Chevalley formula for an arbitrary weight

We now exhibit the Chevalley formula for an arbitrary weight $\lambda = \sum_{i \in I} \lambda_i \varpi_i$. To state the formula, let $\overline{\mathrm{Par}(\lambda)}$ denote the set of I-tuples of partitions $\chi = (\chi^{(i)})_{i \in I}$ such that $\chi^{(i)}$ is a partition of length at most $\max(\lambda_i, 0)$.

Theorem 10. *Let λ be an arbitrary weight, $\Gamma(\lambda)$ an arbitrary λ-chain, and let $x = wt_{\xi} \in W_{\mathrm{af}}^{\geq 0}$. Then, in $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ\lambda)] \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] =$$

$$\sum_{A\in\mathcal{A}(w,\Gamma(\lambda))} \sum_{\chi\in\overline{\mathrm{Par}(\lambda)}} (-1)^{n(A)} q^{-\mathrm{height}(w,A)-\langle\lambda,\xi\rangle-|\chi|} \mathbf{e}^{\mathrm{wt}(w,A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(w,A)t_{\xi+\mathrm{down}(w,A)+\iota(\chi)})}],$$

where $n(A)$, for $A = \{j_1 < \cdots < j_s\}$, is the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

Example 11. Assume that $\mathfrak{g}$ is of type A_2, and $\lambda = \varpi_1 - \varpi_2$. Then, $\Gamma(\lambda) := (\alpha_1, -\alpha_2)$ is a λ-chain of roots. Assume that $w = s_1 = s_{\alpha_1}$. In this case, we see that $\mathcal{A}(s_1, \Gamma(\lambda)) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and we have the following table.

A	$n(A)$	$\mathrm{height}(s_1, A)$	$\mathrm{wt}(s_1, A)$	$\mathrm{end}(s_1, A)$	$\mathrm{down}(s_1, A)$
$\emptyset$	0	0	$s_1\lambda$	s_1	0
$\{1\}$	0	1	λ	e	$\alpha_1^\vee$
$\{2\}$	1	0	$s_1\lambda$	s_1s_2	0
$\{1,2\}$	1	1	λ	s_2	$\alpha_1^\vee$

Also, we can identify $\overline{\mathrm{Par}(\lambda)}$ with $\mathbb{Z}_{\geq 0}$. Therefore, we obtain

$$[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ\lambda)] \cdot [\mathcal{O}_{\mathbf{Q}_G(s_1t_\xi)}] =$$

$$\sum_{m\in\mathbb{Z}_{\geq 0}} q^{-\langle\lambda,\xi\rangle-m} \Big\{ \underbrace{\mathbf{e}^{s_1\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_1t_{\xi+m\alpha_1^\vee})}]}_{A=\emptyset} + \underbrace{q^{-1}\mathbf{e}^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(t_{\xi+\alpha_1^\vee+m\alpha_1^\vee})}]}_{A=\{1\}}$$
$$+ \underbrace{(-1)\mathbf{e}^{s_1\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_1s_2t_{\xi+m\alpha_1^\vee})}]}_{A=\{2\}} + \underbrace{(-1)q^{-1}\mathbf{e}^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_2t_{\xi+\alpha_1^\vee+m\alpha_1^\vee})}]}_{A=\{1,2\}} \Big\}.$$

4 The quantum K-theory of flag manifolds

Y.-P. Lee defined the (small) *quantum K-theory* of a smooth projective variety X, denoted by $QK(X)$ [8]. This is a deformation of the ordinary K-ring of X, analogous to the relation between quantum cohomology and ordinary cohomology. The deformed product is defined in terms of certain generalizations of *Gromov-Witten invariants* (i.e., the structure constants in quantum cohomology), called *quantum K-invariants of Gromov-Witten type*.

In order to describe the (small) T-equivariant quantum K-algebra $QK_T(G/B)$, for the finite-dimensional flag manifold G/B, we associate a variable Q_k to each simple coroot $\alpha_k^\vee$, and let $\mathbb{Z}[Q] := \mathbb{Z}[Q_1, \ldots, Q_r]$. Given $\xi = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ in $Q^{\vee,+}$, let

$Q^\xi := Q_1^{d_1} \cdots Q_r^{d_r}$. Let $\mathbb{Z}[P][Q] := \mathbb{Z}[P] \otimes_{\mathbb{Z}} \mathbb{Z}[Q]$, where the group algebra $\mathbb{Z}[P]$ of P was defined at the beginning of Section 3. By the finiteness theorem of Anderson-Chen-Tseng ([1]), we can define $QK_T(G/B)$ to be the $\mathbb{Z}[P][Q]$-module $K_T(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P][Q] \subset K_T(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P][\![Q]\!]$, where $\mathbb{Z}[P][\![Q]\!] := \mathbb{Z}[P] \otimes_{\mathbb{Z}} \mathbb{Z}[\![Q]\!]$. The algebra $QK_T(G/B)$ has a $\mathbb{Z}[P][Q]$-basis given by the classes $[\mathcal{O}^w]$ of the structure sheaves of the (opposite) Schubert varieties $X^w \subset G/B$ of codimension $\ell(w)$, for $w \in W$.

