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Quadrant Walks Starting Outside the Quadrant

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Abstract. We investigate a functional equation which resembles the functional equation for the generating function of a lattice walk model for the quarter plane. The interesting feature of this equation is that its orbit sum is zero while its solution is not algebraic. The solution can be interpreted as the generating function of lattice walks in $\mathbb{Z}^2$ starting at $(-1,-1)$ and subject to the restriction that the coordinate axes can be crossed only in one direction. We also consider certain variants of the equation, all of which seem to have transcendental solutions. In one case, the solution is perhaps not even D-finite.

Keywords: Restricted lattice walks, Orbit sum, D-finiteness, Transcendence

1 Introduction

The investigation of lattice walks with small steps restricted to a quadrant has made astonishing progress during the past years [7, 11, 5, 17, 3, 9, 1, 2, 10]. The central problem in this context is to decide for a given step set $S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$ whether the generating function $F(x,y,t) = \sum_{n=0}^{\infty}\sum_{i,j=0}^{n} a_{i,j,n}x^iy^jt^n$ counting the number $a_{i,j,n}$ of walks in $\mathbb{N}^2$ starting at $(0,0)$, ending at (i,j), and consisting of exactly n steps, each step taken from S, is D-finite. If so, it is further of interest whether it is even algebraic. Although it is not obvious at first glance, it is meanwhile well understood how the finiteness of a certain group associated to associated to the model implies its algebraicity [12, 16, 19].

For simplicity, let us focus on the step set $S = \{\leftarrow,\rightarrow,\uparrow,\downarrow\}$. If $Q(x,y,t)$ is the generating function for this model, the combinatorial definition translates into the functional equation

$$(1-(x+y+\bar{x}+\bar{y})t)Q(x,y,t) = 1 - \bar{x}tQ(0,y,t) - \bar{y}tQ(x,0,t),$$

where we write $\bar{x} = 1/x$ and $\bar{y} = 1/y$ for short. The group of the model can be used to solve this equation. In the present example, it is generated by the rational maps $\Phi = (\binom{x}{y} \mapsto \binom{\bar{x}}{y})$ and $\Psi = (\binom{x}{y} \mapsto \binom{x}{\bar{y}})$ under composition, i.e., $G = \{\mathrm{id},\Phi,\Psi,\Phi\circ\Psi\}$. The idea for solving the functional equation is to let the elements of the group act on

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it to get four copies of the equation, and then take a linear combination of these four copies with the aim of canceling the terms $Q(\cdots)$ appearing on the right. This leads to

$$\begin{aligned}&(1-(x+y+\bar{x}+\bar{y})t)\big(xyQ(x,y,t)-\bar{x}yQ(\bar{x},y,t)-x\bar{y}Q(x,\bar{y},t)+\bar{x}\bar{y}Q(\bar{x},\bar{y},t)\big)\\&\qquad=xy-\bar{x}y-x\bar{y}+\bar{x}\bar{y}.\end{aligned}$$

The expression on the right is the *orbit sum.* Divide by $1-(x+y+\bar{x}+\bar{y})t$ and observe next that $xyQ(x,y,t)$ is the only term on the left whose exponents with respect to x and y are positive, while for all terms $x^iy^jt^n$ appearing in any of the other terms on the left we have $i<0$ or $j<0$. Therefore, by extracting the positive part, we can eliminate the unwanted terms $Q(\bar{x},y,t), Q(x,\bar{y},t), Q(\bar{x},\bar{y},t)$ and get

$$xyQ(x,y,t)=[x^>][y^>]\frac{xy-\bar{x}y-x\bar{y}+\bar{x}\bar{y}}{1-(x+y+\bar{x}+\bar{y})t}. \tag{Q}$$

Since extracting the positive part preserves D-finiteness [18], $Q(x,y,t)$ is D-finite.

The step sets $\{\leftarrow,\downarrow,\nearrow\}$ (Kreweras), $\{\rightarrow,\uparrow,\swarrow\}$ (reverse Kreweras), $\{\leftarrow,\rightarrow,\downarrow,\uparrow,\nearrow,\swarrow\}$ (double Kreweras), and $\{\leftarrow,\swarrow,\rightarrow,\nearrow\}$ (Gessel) also have a certain finite group of rational maps associated to them, but the approach above for solving the functional equations for the generating functions fails, because the orbit sum turns out to be zero in these cases. Using more sophisticated arguments, it can be shown that the generating functions for these models are not only D-finite but in fact algebraic [7, 4]. In fact, the generating function happens to be algebraic if and only if the orbit sum vanishes.

