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Bijection between trees in Stanley character formula and factorizations of a cycle

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Abstract. Stanley and Féray gave a formula for the irreducible character of the symmetric group related to a *multi-rectangular Young diagram*. This formula shows that the character is a polynomial in the multi-rectangular coordinates and gives an explicit combinatorial interpretation for its coefficients in terms of counting certain decorated maps (i.e., graphs drawn on surfaces). In the current paper we concentrate on the coefficients of the top-degree monomials in the Stanley character polynomial which corresponds to counting certain decorated plane trees. We give an explicit bijection between such trees and minimal factorizations of a cycle.

Keywords: plane trees, minimal factorizations of a cycle, bijection, irreducible characters of the symmetric groups

1 Motivations

1.1 Normalized characters and Stanley polynomials

For a Young diagram λ with $N = |\lambda|$ boxes and a partition $\pi \vdash k$ we denote by

$$\mathrm{Ch}_\pi(\lambda) = \begin{cases} N(N-1)(N-2)\cdots(N-k+1)\,\frac{\chi^\lambda(\pi\cup 1^{N-k})}{\chi^\lambda(1^N)} & \text{for } k \leq N, \\ 0 & \text{otherwise} \end{cases}$$

the *normalized irreducible character of the symmetric group*, where $\chi^\lambda(\rho)$ denotes the value of the usual irreducible character of the symmetric group which corresponds to the Young diagram λ, evaluated on any permutation with the cycle decomposition given by the partition ρ. One of the goals of *asymptotic representation theory* is to understand the behavior of such normalized characters in the scaling when the partition π is fixed and the number of the boxes of the Young diagram λ tends to infinity.

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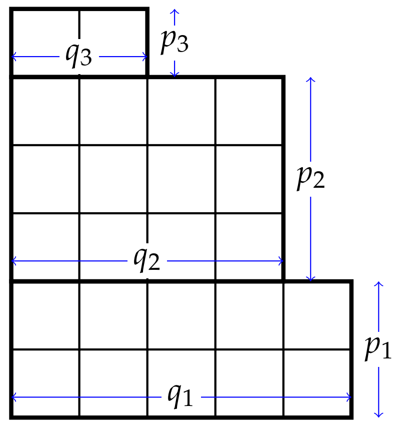


Figure 1: Multi-rectangular Young diagram $\mathbf{p} \times \mathbf{q} = (2,3,1) \times (5,4,2)$.

For a pair of sequences of non-negative integers $\mathbf{p} = (p_1, \dots, p_\ell)$ and $\mathbf{q} = (q_1, \dots, q_\ell)$ such that $q_1 \geq \cdots \geq q_\ell$ we consider the *multi-rectangular Young diagram* $\mathbf{p} \times \mathbf{q}$, see Figure 1. Stanley [7, 8] initiated investigation of the normalized characters evaluated on such multi-rectangular Young diagrams and proved that

$$(\mathbf{p}, \mathbf{q}) \mapsto \mathrm{Ch}_\pi \left(\mathbf{p} \times \mathbf{q}\right) \tag{1.1}$$

is a polynomial (called now *the Stanley character polynomial*) in the variables $p_1, \dots, p_\ell$ and $q_1, \dots, q_\ell$. He also gave a conjectural formula (proved later by Féray [3]) which gives a combinatorial interpretation to the coefficients of this polynomial in terms of certain *maps* (i.e., graphs drawn on surfaces). Stanley also explained how investigation of its coefficients might shed some light on the *Kerov positivity conjecture*, see [6] for more context.

Despite recent progress in this field (for the proof of the Kerov positivity conjecture see [2, 1]) there are several other positivity conjectures related to the normalized characters Ch_π that remain open (see [4, Conjecture 2.4] and [5]) and which suggest the existence of some additional hidden combinatorial structures behind such characters. We expect that such positivity problems are more amenable to bijective methods and the current article is the first step in this direction.

We will concentrate on the special case when $\pi = (k)$ consists of a single part. In this case the degree of the Stanley polynomial (1.1) turns out to be equal to $k+1$. We will also concentrate on the combinatorial interpretation of the coefficients of the Stanley polynomial (1.1) standing at monomials of this maximal degree $k+1$ which turns out to be related to maps of genus zero, i.e., plane trees. Nevertheless, the methods which we present in the current paper for this special case are applicable in much bigger generality and in a forthcoming full version of this paper [9] we discuss the applications to maps with higher genera.