It is proved in [4] (see also [5]) that there exists a $\mathbb{Z}[P]$-module isomorphism from $QK_T(G/B)$ onto $K_T'(\mathbf{Q}_G)$ that respects the quantum multiplication in $QK_T(G/B)$ and the tensor product in $K_T'(\mathbf{Q}_G)$; in particular, it respects the quantum multiplication with the class of the line bundle $[\mathcal{O}_{G/B}(-\varpi_k)]$ and the tensor product with the class of the line bundle $[\mathcal{O}_{\mathbf{Q}_G}(w_\circ \varpi_k)]$, for $k \in I$. Here we remark that in order to translate the Chevalley formula in $K_T(\mathbf{Q}_G)$ for fundamental weights into the one in the quantum K-theory of G/B, we need to consider $K_T(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P][\![Q]\!]$; for example, in type A_{n-1}, the tensor product in $K_T(\mathbf{Q}_G)$ with the class $[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ \varepsilon_k)]$ for $1 \le k \le n$ corresponds to the quantum multiplication with the class $\frac{1}{1-Q_k}[\mathcal{O}_{G/B}(\varepsilon_k)]$, where $\varepsilon_k := \varpi_k - \varpi_{k-1}$, with $\varpi_0 := 0$, $\varpi_n := 0$, and $Q_n := 0$. Also, note that the above isomorphism sends each (opposite) Schubert class $\mathbf{e}^\mu [\mathcal{O}^w] Q^\xi$ in $QK_T(G/B)$ to the corresponding semi-infinite Schubert class $\mathbf{e}^{-\mu}[\mathcal{O}_{\mathbf{Q}_G(wt_\xi)}]$ in $K_T'(\mathbf{Q}_G)$ for $w \in W$, $\xi \in Q^{\vee,+}$, and $\mu \in P$. These results and the formula in Theorem 9 imply an important conjecture in [13]: the Chevalley formula for $QK_T(G/B)$; we also use the equality $[\mathcal{O}^{s_k}] = 1 - \mathbf{e}^{-\varpi_k}[\mathcal{O}_{G/B}(-\varpi_k)]$ in $K_T(G/B)$.

Theorem 12. *Let $k \in I$, and fix a $(-\varpi_k)$-chain of roots $\Gamma(-\varpi_k)$. Then, in $QK_T(G/B)$, we have*

$$[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w] =$$

$$(1 - \mathbf{e}^{w(\varpi_k)-\varpi_k})[\mathcal{O}^w] + \sum_{A \in \mathcal{A}(w,\Gamma(-\varpi_k)) \setminus \{\emptyset\}} (-1)^{|A|-1}\, Q^{\mathrm{down}(w,A)} \mathbf{e}^{-\varpi_k - \mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}].$$

Let us now turn to the type A_{n-1} flag manifold $Fl_n = SL_n/B$ and its (non-equivariant) quantum K-theory $QK(Fl_n)$. In [10], the first author and Maeno defined the so-called *quantum Grothendieck polynomials*. According to [10, Theorem 6.4], whose proof is based on intricate combinatorics, the mentioned polynomials multiply precisely as stated by the above Chevalley formula; note that in the (non-equivariant) K-theory $K(G/B)$, the (opposite) Schubert class $[\mathcal{O}^w]$ is identical to the class of the structure sheaf of the Schubert variety $X_{w_\circ w} \subset G/B$ of codimension $\ell(w)$ for $w \in W$. As this formula determines the multiplicative structure of $QK(Fl_n)$ [2], we derive the following result, settling the main conjecture in [10].

Theorem 13. *The quantum Grothendieck polynomials represent Schubert classes in $QK(Fl_n)$.*

Given a *degree* $d = (d_1, \ldots, d_{n-1})$, let $N_{s_k,w}^{v,d}$ be the coefficient of $Q_1^{d_1} \cdots Q_{n-1}^{d_{n-1}}[\mathcal{O}^v]$ in the expansion of $[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w]$ in $QK(Fl_n)$ for $k \in I = \{1, \ldots, n-1\}$. Based on Theorem 12 and results on a charge statistic due to the first author, we proved the following theorem, which completely determines the Chevalley coefficients $N_{s_k,w}^{v,d}$.

