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Schubert polynomials and the inhomogeneous TASEP on a ring

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Abstract. Consider a lattice of n sites arranged around a ring, with the n sites occupied by particles of weights $\{1, 2, \ldots, n\}$; the possible arrangements of particles in sites thus corresponds to the $n!$ permutations in S_n. The *inhomogeneous totally asymmetric simple exclusion process* (or TASEP) is a Markov chain on the set of permutations, in which two adjacent particles of weights $i < j$ swap places at rate $x_i - y_{n+1-j}$ if the particle of weight j is to the right of the particle of weight i. (Otherwise nothing happens.) In the case that $y_i = 0$ for all i, the stationary distribution was conjecturally linked to Schubert polynomials by Lam-Williams, and explicit formulas for steady state probabilities were subsequently given in terms of multiline queues by Ayyer-Linusson and Arita-Mallick. In the case of general y_i, Cantini showed that n of the $n!$ states have probabilities proportional to double Schubert polynomials. In this paper we introduce the class of *evil-avoiding permutations*, which are the permutations avoiding the patterns 2413, 4132, 4213 and 3214. We show that there are $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$ evil-avoiding permutations in S_n, and for each evil-avoiding permutation w, we give an explicit formula for the steady state probability ψ_w as a product of double Schubert polynomials. We also show that the Schubert polynomials that arise in these formulas are flagged Schur functions, and give a bijection in this case between multiline queues and semistandard Young tableaux.

Keywords: Schubert polynomials, TASEP, multiline queues

1 Introduction

In recent years, there has been a lot of work on interacting particle models such as the *asymmetric simple exclusion process* (ASEP), a model in which particles hop on a one-dimensional lattice subject to the condition that at most one particle may occupy a given site. The ASEP on a one-dimensional lattice with open boundaries has been linked to Askey-Wilson polynomials and Koornwinder polynomials [7, 4, 8], while the ASEP on a ring has been linked to Macdonald polynomials [5, 6]. The *inhomogeneous totally*

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asymmetric simple exclusion process (TASEP) is a variant of the exclusion process on the ring in which the hopping rate depends on the weight of the particles. In this paper we build on works of Lam-Williams [10], Ayyer-Linusson [2], and especially Cantini [3] to give formulas for many steady state probabilities of the inhomogeneous TASEP on a ring in terms of Schubert polynomials.

Definition 1.1. Consider a lattice with n sites arranged in a ring. Let $\mathrm{St}(n)$ denote the $n!$ labelings of the lattice by distinct numbers $1, 2, \ldots, n$, where each number i is called a *particle of weight* i. The *inhomogeneous TASEP on a ring of size* n is a Markov chain with state space $\mathrm{St}(n)$ where at each time t a swap of two adjacent particles may occur: a particle of weight i on the left swaps its position with a particle of weight j on the right with transition rate $r_{i,j}$ given by:

$$r_{i,j} = \begin{cases} x_i - y_{n+1-j} \text{ if } i < j \\ 0 \text{ otherwise.} \end{cases}$$

In what follows, we will identify each state with a permutation in S_n. Following [10, 3], we multiply all steady state probabilities for $\mathrm{St}(n)$ by the same constant, obtaining "renormalized" steady state probabilities ψ_w, so that

$$\psi_{123\ldots n} = \prod_{i<j} (x_i - y_{n+1-j})^{j-i-1}. \tag{1.1}$$

See Figure 1 for the state diagram when $n = 3$.

In the case that $y_i = 0$, Lam and Williams [10] studied this model[1] and conjectured that after a suitable normalization, each steady state probability ψ_w can be written as a monomial factor times a positive sum of Schubert polynomials, see Table 1 and Table 2. They also gave an explicit formula for the monomial factor, and conjectured that under certain conditions on w, ψ_w is a multiple of a particular Schubert polynomial. Subsequently Ayyer and Linusson [2] gave a conjectural combinatorial formula for the stationary distribution in terms of *multiline queues*, which was proved by Arita and Mallick [1]. In [3], Cantini introduced the version of the model given in Definition 1.1[2] with y_i general, and gave a series of *exchange equations* relating the components of the stationary distribution. This allowed him to give explicit formulas for the steady state probabilities for n of the $n!$ states as products of double Schubert polynomials.