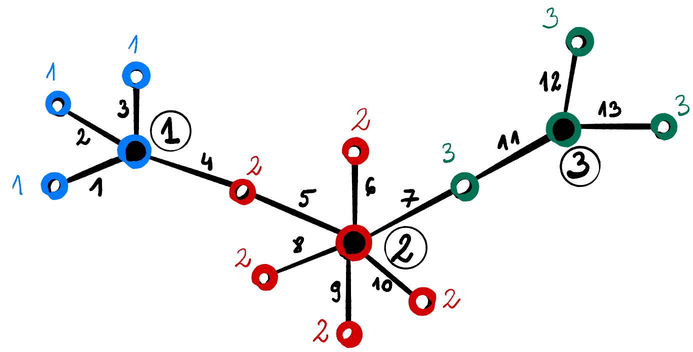


Figure 2: An example of a Stanley tree of type $(3,5,3)$. The circled numbers indicate the labels of the black verices. The black numbers indicate the labels of the edges. The colors (blue for 1, red for 2, green for 3) indicate the values of the function f on white vertices.

1.2 Stanley trees

Let T be a *bicolored* plane tree, i.e., a plane tree with each vertex painted black or white and with edges connecting the vertices of opposite colors. We assume that the tree has k edges labelled with the numbers $1,\dots,k$. We also assume that it has n black vertices labelled with the numbers $1,\dots,n$. We define the function f which to each white vertex associates the maximum of the labels of its black neighbors. We will say that T is a *Stanley tree of type*

$$\Big(\big|f^{-1}(1)\big|,\dots,\big|f^{-1}(n)\big|\Big),$$

i.e., the type gives the information about the number of the white vertices for which the function f takes a specified value. Figure 2 gives an example of a Stanley tree of type $(3,5,3)$. By $\mathcal{T}_{b_1,\dots,b_n}$ we denote the set of Stanley trees of type $(b_1,\dots,b_n)$.

1.3 Coefficients of top-degree p-squarefree monomials

It turns out that in the analysis of Stanley polynomials it is enough to restrict attention to $\mathbf{p}$-*squarefree monomials*, i.e., the monomials of the form $p_1\cdots p_n q_1^{b_1}\cdots q_n^{b_n}$, see [1].

Lemma 1.1. *For all integers $b_1,\dots,b_n \geq 0$ such that $b_1+\cdots+b_n+n = k+1$ the $\mathbf{p}$-squarefree coefficient of the Stanley character polynomial is given by*

$$\left[p_1\cdots p_n q_1^{b_1}\cdots q_n^{b_n}\right]\mathrm{Ch}_k\left(\mathbf{p}\times\mathbf{q}\right) = \pm\frac{1}{(k-1)!}\left|\mathcal{T}_{b_1,\dots,b_n}\right|.$$

In order to be concise we will not discuss the sign on the right-hand side. The above lemma is a special case of a general formula conjectured by Stanley [8] and proved by Féray [3] and therefore we refer to it as Stanley–Féray character formula.

1.4 Minimal factorizations of long cycles

We fix an integer $k \geq 1$ and denote by $\mathfrak{S}_k$ the corresponding symmetric group. We say that a permutation $\pi \in \mathfrak{S}_k$ is a *cycle of length* ℓ if it is of the form $\pi = (a_1, \dots, a_\ell)$.

Let $a_1, \dots, a_n \geq 2$ be integers. We say that a tuple $(\sigma_1, \dots, \sigma_n)$ is a *factorization of a long cycle of type* $(a_1, \dots, a_n)$ if $\sigma_1, \dots, \sigma_n \in \mathfrak{S}_k$ are such that the product $\sigma_1 \cdots \sigma_n$ is a cycle of length k and σ_i is a cycle of length a_i for each choice of $i \in \{1, \dots, n\}$. In the current paper we concentrate on *minimal factorizations* which correspond to the special case when

$$\sum_{i=1}^{n} (a_i - 1) = k - 1.$$

By $\mathcal{C}_{a_1,\dots,a_n}$ we denote the set of such minimal factorizations of a long cycle of type $(a_1, \dots, a_n)$.

