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Non-orientable branched coverings, b-Hurwitz numbers, and positivity for multiparametric Jack expansions (extended abstract)

Guillaume Chapuy[1] * and Maciej Dołęga[2] †

[1]*CNRS, IRIF UMR 8243, Université de Paris,*
[2]*Institute of Mathematics, Polish Academy of Sciences, 00-956 Warszawa, Poland.*

Abstract. We introduce a one-parameter deformation of the 2-Toda tau-function of maps (or more generally, constellations), obtained by deforming Schur functions into Jack symmetric functions. We show that its coefficients are polynomials in the deformation parameter b with nonnegative integer coefficients. These coefficients count generalized constellations on an arbitrary surface, orientable or not, with an appropriate b-weighting that "measures" in some sense their non-orientability. The particular case of bipartite maps gives the best progress so far towards the "b-conjecture" of Goulden and Jackson from 1996.

Our proof consists in showing that the partition function satisfies an infinite set of PDEs. These PDEs have two definitions, one given by Lax equations, the other one following an explicit combinatorial decomposition.

1 Introduction

This paper is an abbreviated version of [1], of which we present some of the main results and ideas. We refer to [1] for complete proofs and many more developments.

1.1 Constellations and Schur functions

Hurwitz numbers and tau-functions. Hurwitz numbers, in their most general sense, count the number of combinatorially inequivalent branched coverings of the sphere by an orientable surface with a given number of ramification points and given ramification profiles. Hurwitz numbers and their variants (dessins d'enfants, weighted, monotone,

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†mdolega@impan.pl. MD is supported from *Narodowe Centrum Nauki*, grant UMO-2017/26/D/ST1/00186.

orbifold Hurwitz numbers) have numerous connections to mathematical physics, combinatorics, and the moduli spaces of curves, see e.g. [6, 10, 4].

Branched coverings can be realized as certain coloured graphs drawn on orientable surfaces called *constellations* [8], a very general model of maps on surfaces which cover in particular *bipartite maps*, or *transposition factorisations*, as special cases. The enumeration of constellations is strongly linked to characters of the symmetric group (or Schur functions) which gives it a very rich structure. A fundamental fact in the field is that a certain generating function of weighted Hurwitz numbers (or constellations) is a tau-function of the KP/2-Toda hierarchies (see for example [4] and references therein). Explicitly, in the case of k-constellations, this generating function has the form

$$\tau^{(k)}(t;\mathbf{p},\mathbf{q},u_1,\dots,u_k) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\left(\frac{f_\lambda}{n!}\right)^2 \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q})\tilde{s}_\lambda(\underline{u_1})\tilde{s}_\lambda(\underline{u_2})\dots\tilde{s}_\lambda(\underline{u_k}), \tag{1.1}$$

where $\tilde{s}_\lambda = \frac{n!}{f_\lambda}\cdot s_\lambda$ is the normalized Schur function indexed by the integer partition λ of n, expressed as a polynomial in the power-sum variables $\mathbf{p} = (p_i)_{i\geq 1}$ or $\mathbf{q} = (q_i)_{i\geq 1}$, where $\underline{u} = (u, u, \dots)$ is the specialization of all power-sum variables to the same parameter u, and where f_λ is the dimension of the irreducible representation of the symmetric group indexed by λ. From this function (or more precisely its logarithm) one can extract the generating functions for surfaces of genus g with n boundaries, which are one of the main subject of interest of the map-enumeration community. In itself, the particular case $k = 1$ of *bipartite maps* has been extensively studied. We refer to [8, 1] for references.

1.2 Jack polynomials, b-deformations, and our main result

In this paper we consider the one-parameter deformation, or *b-deformation*, of the function $\tau^{(k)}$ defined by

$$\tau_b^{(k)}(t;\mathbf{p},\mathbf{q},u_1,\dots,u_k) : = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\frac{1}{j_\lambda^{(\alpha)}} J_\lambda^{(\alpha)}(\mathbf{p})J_\lambda^{(\alpha)}(\mathbf{q})J_\lambda^{(\alpha)}(\underline{u_1})\dots J_\lambda^{(\alpha)}(\underline{u_k}), \tag{1.2}$$

where $J_\lambda^{(\alpha)}$ is the Jack symmetric function of parameter $\alpha = 1 + b$, for a formal variable b, and where $j_\lambda^{(\alpha)} := \mathrm{hook}_\alpha(\lambda)\,\mathrm{hook}'_\alpha(\lambda)$ is a natural b-deformation of $n!^2/f_\lambda^2$, see Section 4.1. Jack functions are obtained as a one-parameter limit of Macdonald polynomials that interpolates between Schur and zonal polynomials, respectively for $b = 0, 1$ [9]. In particular the function $\tau_b^{(k)}$ is equal to $\tau^{(k)}$ for $b = 0$.