The equivalence between zeroness of the orbit sum and algebraicity of the generating function is not an accident, but it can be explained [17, Sec. 8 and Sec. 9.1]. However, as we shall show in this paper, the equivalence does not hold in all circumstances. Consider the following slight variation of the functional equation quoted above for the step set $\{\leftarrow,\rightarrow,\uparrow,\downarrow\}$:

$$(1-(x+y+\bar{x}+\bar{y})t)F(x,y,t)=\bar{x}\bar{y}-\bar{x}tF(0,y,t)-\bar{y}tF(x,0,t). \tag{F}$$

The only difference is that we replaced the inhomogeneous term 1 by $\bar{x}\bar{y}$. If we now multiply the equation by xy, as above, then let the group elements act on the equation, as above, and then take the weighted sum of the resulting equations, as above, we get

$$(1-(x+y+\bar{x}+\bar{y})t)\big(xyF(x,y,t)-\bar{x}yF(\bar{x},y,t)-x\bar{y}F(x,\bar{y},t)+\bar{x}\bar{y}F(\bar{x},\bar{y},t)\big)=0,$$

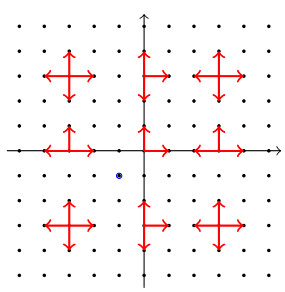
and it is not clear how to proceed from here.

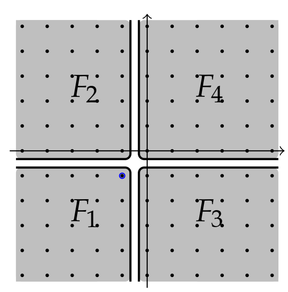
It is clear that any solution $F(x,y,t)\in\mathbb{Q}[x,\bar{x},y,\bar{y}][[t]]$ will have the form $\bar{x}\bar{y}t^0+\cdots$, so before we proceed, we should clarify what we mean by the expressions $F(0,y,t)$ and $F(x,0,t)$ on the right hand side of (F). There are several options. A natural interpretation is $F(0,y,t)=[x^0]F(x,y,t)$ and $F(x,0,t)=[y^0]F(x,y,t)$. We consider this case in

Section 2. Other interpretations also include certain restrictions on the other variable. For example, we could choose to read $F(0,y,t)$ as $[x^0][y^<]F(x,y,t)$ (keeping only negative exponents of y), as $[x^0][y^\geq]F(x,y,t)$ (keeping only nonnegative exponents of y), as $[x^0][y^\leq]F(x,y,t)$ (keeping only nonpositive exponents of y), or as $[x^0][y^>]F(x,y,t)$ (keeping only positive exponents of y), and analogously for $F(x,0,t)$. For some of these interpretations, we can show that the solution of the functional equation is D-finite. For some of them, we can show that the correctness of a guessed differential equation for the specialization $F(1,1,t)$ of the solution $F(x,y,t)$ implies the transcendence of the solution (Sections 3 and 4). One case seems to be more complicated. In this case, we conjecture that the solution is not even D-finite (Section 5).

2 Four Compartments

With the interpretation $F(0,y,t) = [x^0]F(x,y,t)$ and $F(x,0,t) = [y^0]F(x,y,t)$, the solution $F(x,y,t)$ of the functional equation (F) counts walks that start at $(-1,-1)$ and move through the plane $\mathbb{Z}^2$ subject to the restriction that the axes of the coordinate system can be passed only in one direction (west to east or south to north, respectively). We claim that the generating function $F(x,y,t)$ counting walks in this model is D-finite. To show this, define $F_1 = [x^<][y^<]F$, $F_2 = [x^<][y^\geq]F$, $F_3 = [x^\geq][y^<]F$, and $F_4 = [x^\geq][y^\geq]F$, so that $F = F_1 + F_2 + F_3 + F_4$.