2 The main result: bijection between Stanley trees and minimal factorizations of long cycles

Theorem 2.1. *Let $n \geq 2$ and $b_1, \dots, b_n \geq 1$ be integers. We define the integers $a_1, \dots, a_n$ by*

$$a_i = \begin{cases} b_i + 1 & \text{if } i \in \{1, n\}, \\ b_i + 2 & \text{otherwise.} \end{cases}$$

Then the algorithm presented below gives a bijection between the set $\mathcal{C}_{a_1,\dots,a_n}$ and the set $\mathcal{T}_{b_1,\dots,b_n}$.

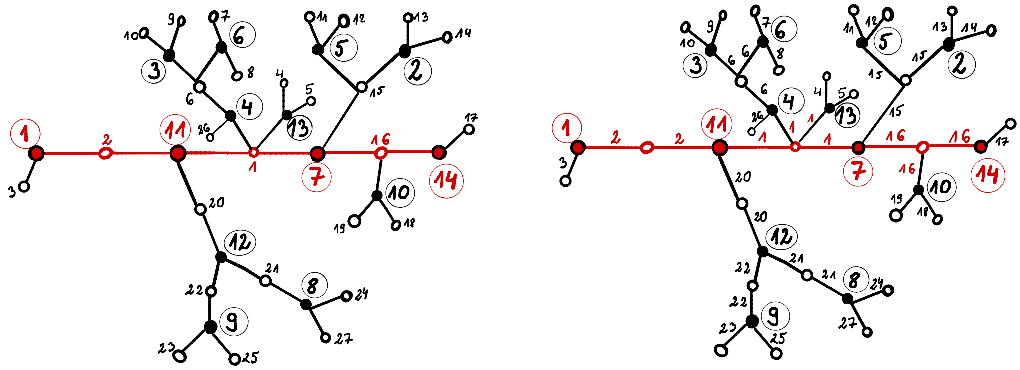
2.1 The first step: from a factorization to a tree with repeated edge labels

In the first step of our algorithm to a given minimal factorization $(\sigma_1, \dots, \sigma_n) \in \mathcal{C}_{a_1,\dots,a_n}$ we will associate a bicolored plane tree T_1 with labelled black vertices and labelled edges. The remaining part of the current section is devoted to the details of this construction.

2.1.1 The tree T_0

We start by creating a tree T_0 with n black vertices labelled $1, \dots, n$ and with k white vertices labelled $1, \dots, k$. Each black vertex i corresponds to the cycle $\sigma_i = (\sigma_{i,1}, \dots, \sigma_{i,a_i})$ and so we connect the black vertex i with the white vertices $\sigma_{i,1}, \dots, \sigma_{i,a_i}$.

In order to give this tree the structure of a *plane tree* we need to specify the cyclic order of the edges around each vertex. We declare that going clockwise around the black vertex i the cyclic order of the labels of the white neighbours should correspond to



(a) (b)

Figure 3: (a) The output T_0 of the first step of the algorithm applied to the minimal factorization (2.1). The spine is drawn in red. (b) The tree T_1 which is the starting point of the second step of our algorithm.

the cyclic order $\sigma_{i,1}, \dots, \sigma_{i,a_i}$. The cyclic order around the white vertices is more involved and we present it in the following.

The path between the two black vertices with the labels 1 and n will be called *the spine*; on Figure 3a it is drawn in red. There will be two separate rules which determine the cyclic order of the edges around a given white vertex, depending whether the vertex belongs to the spine or not.

For each white vertex which is *not* on the spine we declare that going counterclockwise around it, the labels of its black neighbors should be arranged in the increasing way (for example, the neighbors of the white vertex 6 on Figure 3a listed in the counterclockwise order are $3, 4, 6$).

For each white vertex v which belongs to the spine there are exactly two black neighbors which belong to the spine; we denote their labels by x_1 and x_2 with $x_1 < x_2$. Going counterclockwise around v, all non-spine edges should be inserted after x_1 and before x_2. Their order is determined by the requirement that—after neglecting the vertex x_2—the cyclic counterclockwise order of the remaining vertices should be increasing. For example, for the white vertex 1 on Figure 3a we have $x_1 = 7$, $x_2 = 11$ and the counterclockwise cyclic order of the non-x_2 black neighbors is $4, 7, 13$.