In this paper we build on [3, 2, 1], and give many more explicit formulas for steady state probabilities in terms of Schubert polynomials: in particular, we give a formula for ψ_w as a product of (double) Schubert polynomials whenever w is *evil-avoiding*, that is, it

[1] However the convention of [10] was slightly different; it corresponds to labeling states by the inverse of the permutations we use here.

[2] We note that in [3], the rate $r_{i,j}$ was $x_i - y_j$ rather than $x_i - y_{n+1-j}$ as we use in Definition 1.1.

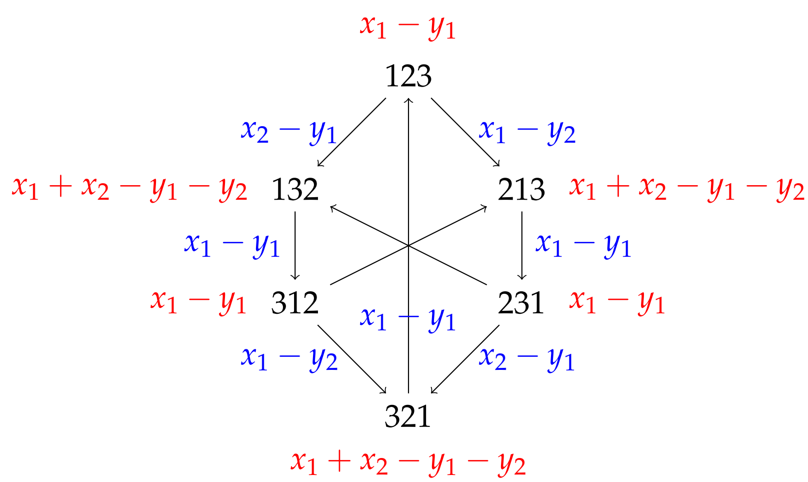

Figure 1: The state diagram for the inhomogeneous TASEP on St(3), with transition rates shown in blue, and steady state probabilities ψ_w in red. Though not shown, the transition rate $312 \to 213$ is $x_2 - y_1$ and the transtition rate $231 \to 132$ is $x_1 - y_2$.

State w	Probability ψ_w
1234	$(x_1 - y_1)^2(x_1 - y_2)(x_2 - y_1)$
1324	$(x_1 - y_1)\mathfrak{S}_{1432}$
1342	$(x_1 - y_1)(x_2 - y_1)\mathfrak{S}_{1423}$
1423	$(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)\mathfrak{S}_{1243}$
1243	$(x_1 - y_2)(x_1 - y_1)\mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423}\mathfrak{S}_{1342}$

Table 1: The renormalized steady state probabilities for $n = 4$.

avoids the patterns $2413, 4132, 4213$ and 3214.[3] We show that there are $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$ evil-avoiding permutations in S_n, so this gives a substantial generalization of Cantini's previous result [3] in this direction. We also prove the monomial factor conjecture from [10]. Finally, we show that the Schubert polynomials that arise in our formulas are flagged Schur functions, and give a bijection in this case between multiline queues and semistandard Young tableaux.

In order to state our main results, we need a few definitions. First, we say that two states w and w' are *equivalent*, and write $w \sim w'$, if one state is a cyclic shift of the

[3] We call these permutations *evil-avoiding* because if one replaces i by 1, e by 2, l by 3, and v by 4, then *evil* and its anagrams *vile, veil* and *leiv* become the four patterns $2413, 4132, 4213$ and 3214. Note that Leiv is a name of Norwegian origin meaning "heir."