The equation for F translates into the following system of functional equations for F_1, F_2, F_3, F_4, where we write $S = x + y + \bar{x} + \bar{y}$:

$$\begin{aligned}
F_1 &= \bar{x}\bar{y} && + StF_1 - t[\bar{x}]F_1 - t[\bar{y}]F_1 \\
F_2 &= t[\bar{y}]F_1 && + StF_2 - t[\bar{x}]F_2 - \bar{y}t[y^0]F_2 \\
F_3 &= t[\bar{x}]F_1 && + StF_3 - t[\bar{y}]F_3 - \bar{x}t[x^0]F_3 \\
F_4 &= \underbrace{t[\bar{x}]F_2 + t[\bar{y}]F_3}_{\text{initial conditions}} && + \underbrace{StF_4}_{\text{recurrence}} \underbrace{- \bar{x}t[x^0]F_4 - \bar{y}t[y^0]F_4}_{\text{boundary conditions}}.
\end{aligned}$$

The equation for F_1 does not depend on F_2, F_3, F_4 and can therefore be solved directly. In fact, we have $F_1(x,y,t) = \bar{x}\bar{y}Q(\bar{x},\bar{y},t)$ for the $Q(x,y,t)$ from equation (Q).

Knowing F_1, we can solve the second equation for F_2 by the same technique. Noting that $[\bar{y}]F_1$ is independent of y, the result is

$$F_2(x,y,t) = \bar{y}[x^<][y^>]\frac{t(y[\bar{y}]F_1(x,y,t) - y[\bar{y}]F_1(\bar{x},y,t) - \bar{y}[\bar{y}]F_1(x,y,t) + \bar{y}[\bar{y}]F_1(\bar{x},y,t))}{1-St}$$
$$= t\bar{y}[x^<][y^>]\frac{(y-\bar{y})[\bar{y}](F_1(x,y,t) - F_1(\bar{x},y,t))}{1-St},$$

so F_2 is D-finite because it is the positive/negative part of a D-finite series. Moreover, using $F_1(x,y,t) = \bar{x}\bar{y}Q(\bar{x},\bar{y},t)$ and (Q) we get

$$F_1(x,y,t) - F_1(\bar{x},y,t) = [y^<]\frac{xy - \bar{x}y - x\bar{y} + \bar{x}\bar{y}}{1-St},$$

which can be used to simplify the expression for $F_2(x,y,t)$ further to

$$F_2(x,y,t) = t\bar{y}[x^<]\Big(\big([y^>]\frac{y-\bar{y}}{1-St}\big)\big([\bar{y}]\frac{xy-\bar{x}y-x\bar{y}+\bar{x}\bar{y}}{1-St}\big)\Big).$$

Because of symmetry, we have $F_3(x,y,t) = F_2(y,x,t)$, so this one is D-finite too, and we can directly proceed to the equation for F_4, which we can now solve in terms of the known functions F_2, F_3, again by letting the group elements act, forming a weighted sum, dividing by $1-St$ and extracting the positive part. The result is

$$F_4(x,y,t) = \bar{x}\bar{y}t[x^>][y^>]\frac{G(x,y,t) - G(\bar{x},y,t) - G(x,\bar{y},t) + G(\bar{x},\bar{y},t)}{1-St},$$

with $G(x,y,t) = xy[\bar{x}]F_2(x,y,t) + xy[\bar{y}]F_3(x,y,t)$. We already see at this point that F_4 is D-finite, because it is the positive part of a D-finite series, so we can conclude that $F = F_1 + F_2 + F_3 + F_4$ is D-finite, because it is the sum of four D-finite series. Moreover, using the expression for F_2 derived above, we can state F_4 explicitly as

$$F_4(x,y,t) = \bar{x}\bar{y}t^2[y^>]\Big(\big([\bar{x}]\frac{(y-\bar{y})[\bar{y}]\frac{xy-\bar{x}y-x\bar{y}+\bar{x}\bar{y}}{1-St}}{1-St}\big)\big([x^>]\frac{x-\bar{x}}{1-St}\big)\Big)$$
$$+\bar{x}\bar{y}t^2[x^>]\Big(\big([\bar{y}]\frac{(x-\bar{x})[\bar{x}]\frac{xy-\bar{x}y-x\bar{y}+\bar{x}\bar{y}}{1-St}}{1-St}\big)\big([y^>]\frac{y-\bar{y}}{1-St}\big)\Big).$$