For example, Figure 3a gives the tree T_0 which corresponds to the minimal factorization $(\sigma_1, \dots, \sigma_{14}) \in \mathcal{C}_{2,3^{12},2}$ with

$$\sigma_1 = (2,3), \quad \sigma_2 = (13,14,15), \quad \sigma_3 = (6,10,9), \quad \sigma_4 = (1,26,6), \quad \sigma_5 = (11,12,15),$$
$$\sigma_6 = (6,7,8), \quad \sigma_7 = (1,15,16) \quad \sigma_8 = (21,24,27), \quad \sigma_9 = (22,25,23), \quad \sigma_{10} = (16,18,19),$$
$$\sigma_{11} = (2,1,20), \quad \sigma_{12} = (20,21,22), \quad \sigma_{13} = (1,4,5), \quad \sigma_{14} = (16,17). \quad (2.1)$$

The product $\sigma = \sigma_1 \dots \sigma_{14}$ is a cycle of length 27 and

$$\sigma = (11,12,13,14,15,16,17,18,19,26,10,9,6,7,8,1,4,5,20,24,27,21,25,23,22,3,2). \quad (2.2)$$

2.1.2 Information about the initial tree T_0

In the current section we will define certain sets and functions which describe the shape of the initial tree T_0. In the language of programmers: we will create variables B_x, p_x, S_B, S_E which will not change their values during the execution of the algorithm.

For each white vertex label $x \in \{1, \dots, k\}$ we denote by $B_x \subseteq \{1, \dots, n\}$ the set of labels of its black neighbors. We can orient the non-spine edges of the tree so that the arrows point towards the spine. For a white non-spine vertex label $x \in \{1, \dots, k\}$ we denote by $p_x \in \{1, \dots, n\}$ the label of the black neighbor of the white vertex x which is in the direction of the spine. By $S_B \subseteq \{1, \dots, n\}$ we denote the set of the labels of black spine vertices. By $S_E \subseteq \{1, \dots, k\}$ we denote the set of labels of white spine vertices.

For the example from Figure 3a we have:

$$B_1 = \{4,7,11,13\}, \quad B_2 = \{1,11\}, \quad B_6 = \{3,4,6\}, \quad B_{16} = \{7,14,10\},$$
$$B_{15} = \{2,5,7\}, \quad B_{20} = \{11,12\}, \quad B_{22} = \{9,12\}, \quad B_{21} = \{8,12\},$$
$$p_6 = 4, \quad p_{15} = 7, \quad p_{20} = 11, \quad p_{22} = 12, \quad p_{21} = 12,$$
$$S_B = \{1,7,11,14\}, \quad S_E = \{1,2,16\}. \quad (2.3)$$

Let L be the sequence of labels of the vertices which are white, non-spine, and non-leaf, arranged in the following order. We start from the unique edge connecting the black vertex labelled 1 with a white spine vertex from the set S_E. Then we traverse the tree holding it with the right hand and order the vertices in L according to the time of the first visit ("depth-first search"). For example, for the tree T_0 shown on Figure 3a we have $L = (6,15,20,21,22)$.

We denote by T_1 the tree T_0 in which each edge is labelled by its white endpoint and then all labels of the white vertices are removed, see Figure 3b. The tree T_1 is the output of the first step of the algorithm.