State w	Probability ψ_w
12345	$\mathbf{x}^{(6,3,1)}$
12354	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$\mathbf{x}^{(2,1,0)}(\mathfrak{S}_{164325}+\mathfrak{S}_{25431})$
13542	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$\mathbf{x}^{(1,0,0)}(\mathfrak{S}_{1753246}+\mathfrak{S}_{265314}+\mathfrak{S}_{2743156}+\mathfrak{S}_{356214}+\mathfrak{S}_{364215}+\mathfrak{S}_{365124})$
14352	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$\mathbf{x}^{(3,1,0)}(\mathfrak{S}_{146325}+\mathfrak{S}_{24531})$
15324	$\mathbf{x}^{(2,1,1)}(\mathfrak{S}_{15432}+\mathfrak{S}_{164235})$
15342	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Table 2: The renormalized steady state probabilities for $n = 5$, when each $y_i = 0$. In the table, $\mathbf{x}^{(a,b,c)}$ denotes $x_1^a x_2^b x_3^c$.

other, e.g. $(w_1, \ldots, w_n) \sim (w_2, \ldots, w_n, w_1)$. Because of the cyclic symmetry inherent in the definition of the TASEP on a ring, it is clear that the probabilities of states w and w' are equal whenever $w \sim w'$. We will therefore often assume, without loss of generality, that $w_1 = 1$. Note that up to cyclic shift, $\mathrm{St}(n)$ contains $(n-1)!$ states.

Definition 1.2. Let $w = (w_1, \ldots, w_n) \in \mathrm{St}(n)$. We say that w is a *k-Grassmannian permutation*, and we write $w \in \mathrm{St}(n, k)$ if: $w_1 = 1$; w is *evil-avoiding*, i.e. w avoids the patterns 2413, 3214, 4132, and 4213; and w^{-1} has exactly *k descents*, equivalently, there are exactly k letters b in w such that $b + 1$ appears to the left of b in w.

Definition 1.3. We associate to each $w \in \mathrm{St}(n, k)$ a sequence of partitions $\Psi(w) = (\lambda^1, \ldots, \lambda^k)$ as follows. Write the Lehmer code of w^{-1} as $\mathrm{code}(w^{-1}) = c = (c_1, \ldots, c_n)$;

since w^{-1} has k descents, c has k *descents* in positions we denote by $a_1, \ldots, a_k$. We also set $a_0 = 0$. For $1 \leq i \leq k$, we define $\lambda^i = (n-a_i)^{a_i} - (\underbrace{0, \cdots, 0}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \ldots, c_{a_i})$.

See Table 3 for examples of the map $\Psi(w)$.

Definition 1.4. Given a positive integer n and a partition λ of length $\leq (n-2)$, we define an integer vector $g_n(\lambda) = (v_1, \ldots, v_n)$ of length n as follows. Write $\lambda = (\mu_1^{k_1}, \cdots, \mu_l^{k_l})$ where $k_i > 0$ and $\mu_1 > \cdots > \mu_l$. We assign values to the entries $(v_1, \ldots, v_n)$ by performing the following step for i from 1 to l.

- (Step i) Set $v_{n-\mu_i}$ equal to μ_i. Moving to the left, assign the value μ_i to the first $(k_i - 1)$ unassigned components.

After performing Step l, we assign the value 0 to any entry v_j which has not yet been given a value.

Note that in Step 1, we set $v_{n-\mu_1}, v_{n-\mu_1-1}, \cdots, v_{n-\mu_1-k_1+1}$ equal to μ_1.

Example 1.5.

$$\begin{aligned} g_5((2,1,1)) &= (0,1,2,1,0) \\ g_6((3,2,2,1)) &= (0,2,3,2,1,0) \\ g_6((3,1,1)) &= (0,0,3,1,1,0). \end{aligned}$$

The main result of this paper is Theorem 3.1. We state here our main result in the case that each $y_i = 0$. The definition of Schubert polynomial can be found in Section 2.

Theorem 1.6. *Let $w \in \mathrm{St}(n,k)$ be a k-*Grassmannian *permutation, as in Definition 1.2, and let $\Psi(w) = (\lambda^1, \ldots, \lambda^k)$. Adding trailing 0's if necessary, we view each partition λ^i as a vector in $\mathbb{Z}_{\geq 0}^{n-2}$, and set $\mu := (\binom{n-1}{2}, \binom{n-2}{2}, \ldots, \binom{2}{2}) - \sum_{i=1}^{k} \lambda^i$. Then when each $y_i = 0$, the renormalized steady state probability ψ_w is given by*

$$\psi_w = \mathbf{x}^\mu \prod_{i=1}^{k} \mathfrak{S}_{g_n(\lambda^i)},$$

where $\mathfrak{S}_{g_n(\lambda^i)}$ is the Schubert polynomial associated to the permutation with Lehmer code $g_n(\lambda^i)$, and g_n is given by Definition 1.4.