The expressions we found for F_1, F_2, F_3, F_4 are small enough that we succeeded to use the techniques from [3] and Koutschan's package [15] to construct a certified annihilating operator for $F(1,1,t)$. We suppress the computational details here and refer the interested reader to the Mathematica session posted on the website of this paper [8]. The bottom line is that the coefficient sequence of $F(1,1,t)$ satisfies the recurrence

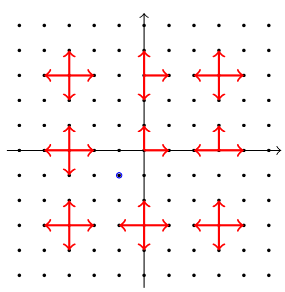
$$(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2}$$
$$-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1}$$
$$-16(1+n)(2+n)(3+n)(2+4n+n^2)a_n = 0.$$

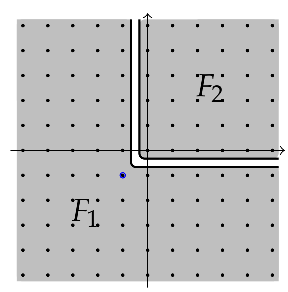
This recurrence has only asymptotic solutions of the form $4^n n^{-1}$ and $(-4)^n n^{-3}$ (as can be found using [13, 14]). Neither of these forms can arise from an algebraic series, so $F(1,1,t)$ must be transcendental.

In summary, we have shown that for the interpretation $F(0,y,t) = [x^0]F(x,y,t)$ and $F(x,0,t) = [y^0]F(x,y,t)$, the solution $F(x,y,t)$ of the functional equation (F) is D-finite but not algebraic.

3 A Large and a Small Compartment

We now turn to the variant of (F) in which $F(0,y,t)$ is interpreted as $[x^0][y^{\geq}]F(x,y,t)$, and $F(x,0,t)$ likewise. In this case, the equation describes a model in which only the nonnegative part of each axis forms a semipermeable barrier in the sense that walks can enter the non-negative quadrant, but they can not leave it. Walks in this model can freely move around in the complement of the north-east quadrant, which is a three quarter plane, and once they leave this area, they are locked in the north-east quadrant. It is therefore natural to write the generating function $F(x,y,t)$ for this model as $F = F_1 + F_2$ where $F_1 = [x^<]F + [x^{\geq}][y^<]F$ keeps track of the three quarter plane and $F_2 = [x^{\geq}][y^{\geq}]F$ takes care of the remaining quarter plane.





It is known [6, 20] that the generating function $C(x,y,t)$ for simple walks avoiding the positive quadrant is D-finite. Hence also $F_1(x,y,t) = \bar{x}\bar{y}C(x,y,t)$ is D-finite. The series F_2 counts walks in the quarter plane with initial conditions prescribed by the sections of F_1:

$$F_2 = \underbrace{t[\bar{x}][y^{\geq}]F_1 + t[\bar{y}][x^{\geq}]F_1}_{\text{initial conditions}} + \underbrace{StF_2}_{\text{recurrence}} \underbrace{- \bar{x}t[x^0]F_2 - \bar{y}t[y^0]F_2}_{\text{boundary conditions}}.$$

This is again a functional equation which we can solve like in the introduction, the result being a positive part expression in terms of F_1:

$$F_2 = \bar{x}\bar{y}t[x^>][y^>]\frac{H(x,y,t) + H(y,x,t)}{1 - St},$$

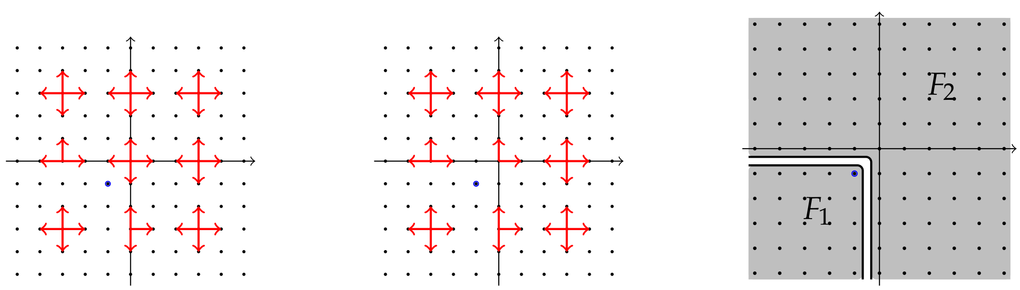
with $H(x,y,t) = (\bar{x} - x)[\bar{x}]\big(\bar{y}[y^{\leq}]F_1(x,\bar{y},t) - y[y^{\geq}]F_1(x,y,t)\big)$. This implies that F_2 is D-finite, so $F = F_1 + F_2$ is D-finite as well.