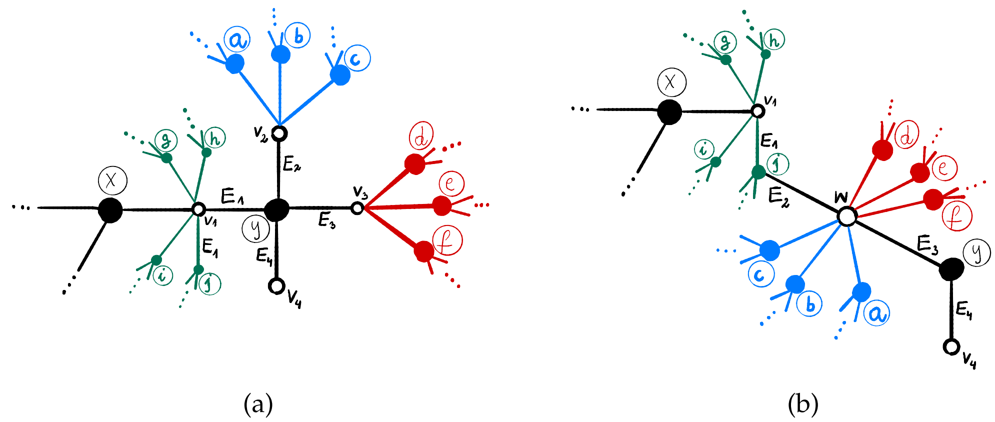


Figure 4: (a) The initial configuration of the tree. (b) The output of $\mathbb{J}_{x,y}$.

2.2 The second step: from a tree with repeated edge labels to a tree with unique edge labels

The starting point of the second step of our algorithm is the bicolored plane tree T_1 with black vertices labelled $1, \dots, n$ and with edges labelled with the numbers $1, \dots, k$; note that the edge labels are repeated. Our goal in this second step of the algorithm is to transform the tree so that the edge labels are not repeated. As an auxiliary tool we will use two operations, called *jump* and *exchange*.

2.2.1 The building blocks of the second step: jump $\mathbb{J}_{x,y}$

The input of $\mathbb{J}_{x,y}$. The operation $\mathbb{J}_{x,y}$ takes as an input a bicolored tree T together with a choice of two of black vertices x, y which are assumed to be at the distance 2, see Figure 4a. We denote their common white neighbor by v_1.

We denote by j the black neighbor of v_1 which—going clockwise around the vertex v_1—is immediately after y (note that it might happen that $j = x$). We also assume that the two edges which form the path between y and j are labelled by the same symbol denoted by E_1, see Figure 4a.

We also assume that the black vertex y has degree at least 3; we denote the edges around the vertex y by $E_1, \dots, E_d$ (going clockwise, starting from the edge E_1) with $d \geq 3$. We denote the white endpoint of the edge E_i by v_i.

The output of $\mathbb{J}_{x,y}$ is defined as follows. We remove the three edges connecting y with v_1, v_2, v_3. We create a new white vertex denoted w and we connect it to the vertex j by a new edge which we label E_2. More specifically, going clockwise around j the newly created edge E_2 is immediately after the edge E_1.

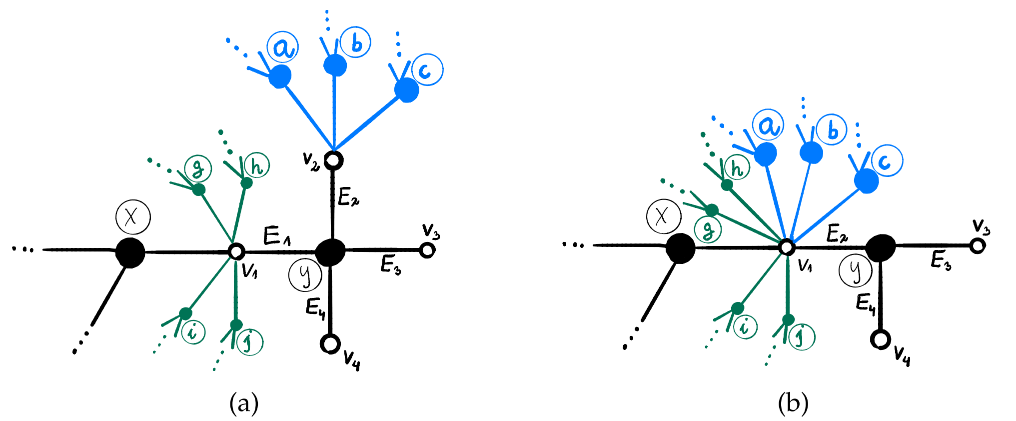


Figure 5: (a) The initial configuration of the tree. (b) The output of $\mathbb{E}_{x,y}$.