Equivalently, writing $\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots)$, we have that

$$\psi_w = \mathbf{x}^\mu \prod_{i=1}^{k} s_{\lambda^i}(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \ldots),$$

where $s_{\lambda^i}(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \ldots)$ denotes the flagged Schur polynomial associated to shape λ^i, where the semistandard tableaux entries in row j are bounded above by $n - \lambda^i_j$.

k	$w \in \mathrm{St}(5,k)$	$\Psi(w)$	probability ψ_w
0	12345	$\emptyset$	$\mathbf{x}^{(6,3,1)}$
1	12354	$(1,1,1)$	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$
1	12435	$(2,2,1)$	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$
1	12453	$(2,2)$	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$
1	12534	$(1,1)$	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$
1	13245	$(3,2)$	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$
1	13425	$(3,1)$	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$
1	13452	(3)	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$
1	14235	$(2,1,1)$	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$
1	14253	$(2,1)$	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$
1	14523	(2)	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$
1	15234	(1)	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$
2	12543	$(2,2),(1,1,1)$	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
2	13254	$(3,2),(1,1,1)$	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
2	13542	$(3),(1,1,1)$	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
2	14352	$(3),(2,2,1)$	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
2	14532	$(3),(2,2)$	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
2	15342	$(3),(1,1)$	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
2	15423	$(2),(1,1,1)$	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
3	15432	$(3),(2,2),(1,1,1)$	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Table 3: Special states $w \in \mathrm{St}(5,k)$ and the corresponding sequences of partitions $\Psi(w)$, together with steady state probabilities ψ_w.

We illustrate Theorem 1.6 in Table 3 in the case that $n = 5$.

Proposition 1.7. *The number of evil-avoiding permutation in S_n satisfies the recurrence $e(1) = 1, e(2) = 2, e(n) = 4e(n-1) - 2e(n-2)$ for $n \geq 3$, and is given explicitly as*

$$e(n) = \frac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}. \tag{1.2}$$

This sequence begins as $1, 2, 6, 20, 68, 232$, and occurs in Sloane's encyclopedia as sequence A006012. The cardinalities $|\mathrm{St}(n,k)|$ also occur as sequence A331969.

Remark 1.8. Let $w(n,h) := (h, h-1, \ldots, 2, 1, h+1, h+2, \ldots, n) \in \mathrm{St}(n)$. In [3, Corollary 16], Cantini gives a formula for the steady state probability of state $w(n,h)$, as a trivial factor times a product of certain (double) Schubert polynomials. Note that our main result is a significant generalization of [3, Corollary 16]. For example, for $n = 4$, Cantini's result gives a formula for the probabilities of three states – $(1,2,3,4),(1,3,4,2)$, and $(1,4,3,2)$. For $n = 5$, his result gives a formula for four states – $(1,2,3,4,5),(1,3,4,5,2)$, $(1,4,5,3,2)$, and $(1,5,4,3,2)$. On the other hand, Theorem 1.6 gives a formula for all six states when $n = 4$ (see Table 1) and 20 of the 24 states when $n = 5$. Asymptotically, since the number of special states in S_n is given by (1.2), Theorem 1.6 gives a formula for roughly $\frac{(2+\sqrt{2})^{n-1}}{2}$ out of the $(n-1)!$ states of $\mathrm{St}(n)$.

Another point worth mentioning is that the Schubert polynomials that occur in the formulas of [3] are all of the form $\mathfrak{S}_{\sigma(b,n)}$, where $\sigma(b,n)$ denotes the permutation $(1, b+1, b+2, \dots, n, 2, 3, \dots, b)$. However, many of the Schubert polynomials arising as (factors) of steady probabilities are not of this form. Already we see for $n = 4$ the Schubert polynomials $\mathfrak{S}_{1432}$ and $\mathfrak{S}_{1243}$, which are not of this form.

Note that it is common to consider a version of the inhomogeneous TASEP in which one allows multiple particles of each weight i. This is the version studied in several of the previous references, and also in [11] (which primarily considers particles of types 0, 1 and 2). We plan to work in this generality in our subsequent work. However, since our focus here is on Schubert polynomials, we restrict to the case of permutations.