Unfortunately, in this case we were not able to derive a certified annihilating operator for $F(1,1,t)$ from this expression. Owing to the size of the equations describing F_1, the computations were too expensive. However, it is easy to guess an annihilating operator L from the first few terms of $F(1,1,t)$. We found a convincing candidate of order 11 with polynomial coefficients of degree 89. It is posted on our website [8]. Assuming that this guessed operator L is correct, we can show that $F(1,1,t)$ is transcendental. The key observation is that the guessed operator can be written as $L = \operatorname{lclm}(L_1,\dots,L_6)$, where $L_1,\dots,L_6$ are certain irreducible operators. The factors $L_1,\dots,L_6$ are also posted on our website [8]. It turns out that $L_1,\dots,L_5$ only have algebraic solutions, while L_6 has a logarithmic singularity and therefore can not have any nonzero algebraic solution. The factorization $L = \operatorname{lclm}(L_1,\dots,L_6)$ means that we have $F(1,1,t) = f_1 + \cdots + f_6$ where each f_i is a solution of L_i. Since $f_1,\dots,f_5$ are algebraic and f_6 is not, it follows that $F(1,1,t)$ is not algebraic unless the term f_6 is zero. In this case however $F(1,1,t)$ would already be annihilated by $\operatorname{lclm}(L_1,\dots,L_5)$, and it can be checked that this is not the case.

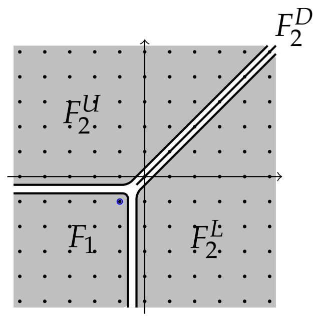
In summary, we have shown that for the interpretation $F(0,y,t) = [x^0][y^{\geq}]F(x,y,t)$ and $F(x,0,t) = [y^0][x^{\geq}]F(x,y,t)$, the solution $F(x,y,t)$ of the functional equation (F) is D-finite. Moreover, under the hypothesis that a guessed annihilating operator for $F(1,1,t)$ is correct, we can also show that $F(1,1,t)$ is transcendental.

4 A Small and a Large Compartment

We now turn to the two variants of (F) where $F(x,0,t)$ and $F(0,y,t)$ are interpreted as $[x^<][y^0]F(x,y,t)$ and $[x^0][y^<]F(x,y,t)$, and as $[x^{\leq}][y^0]F(x,y,t)$ and $[x^0][y^{\leq}]F(x,y,t)$, respectively. In these models the negative and non-positive part, respectively, of each axis forms a semipermeable barrier for the walks. In both of these models walks start in the south-west quadrant, they may leave it, but once left, they do not enter it again. Only in the second model walks that end on the origin can neither be extended by a west step nor a south step. Let $F_1 = [x^< y^<]F$ and $F_2 = F - F_1$.



As in Section 2, we have $F_1(x,y,t) = \bar{x}\bar{y}Q(\bar{x},\bar{y})$ for the $Q(x,y,t)$ from equation (Q). If $F(x,0,t) = [x^< y^0]F(x,y,t)$ and $F(0,y,t) = [x^0 y^<]F(x,y,t)$ (the left-most case on the previous figure) we can show D-finiteness of F_2 using the analytic method [19, 20]. The argument presented in [20] is based on the decomposition $F_2 = F_2^U + F_2^D + F_2^L$ with $F_2^D = \sum_{i\geq 0} x^i y^i [x^i y^i] F_2$ and $F_2^L = \sum_{i\geq 0, j\leq i-1} x^i y^j [x^i y^j] F_2$ and $F_2^U(x,y,t) = F_2^L(y,x,t)$.