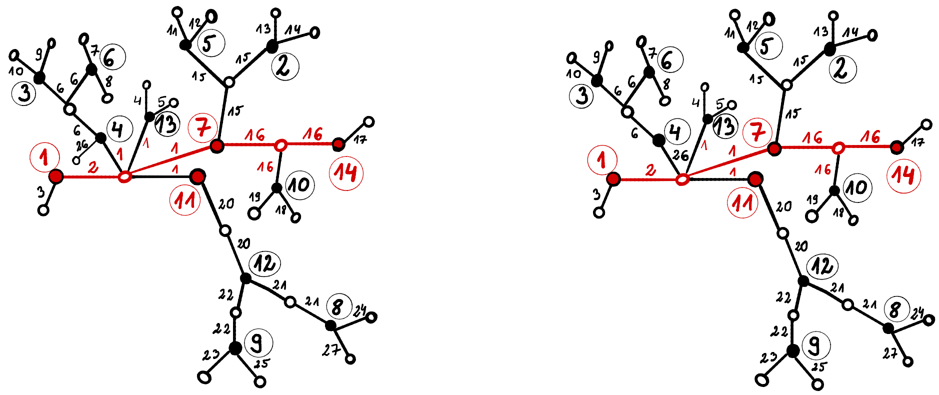
We also connect the vertex w to the vertex y by a new edge which we label E_3. The position of the edge E_3 in the vertex y replaces the three edges which were removed from y.

We merge the vertices v_2 and v_3 to the vertex w. More specifically, the clockwise cyclic order of the edges around the vertex w is as follows: the edge E_2, then the edges from the vertex V_3 (listed in the clockwise order starting from the removed edge E_3; on Figure 4 these edges are marked red), the edge E_3, then the edges from the vertex v_2 (listed in the clockwise order starting from the removed edge E_2; on Figure 4 these edges are marked blue), see Figure 4b.

2.2.2 The building blocks of the second step: exchange $\mathbb{E}_{x,y}$

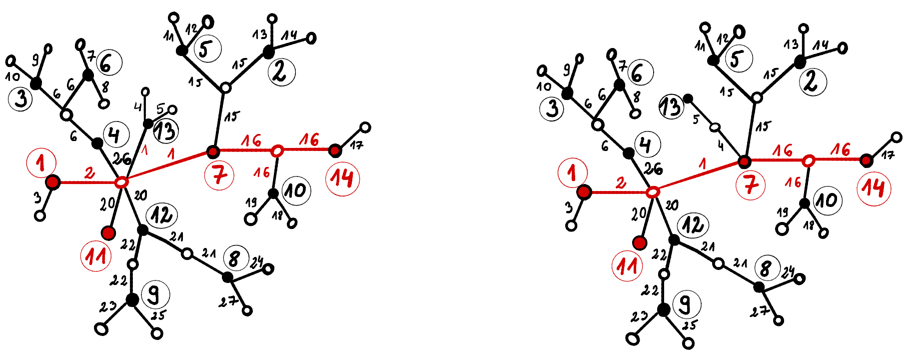
The input of $\mathbb{E}_{x,y}$. The operation $\mathbb{E}_{x,y}$ takes as an input a bicolored tree T together with a choice of two of black vertices x, y which are assumed to be at the distance 2, see Figure 5a. We denote their common white neighbor by v_1. We also assume that the black vertex y has degree at least 2; we denote the edges around the vertex y by $E_1, \dots, E_d$ (going clockwise, starting from the edge E_1) with $d \geq 2$. We denote the white endpoint of the edge E_i by v_i.

The output of $\mathbb{E}_{x,y}$ is defined as follows. We remove the edge between y and v_2. The label of the edge between v_1 and y is changed to E_2. Then we merge the vertex v_2 with the vertex v_1 in such a way that going clockwise around v_1 the newly attached edges are immediately before the edge E_2 (these edges are marked blue on Figure 5).



(a) (b)

Figure 6: (a) The output of $\mathbb{E}_{1,11}$. (b) The output of $\mathbb{E}_{7,4}$.



(a) (b)

Figure 7: (a) The output of $\mathbb{E}_{7,11}$. (b) The output of $\mathbb{J}_{7,13}$.