2 Background on permutations and Schubert polynomials

We let S_n denote the symmetric group on n letters, which is a Coxeter group generated by the simple reflections $s_1, \dots, s_{n-1}$, where s_i is the simple transposition exchanging i and $i+1$. We let $w_0 = (n, n-1, \dots, 2, 1)$ denote the longest permutation.

For $1 \le i < n$, we have the *divided difference operator* ∂_i which acts on polynomials $P(x_1, \dots, x_n)$ as follows:

$$(\partial_i P)(x_1, \dots, x_n) = \frac{P(\dots, x_i, x_{i+1}, \dots) - P(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

If $s_{i_1} \dots s_{i_m}$ is a reduced expression for a permutation w, then $\partial_{i_1} \dots \partial_{i_m}$ depends only on w, so we denote this operator by ∂_w.

Definition 2.1. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two sets of variables, and let

$$\Delta(\mathbf{x}, \mathbf{y}) = \prod_{i+j \le n} (x_i - y_j).$$

To each permutation $w \in S_n$ we associate the *double Schubert polynomial*

$$\mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = \partial_{w^{-1}w_0} \Delta(\mathbf{x}, \mathbf{y}),$$

where the *divided difference operator* acts on the x-variables.

Definition 2.2. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a weakly decreasing sequence of positive integers. We say that r is the *length* of λ, and denote it $r = \text{length}(\lambda)$.

Definition 2.3. The *diagram* or *Rothe diagram* of a permutation w is

$$D(w) = \{(i,j) \mid 1 \leq i,j \leq n, w(i) > j, w^{-1}(j) > i\}.$$

The sequence of the numbers of the points of the diagram in successive rows is called the *Lehmer code* or *code* $c(w)$ of the permutation. We also define $c^{-1}(l)$ to be the permutation whose Lehmer code is l. The partition obtained by sorting the components of the code is called the *shape* $\lambda(w)$ of w.

Example 2.4. If $w = (1,3,5,4,2)$ then $c(w) = (0,1,2,1,0)$ and $\lambda(w) = (2,1,1)$.

Definition 2.5. We say that a permutation w is *vexillary* if and only if there does not exist a sequence $i < j < k < \ell$ such that $w(j) < w(i) < w(\ell) < w(i)$. Such a permutation is also called 2143-*avoiding*.

Definition 2.6. We define the *flag* of a vexillary permutation w, starting from its code $c(w)$, in the following fashion. If $c_i(w) \neq 0$, let e_i be the greatest integer $j \geq i$ such that $c_j(w) \geq c_i(w)$. The flag $\phi(w)$ is then the sequence of integers e_i, ordered to be increasing.

Definition 2.7. Let X_i denote the family of indeterminates $x_1, \ldots, x_i$. For $d_1, \ldots, d_n$ a weakly increasing sequence of n integers, we define the *flagged Schur function*

$$s_\lambda(X_{d_1}, \ldots, X_{d_n}) = \sum_T \mathbf{x}^{\text{type}(T)},$$

where the sum runs over the set of semistandard tableaux T with shape λ for which the entries in the ith row are bounded above by d_i.

There is also a notion of flagged double Schur polynomials. One can define them in terms of tableaux or via a Jacobi-Trudi type formula [12, Section 2.6.5].

Theorem 2.8 ([12, Corollary 2.6.10])**.** *If w is a vexillary permutation with shape $\lambda(w)$ and with flags $\phi(w) = (f_1, \ldots, f_m)$ and $\phi(w^{-1}) = (h_1, \ldots, h_m)$, then we have*

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = s_{\lambda(w)}(X_{f_1} - Y_{h_m}, \ldots, X_{f_m} - Y_{h_1}),$$

i.e. the double Schubert polynomial of w is a flagged double Schur polynomial.