The functions F_2^D and F_2^L satisfy the equations

$$F_2^D = 2t(\bar{x}+y)\sum_{i\geq 0} x^i y^{i-1}[x^i y^{i-1}]F_2^L \qquad - 2t\bar{x}\bar{y}[x^0\bar{y}]F_2^L$$

$$F_2^L = \underbrace{t[\bar{x}]F_1 + t(x+\bar{y})F_2^D}_{\text{initial conditions}} + \underbrace{StF_2^L}_{\text{recurrence}} \underbrace{- t(\bar{x}+y)\sum_{i\geq 0} x^i y^{i-1}[x^i y^{i-1}]F_2^L + t\bar{x}\bar{y}[x^0\bar{y}]F_2^L - t\bar{x}[x^0]F_2^L}_{\text{boundary conditions}}.$$

Eliminating the term involving the infinite sum gives the equation

$$(1-St)F_2^L = t[\bar{x}]F_1 + (tx + t\bar{y} - \tfrac{1}{2})F_2^D - t\bar{x}[x^0]F_2^L. \qquad (*)$$

From here on, we can follow the derivation in [20] step by step and obtain an expression of the form

$$F_2^D(xy,\bar{x},t) = A(y)\oint B(y,z)[\bar{y}]F_1\big(yC(z), 1/C(z), t\big)dz,$$

where A, B, and C are certain algebraic functions, the integral is taken around the unit circle, and t is viewed as a fixed small positive real number. The interested reader will find on our website [8] a Maple session in which this derivation is worked out in full detail, and where also explicit expressions for A, B, and C are provided. What matters here is that the D-finiteness of F_1 together with the algebraicity of A, B, C implies the D-finiteness of F_2^D. Knowing this we can solve the functional equation ($*$) for $[x^0]F_2^L$ after setting x to a root $X(y,t) \in \mathbb{Q}[y,\bar{y}][[t]]$ of the polynomial $1 - St \in \mathbb{Q}[x,\bar{x},y,\bar{y},t]$ in order to eliminate the left hand side. The resulting expression

$$[x^0]F_2^L = X(y,t)[\bar{x}]F_1 + X(y,t)(X(y,t) + \bar{y} - \frac{1}{2t})F_2^D(X(y,t),y,t)$$

certifies that $[x^0]F_2^L$ is D-finite. With the knowledge that F_1, F_2^D, and $[x^0]F_2^L$ are D-finite, it follows from ($*$) that F_2^L is D-finite. Then $F_2^U(x,y,t) = F_2^L(y,x,t)$ is D-finite as well, and it finally follows that $F_2 = F_2^D + F_2^L + F_2^U$ and $F = F_1 + F_2$ are D-finite, as claimed.

Like in the previous section, we were not able to construct a certified annihilating operator for the series $F(1,1,t)$ but only have a convincing guess. Assuming however that this guess is correct, we can again show that the series $F(1,1,t)$ must be transcendental. The reasoning is like in the previous section: the guessed operator now has order 10 and can be written as the least common left multiple of four irreducible operators, exactly one of them admits transcendental solutions and therefore only has transcendental solutions. The lclm of the remaining operators does not annihilate $F(1,1,t)$, so $F(1,1,t)$ must be transcendental. The operators are available on our website [8].

Unfortunately, we are not able to prove D-finiteness of the solution for the interpretation of $F(0,y,t)$ and $F(x,0,t)$ as $[x^0][y^{\leq}]F(x,y,t)$ and $[y^0][x^{\leq}]F(x,y,t)$, respectively. If we proceed as above, we are led to an expression of the form

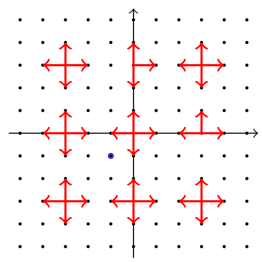
$$F_2^D(xy,\bar{x},t) = A(y) \oint B(y,z)\Big([\bar{y}]F_1\big(yC(z),1/C(z),t\big) - [y^0]F_2^D(xy,\bar{x},t)\Big)dz,$$