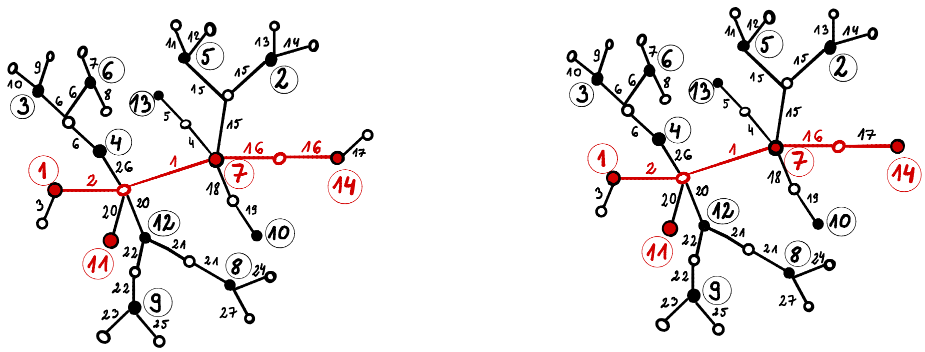
2.2.3 Spine treatment

For each $i \in S_E$ we apply the following procedure (the final output will not depend on the order in which we choose the elements of S_E). Since the spine in T_0 is a path, the intersection $B_i \cap S_B$ corresponds to the labels of the two black spine neighbors of the white vertex i in the tree T_0. We denote $B_i \cap S_B = \{x, y_1\}$ with $x < y_1$. We run the following loop over $y \in B_i \setminus \{x\}$ (with the ascending order). If $y = y_1$ or $y < x$ then we apply $\mathbb{E}_{x,y}$; otherwise we apply $\mathbb{J}_{x,y}$.

Example. We continue the example from Figure 3b. We recall that $S_E = \{1, 2, 16\}$.

For $i = 2$ we have $x = 1$ and $y_1 = 11$. Since $B_2 \setminus \{1\} = \{11\}$, the internal loop is applied once with $y = 11$. As a result we apply $\mathbb{E}_{1,11}$, see Figure 6a.

For $i = 1$ we have $x = 7$ and $y_1 = 11$. Since $B_1 \setminus \{7\} = \{4, 11, 13\}$ the loop runs over:



(a) (b)

Figure 8: (a) The output of $\mathbb{J}_{7,10}$. (b) The output of $\mathbb{E}_{7,14}$.

• $y = 4$ and we apply $\mathbb{E}_{7,4}$, see Figure 6b; • $y = 11$ and we apply $\mathbb{E}_{7,11}$, see Figure 7a; • $y = 13$ and we apply $\mathbb{J}_{7,13}$, see Figure 7b.

For $i = 16$ we have $x = 7$ and $y_1 = 14$. Since $B_{16} \setminus \{7\} = \{10, 14\}$ the loop runs over: • $y = 10$ and we apply $\mathbb{J}_{7,10}$, see Figure 8a; • $y = 14$ and we apply $\mathbb{E}_{7,14}$, see Figure 8b.

2.2.4 Rib treatment

For each successive edge label i from the set L we apply the following procedure (the final output depends on the order in which we choose the label of edges).

We run the following loop over $y \in B_i \setminus \{p_i\}$ (with the ascending order). If $y < p_i$ we apply $\mathbb{E}_{p_i,y}$; otherwise we apply $\mathbb{J}_{p_i,y}$.

Example. We continue the example from Figure 8b. We recall that $L = (6, 15, 20, 21, 22)$.

For $i = 6$ we have $p_6 = 4$. Since $B_6 \setminus \{4\} = \{3, 6\}$ the loop runs over: • $y = 3$ and we apply $\mathbb{E}_{4,3}$, see Figure 9a; • $y = 6$ and we apply $\mathbb{J}_{4,6}$, see Figure 9a.

For $i = 15$ we have $p_{15} = 7$. Since $B_{15} \setminus \{7\} = \{2, 5\}$ the loop runs over: • $y = 2$ and we apply $\mathbb{E}_{7,2}$, see Figure 9b; • $y = 5$ and we apply $\mathbb{E}_{7,5}$, see Figure 9b.

For $i = 20$ we have $p_{20} = 11$. Since $B_{20} \setminus \{11\} = \{12\}$, the internal loop is applied once with $y = 12$. As a result we apply $\mathbb{J}_{11,12}$, see Figure 10a.

For $i = 21$ we have $p_{21} = 12$. Since $B_{21} \setminus \{12\} = \{8\}$, the internal loop is applied once with $y = 8$. As a result we apply $\mathbb{E}_{12,8}$, see Figure 10b.