3 Main results

Let $w \in S_n$ be a state. In what follows, we write $a \to b \to c$ if the letters a, b, c appear in cyclic order in w. So for example, if $w = 1423$, we have that $1 \to 2 \to 3$ and $2 \to 3 \to 4$, but it is not the case that $3 \to 2 \to 1$ or $4 \to 3 \to 2$.

$$xyFact(w) = \prod_{i=1}^{n-2} \prod_{\substack{k>i+1 \\ i \to i+1 \to k}} (x_1 - y_{n+1-k}) \cdots (x_i - y_{n+1-k}). \tag{3.1}$$

The following is our main theorem; when each $y_i = 0$, it reduces to Theorem 1.6. The proof uses z-deformation of ψ_w and exchange equations introduced in [3].

Theorem 3.1. *Let $w \in \mathrm{St}(n,k)$, and write $\Psi(w) = (\lambda^1, \cdots, \lambda^k)$. Then the (renormalized) steady state probability is given by*

$$\psi_w = xyFact(w) \prod_{i=1}^{k} \mathfrak{S}_{g_n(\lambda^i)}, \tag{3.2}$$

where $\mathfrak{S}_{g_n(\lambda^i)}$ is the double Schubert polynomial associated to the permutation with Lehmer code $g_n(\lambda^i)$, and g_n is given by Definition 1.4.

We also prove the *monomial factor conjecture* from [10]. Suppose that $y_i = 0$ for all i. Given a state w, let $a_i(w)$ be the number of integers greater than $(i+1)$ on the clockwise path from $(i+1)$ to i. Let $\eta(w)$ be the largest monomial that can be factored out of ψ_w. The following statement was conjectured in [10, Conjecture 2].

Theorem 3.2. *Let $w \in \mathrm{St}(n)$. Then*

$$\eta(w) = \prod_{i=1}^{n-2} x_i^{a_i(w)+\cdots+a_{n-2}(w)}.$$

4 Multiline queues and semistandard tableaux

It was proved in [1] that when each $y_i = 0$, the steady state probabilities ψ_w for the TASEP on a ring can be expressed in terms of the *multiline queues* of Ferrari and Martin [9]. On the other hand, we know from Theorem 1.6 that when $w \in \mathrm{St}(n,1)$ (i.e. w^{-1} is a Grassmann permutation and $w_1 = 1$), ψ_w equals a monomial times a single flagged Schur polynomial. In this section we will explain that result by giving a bijection between the relevant multiline queues and the corresponding semistandard tableaux.

Definition 4.1. Fix positive integers L and n. A *multiline queue* Q is an $L \times n$ array in which each of the Ln positions is either vacant or occupied by a ball. We say it has *content* $\mathbf{m} = (m_1, \ldots, m_n)$ if it has $m_1 + \cdots + m_i$ balls in row i for $1 \leq i \leq n$. We number the rows from top to bottom from 1 to L, and the columns from right to left from 1 to n.

Definition 4.2. Given an $L \times n$ multiline queue Q, the *bully path projection* on Q is, for each row r with $1 \le r \le L-1$, a particular matching of balls from row r to row $r+1$, which we now define. If ball b is matched to ball b_0 in the row below then we connect b and b_0 by the shortest path that travels either straight down or from left to right (allowing the path to wrap around the cylinder if necessary). Here each ball is assigned a *class*, and matched according to the following algorithm:

- All the balls in the first row are defined to be of class 1.
- Suppose we have matched all the balls in rows $1, 2, \dots, r-1$ and have assigned a class to all balls in rows $1, 2, \dots, r$. We now consider the balls in rows r.
- Pick any order of the balls in row r such that balls with smaller labels come before balls with larger labels. Consider the balls in this order; suppose we are considering a ball b of class i in row r. If there is an unmatched ball directly below b in row $r+1$, we let $M(b)$ be that ball; otherwise we move to the right in row $r+1$ and let $M(b)$ be the first unmatched ball that we find (wrapping around from column 1 to n if necessary). We match b to ball $M(b)$ and say that $M(b)$ is of class i.
- The previous step gives a matching of all balls in row r to balls below in row $r+1$. We assign class $r+1$ to any balls in row $r+1$ that were not yet assigned a class. We now repeat the process and consider the balls in row $r+1$.

After completing the bully path projection for Q, let $w = (w_1, \cdots, w_n)$ be the labeling of the balls read from right to the left in row L (where a vacancy is denoted by $L+1$). We say that Q is a multiline queue of *type* w and let $MLQ(w)$ denote the set of all multiline queues of type w. We also consider a type of row r in Q to be the labeling of the balls read from right to the left in row r (where a vacancy is denoted by $r+1$).