Theorem 14. *For every k, v and parabolic coset $\sigma W_{I\setminus\{k\}}$ not containing v, there are unique d and $w \in \sigma W_{I\setminus\{k\}}$ (they can be constructed explicitly), such that $N_{s_k,w}^{v,d} = \pm 1$ (the sign is as in Theorem 12). All the other coefficients $N_{s_k,w}^{v,d}$ are 0. Moreover, in the expansion of $[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^{w}]$ there is a minimum and a maximum degree (with respect to the componentwise order), which are constructed explicitly.*

5 The quantum K-theory of partial flag manifolds

As an application of Theorem 12, we give a Chevalley formula for partial flag manifolds corresponding to minuscule weights in types A, B, D, and E; also, in type C, we give a Chevalley formula for partial flag manifolds corresponding to all fundamental weights.

Let $k \in I$ be such that ϖ_k is a minuscule fundamental weight in types A, B, D, and E; also, let $k \in I$ be arbitrary in type C. Let $P_J \supset B$ be the maximal (standard) parabolic subgroup of G associated to the subset $J := I \setminus \{k\}$. The T-equivariant quantum K-theory $QK_T(G/P_J)$ of the partial flag manifold G/P_J is defined as $K_T(G/P_J) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P][Q_k]$, where $K_T(G/P_J)$ is the T-equivariant K-theory of G/P_J, and $\mathbb{Z}[Q_k]$ is the polynomial ring in the single (Novikov) variable $Q_k = Q^{\alpha_k^\vee}$ corresponding to the simple coroot $\alpha_k^\vee$. The (opposite) Schubert classes $[\mathcal{O}_J^y]$, for $y \in W^J$, form a $\mathbb{Z}[P][Q_k]$-basis. It is proved in [5] that there exists a $\mathbb{Z}[P]$-module surjection Φ_J from $QK_T(G/B)$ to $QK_T(G/P_J)$ such that $\Phi_J([\mathcal{O}^w]) = [\mathcal{O}_J^{\lfloor w \rfloor}]$ for each $w \in W$, and $\Phi_J(Q^\xi) = Q^{[\xi]^J}$ for each $\xi \in Q^{\vee,+}$, where $[\xi]^J := c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$. Also, it is proved in [5] that $\Phi_J([\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_{G/P_J}(-\varpi_k)]$. As is shown in [2], the quantum multiplication in $QK_T(G/P_J)$ is uniquely determined by its $\mathbb{Z}[P]$-module structure and the quantum multiplication with the class of the line bundle $[\mathcal{O}_{G/P_J}(-\varpi_k)]$. Therefore, the $\mathbb{Z}[P]$-module surjection Φ_J respects the quantum multiplications in $QK_T(G/B)$ and $QK_T(G/P_J)$.

Based on the facts above, we obtain a formula for the quantum multiplication with $[\mathcal{O}_{G/P_J}(-\varpi_k)]$ in $QK_T(G/P_J)$ from Theorem 12 in $QK_T(G/B)$ by applying Φ_J; this argument works for an arbitrary fundamental weight ϖ_k of G of any type. However, upon applying Φ_J, there are many terms to be cancelled in the corresponding formula in $QK_T(G/P_J)$. For a minuscule fundamental weight ϖ_k of G of type A, B, D, or E, and for an arbitrary fundamental weight ϖ_k of G of type C, we cancel out all these terms via a sign-reversing involution, and obtain a cancellation-free formula. Below we give such a formula in type C, which is not included in [2]. For each $k \in I$, consider a specific $(-\varpi_k)$-chain $\Gamma(k)$ whose initial segment $\Gamma^1(k)$ is of the following form:

$$\varepsilon_1 + \varepsilon_2; \varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_3; \ldots; \varepsilon_1 + \varepsilon_k, \varepsilon_2 + \varepsilon_k, \ldots, \varepsilon_{k-1} + \varepsilon_k.$$

For a w-admissible subset $A \in \mathcal{A}(w, \Gamma(k))$, with $w \in W^J$, we set $A^1 := A \cap \Gamma^1(k)$.

Theorem 15. *Let G be of type C_r, and $k \in I$ arbitrary. Then, for a given $w \in W^J$, we have the following cancellation-free formula in $QK_T(G/P_J)$, where $\mathcal{A}_{\lessdot}(w, \Gamma(k))$ consists of all the w-admissible subsets A with $\Pi(w, A)$ a saturated chain in Bruhat order:*

$$[\mathcal{O}_J^w] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)] = \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k))} (-1)^{|A|} \mathbf{e}^{\mathrm{wt}(w,A)} [\mathcal{O}_J^{\mathrm{end}(w,A)}] - Q_k \sum_{\substack{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k)) \\ \mathrm{end}(w,A^1) \geq \lfloor s_\theta \rfloor}} (-1)^{|A|} \mathbf{e}^{\mathrm{wt}(w,A)} [\mathcal{O}_J^{\lfloor \mathrm{end}(w,A) s_{2\varepsilon_k} \rfloor}].$$

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