For $i = 22$ we have $p_{22} = 12$. Since $B_{22} \setminus \{12\} = \{9\}$, the internal loop is applied once with $y = 9$. As a result we apply $\mathbb{E}_{12,9}$, see Figure 11a.

Figure 11b gives the output of our algorithm applied to the minimal factorization (2.1). The result is a Stanley tree of type (1^{14}).

Due to space restrictions, the explicit construction of the inverse of the bijection from

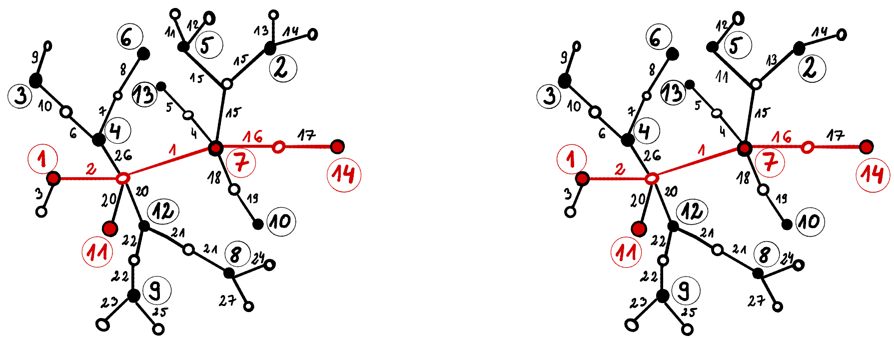


Figure 9: (a) The output of $\mathbb{E}_{4,3}$ and $\mathbb{J}_{4,6}$. (b) The output of $\mathbb{E}_{7,2}$ and $\mathbb{E}_{7,5}$.

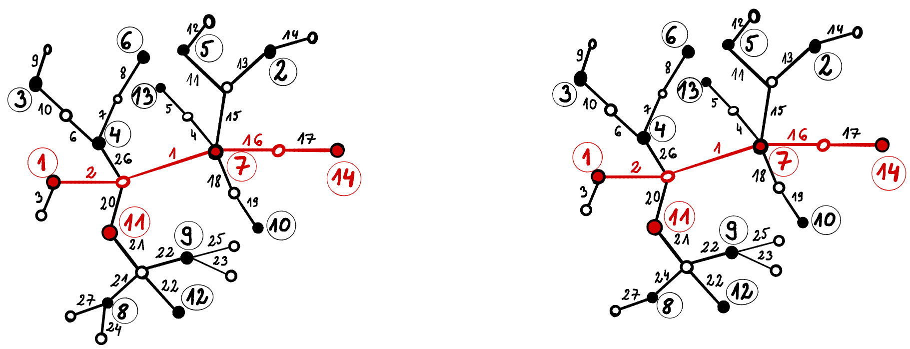


Figure 10: (a) The output of $\mathbb{J}_{11,12}$. (b) The output of $\mathbb{E}_{12,8}$.

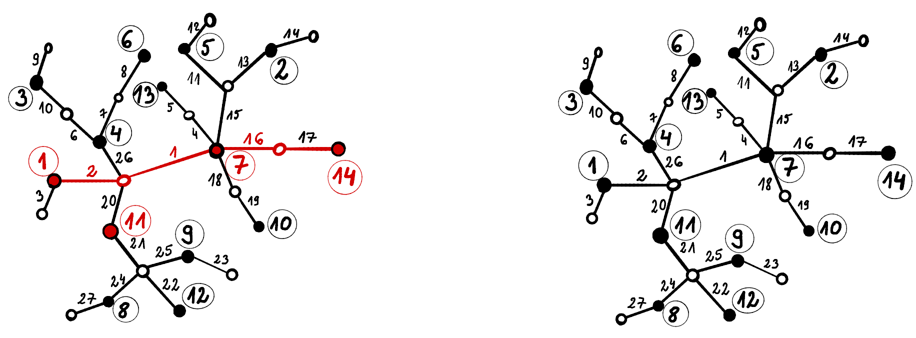


Figure 11: (a) The output of $\mathbb{E}_{12,9}$. (b) The output of our algorithm applied to the minimal factorization (2.1).

Sections 2.1 to 2.2, as well as the proof of its correctness, is postponed to the forthcoming paper [9].

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