A vacancy in Q is called $i-$*covered* if it is traversed by a path starting on row i, but not traversed by any path starting on row i' such that $i' < i$.

See Figure 2 for an example.

Figure 2: A multiline queue of type $(1,2,4,3,5)$, and the corresponding semistandard tableau under the bijection in Proposition 4.7.

We define a weight $wt(Q)$ for multiline queues. It was first introduced in [2].

Definition 4.3. Given an $L \times n$ multiline queue Q, let v_r be the number of vacancies in row r and let $z_{r,i}$ be the number of $i-$ *covered* vacancies in row r. Set $V_i = \sum_{j=i+1}^{L} v_j$. We define

$$wt(Q) = \prod_{i=1}^{L-1}(x_i^{V_i}) \prod_{1 \leq i < r \leq L} (\frac{x_r}{x_i})^{z_{r,i}}.$$

Example 4.4. The multiline queue Q in Figure 2 has a $1-$ *covered* vacancy in row 2, a $2-$ *covered* vacancy in row 3 and a $3-$ *covered* vacancy in row 4. The weight of Q is

$$wt(Q) = x_1^{3+2+1} x_2^{2+1} x_3^{1} (\frac{x_2}{x_1})(\frac{x_3}{x_2})(\frac{x_4}{x_3}) = x_1^5 x_2^3 x_3 x_4.$$

The following result was conjectured in [2] and proved in [1].

Theorem 4.5 ([1])**.** *Consider the inhomogeneous TASEP on a ring (with each $y_i = 0$). We have*

$$\psi_w = \sum_{Q \in MLQ(w)} wt(Q).$$

We now give a (weight-preserving up to a constant factor) bijection between multiline queues in $MLQ(w)$ and certain semistandard tableaux, when $w \in \mathrm{St}(n,1)$, i.e. w^{-1} is a Grassmann permutation and $w_1 = 1$.

Definition 4.6. Given a partition $\lambda = (\mu_1^{b_1}, \cdots, \mu_k^{b_k}, 0^c)$, such that $\mu_1 > \cdots > \mu_k > 0$ and $b_i > 0, c \geq 0$, we define a permutation $w(\lambda)$ as follows. Identify λ with the lattice path from $(\mu_1, \sum_{i=1}^{k} b_i + c)$ to $(0,0)$ that defines the southeast border of its Young diagram. Label the vertical steps of the lattice path from 1 to k from top to bottom, and then the horizontal steps in increasing order from right to left starting from $k+1$. Reading off the numbers along the lattice path gives $w(\lambda)$. See Figure 3.

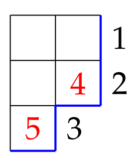

Figure 3: The partition $\lambda = (2,2,1)$ and $w(\lambda) = (1,2,4,3,5)$.

Proposition 4.7. *Given a partition $\lambda = (\mu_1^{b_1}, \cdots, \mu_k^{b_k}, 0^c)$ as in Definition 4.6, let the vector $d = (d_1, \cdots, d_k)$ be the numbers assigned to horizontal steps right after vertical steps in the construction of $w(\lambda)$. For example, in Figure 3, $d = (4,5)$. Let d' be the vector*

$$d' = (\underbrace{d_1 - b_1, \cdots, d_1 - 1}_{b_1}, \underbrace{d_2 - b_2, \cdots, d_2 - 1}_{b_2}, \ldots, \underbrace{d_k - b_k, \cdots, d_k - 1}_{b_k}).$$

Then there exists a bijection $f : MLQ(w) \to SSYT(\lambda, d')$ such that $wt(Q) = Kx^{type(f(Q))}$ for some monomial K, where $SSYT(\lambda, d')$ is the set of semistandard tableaux with shape λ for which the entries in the i th row are bounded above by d'_i. In particular, we have

$$\psi_{w(\lambda)} = \sum_{Q \in MLQ(w(\lambda))} wt(Q) = K \sum_{T \in SSYT(\lambda, d')} x^{type(T)} = Ks_\lambda(X_{d'_1}, X_{d'_2}, \dots).$$

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