where A, B, and C are again algebraic functions, the integral is taken around the unit circle, and t is viewed as a fixed small positive real number. As we do not know whether $[y^0]F_2^D(xy,\bar{x},t)$ is D-finite, we are stuck at this point. It does seem however that the solution $F(x,y,t)$ is D-finite also in this case, at least for $x = y = 1$. We have found a convincing candidate for an annihilating operator of order 13 by guessing. Again, this operator can be written as the least common left multiple of irreducible operators $L_1,\dots,L_6$, available on our website, so we can write $F(1,1,t) = f_1 + \cdots + f_6$ for certain solutions f_i of the irreducible right factors L_i. The difference to the earlier cases is that now several of the L_i have transcendental solutions, so in order to show that $F(1,1,t)$ is not algebraic, we must show that their sum is not algebraic. This can be done by constructing an operator P which annihilates all but one of the transcendental summands f_i, so that $P \cdot F(1,1,t)$ can be written as a sum of one transcendental series and some algebraic series, which asserts that $P \cdot F(1,1,t)$ is transcendental. But then $F(1,1,t)$ must be transcendental as well, because if it were algebraic, so would be $P \cdot F(1,1,t)$.

In summary, we have shown that for the interpretation $F(0,y,t) = [x^0][y^{<}]F(x,y,t)$ and $F(x,0,t) = [y^0][x^{<}]F(x,y,t)$, the solution $F(x,y,t)$ of the functional equation (F) is D-finite. Moreover, under the hypothesis that guessed annihilating operators for the solutions at $x = y = 1$ are correct, we can also show that these series are transcendental. For the interpretation $F(0,y,t) = [x^0][y^{\leq}]F(x,y,t)$ and $F(x,0,t) = [y^0][x^{\leq}]F(x,y,t)$, we have no proof that the solution $F(x,y,t)$ is D-finite, but we have a guessed equation for $F(1,1,t)$ whose correctness implies the transcendence of the solution.

5 No Compartments

One other variant of (F) is when we read $F(0,y,t)$ as $[x^0][y^>]F(x,y,t)$, and $F(x,0,t)$ likewise. In this model the positive part of each axis forms a semipermeable barrier for the walks.



In contrast to the models before there is no natural division of the domain into compartments that cannot be left once entered. Even though a walk may pass through the positive part of either axis only in one direction, it can still escape from the first quadrant through the origin. It is arguably for this reason that this model appears to be more difficult than the others. Indeed, we have not been able to solve it.

Computer experiments with the first 2000 series terms suggest that the coefficient sequence of $F(1,1,t)$ grows asymptotically like $c4^n n^{-1/3}$ for $n \to \infty$ and some constant $c \approx 1.91$. Moreover, for the number a_n of walks of length $2n$ starting and ending at $(-1,-1)$, we find, based on 6300 sequence terms, a conjectured asymptotic behaviour of the form $c4^n n^{-5/3}$ for $n \to \infty$ and some nonzero constant c. Even if these growth rates are correct, they cannot even be used to exclude algebraicity of the generating functions.

We have also searched for candidates for algebraic and differential equations by guessing based on almost 98000 sequence terms (modulo 45007), but did not find any. This implies that such equations, if they exist, must be extraordinarily large. We are tempted to conjecture that they do not exist, i.e., that the solution $F(x,y,t)$ of (F) is not D-finite for the interpretation under consideration. The terms we computed can be found on our website.

6 Conclusion

We investigated the functional equation

$$(1-(x+y+\bar{x}+\bar{y})t)F(x,y,t) = \bar{x}\bar{y} - \bar{x}tF(0,y,t) - \bar{y}tF(x,0,t)$$

and its solution $F(x,y,t)$ in $\mathbb{Q}[x,\bar{x},y,\bar{y}][[t]]$ for different interpretations of $F(0,y,t)$ and $F(x,0,t)$. For $F(0,y,t) = [x^0]F$ and $F(x,0,t) = [y^0]F$ we answered the main questions: we proved that F is D-finite, and we showed that it is not algebraic. We point out that in [17, Sec. 8] it was shown that the orbit-sum is zero if and only if the generating function is algebraic. The reasoning is based on the assumption that the generating function is analytic in a neighbourhood of the origin. Since this is not the case for the examples discussed here, there is no conflict with our observations.

For other interpretations there are several open points which would deserve further consideration. One point is the pending proof of the guessed operators on which the transcendence arguments of Sections 3–4 rely. In the second case considered in Section 4, not only the guessed operator but also D-finiteness in general remains to be proven. Another open issue is the clarification of the nature of the solution in Section 5: is it really non-D-finite? Besides answering these open questions, there are some natural extensions and generalizations which could be addressed. For example, we have only considered analogous interpretations for $F(x, 0, t)$ and $F(0, y, t)$ in this paper, but mixed cases such as $F(x, 0, t) = [y^0][x^>]F(x, y, t)$ and $F(0, y, t) = [x^0][y^\leq]F(x, y, t)$ might also be interesting. First experiments suggest that some combinations are D-finite. Another possible variation concerns the starting point. There are other points besides $(-1, -1)$ which lead to a zero orbit sum, for example $(-1, 1)$. Can the starting point affect the nature of the solution? Finally, we have restricted ourselves to the case of simple walks, and it would be interesting to see what happens for other step sets.

References

[1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. *Counting quadrant walks via Tutte's invariant method*. Tech. rep. 2017. arXiv:1708.08215.

[2] A. Bostan. "Calcul Formel pour la Combinatoire des Marches". Habilitation a Diriger des Recherches, Universite Paris 13. 2017.

[3] A. Bostan, F. Chyzak, M. van Hoeij, M. Kauers, and L. Pech. "Hypergeometric Expressions for Generating Functions of Walks with Small Steps in the Quarter Plane". *European Journal of Combinatorics* **61** (2017), pp. 242–275.

[4] A. Bostan and M. K. with an appendix by Mark van Hoeij. "The Complete Generating Function for Gessel Walks is algebraic". *Proceedings of the AMS* **138**.9 (2010), pp. 3063–3078.

[5] A. Bostan, K. Raschel, and B. Salvy. "Non-D-finite excursions in the quarter plane". *Journal of Combinatorial Theory, Series A* **121** (2014), pp. 45–63.

[6] M. Bousquet-Mélou. "Plane lattice walks avoiding a quadrant". *Journal of Combinatorial Theory A* **144** (2016), pp. 37–79.

[7] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". *Contemporary Mathematics* **520** (2010), pp. 1–40.

[8] M. Buchacher, M. Kauers, and A. Trotignon. "Quadrant Walks Starting Outside the Quadrant – Supplementary Material". 2020. Link.

[9] J. Courtiel, S. Melczer, M. Mishna, and K. Raschel. "Weighted Lattice Walks and Universality Classes". *Journal of Combinatorial Theory, Series A* **152** (2017), pp. 255–302.

[10] T. Dreyfus, C. Hardoin, J. Roques, and M. F. Singer. "On the Nature of the Generating Series of Walks in the Quarter Plane". *Inventiones mathematicae* **213**.1 (2018), pp. 205–236.

[11] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random Walks in the quarter-plane*. Springer, 1999.

[12] G. Fayolle and K. Raschel. "On the holonomy or algebraicity of generating functions counting lattice walks in the quarter-plane". *Markov Processes and Related Fields* **16**.3 (2010), pp. 485–496.

[13] M. Kauers. *A Mathematica Package for Computing Asymptotic Expansions of Solutions of P-Finite Recurrence Equations*. Tech. rep. 11-04. RISC-Linz, 2011.

[14] M. Kauers, M. Jaroschek, and F. Johansson. "Ore Polynomials in Sage". *Computer Algebra and Polynomials*. LNCS 8942. Springer, 2014, pp. 105–125.

[15] C. Koutschan. *HolonomicFunctions (User's Guide)*. Tech. rep. 10-01. RISC Report Series, University of Linz, Austria, 2010.

[16] I. Kurkova and K. Raschel. "On the functions counting walks with small steps in the quarter plane". *Publications Mathématiques de l'IHÉS* **116** (2012), pp. 109–124.

[17] I. Kurkova and K. Raschel. "New steps in walks with small steps in the quarter plane". *Annals of Combinatorics* **19** (2015), pp. 461–511.

[18] L. Lipshitz. "The diagonal of a D-finite power series D-finite." *J. Algebra* **113(2)** (1988), pp. 373–378.

[19] K. Raschel. "Counting walks in a quadrant: a unified approach via boundary value problems". *Journal of the EMS* **14** (2012), pp. 749–777.

[20] K. Raschel and A. Trotignon. "On walks avoiding a quadrant". *Electronic Journal of Combinatorics* **26**.P3.31 (2019), pp. 1–34.