The deformation (1.2) was introduced by Goulden and Jackson [3] in the case $k = 1$ of bipartite maps (in fact [3] considers a more general function where the sequence $\underline{u_1}$ is replaced by a third arbitrary sequence of parameters). Using the connection between zonal polynomials and representation theory of the Gelfand pair $(\mathfrak{S}_{2n}, \mathbb{H}_n)$, they proved

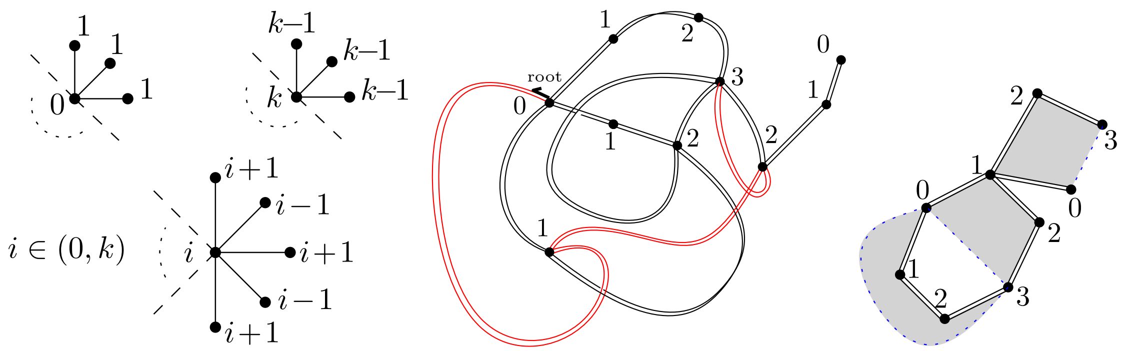

Figure 1: *Left:* the local colour-constraints in a k-constellation. *Center:* a rooted 3-constellation on the Klein bottle, in ribbon-graph representation. The right-path of the root (Section 3.2) is highlighted in red. *Right:* A planar 3-constellation. By adding the dotted edges and considering the greyed areas as hyperedges, one recovers the usual (orientation-depending) representation of constellations [8] familiar to connoisseurs of the orientable case. Moreover, what we call k-constellations here are often called $(k+1)$-constellations in the orientable literature.

that for $b = 1$ this function enumerates analogues of bipartite maps on general surfaces (orientable or not). In the same paper they formulate the "b–conjecture" and the related "Matching-Jack conjecture", among the most remarkable open problems in algebraic combinatorics. They assert that the coefficients have an interpretation for arbitrary b: they count bipartite maps on general surfaces, with a weight which is a polynomial in b with nonnegative coefficients. The representation theoretic tools used in the case $b \in \{0, 1\}$ do not apply for general b, and this conjecture is still wide open despite many partial results [3, 7, 2, 5].

Our main result, in the case $k = 1$, goes much further than these results and establishes the b-conjecture in the case of *two full sets of variables* ((p_i) and (q_i) in our notation). In the case $k > 1$, it leads us to introduce a non-orientable generalization of constellations, and to a more general result, not directly comparable with the conjectures of [3].

1.3 Main result.

Our main result gives a combinatorial (and in fact, geometric) meaning to the coefficients of $\tau_b^{(k)}$. Define a k-constellation as a graph embedded on a compact surface (orientable or not), with simply connected faces, whose vertices are colored with colours in $[0..k]$ subject to the local constraints of Figure 1–Left (see Section 2 for more details). We let $F(\mathbf{M})$ and $V_i(\mathbf{M})$ be the sets of faces and i-coloured vertices of $\mathbf{M}$, respectively.

Theorem 1.1 (Main result)**.** *For every $k \geq 1$, we have*

$$(1+b)\frac{t\partial}{\partial t}\ln \tau_b^{(k)}(t;\mathbf{p},\mathbf{q},u_1,\ldots,u_k) = \sum_{(\mathbf{M},c)} \kappa(\mathbf{M}) t^{|\mathbf{M}|} b^{\nu_\rho(\mathbf{M},c)}, \tag{1.3}$$

where the sum is taken over all rooted k-constellations (orientable or not). Here $|\mathbf{M}|$ is the size of $\mathbf{M}$ (defined in Section 2) and the monomial $\kappa(\mathbf{M})$ keeps track of the degrees of faces, 0-coloured vertices, and of the number of i-coloured vertices for $1 \leq i \leq k$ as follows

$$\kappa(\mathbf{M}) := \prod_{f\in F(\mathbf{M})} p_{\deg(f)} \prod_{v\in V_0(\mathbf{M})} q_{\deg(v)} \prod_{i=1}^{k} u_i^{|V_i(\mathbf{M})|}. \tag{1.4}$$

Moreover $\nu_\rho(\mathbf{M},c)$ is a nonnegative integer which is zero if and only if $\mathbf{M}$ is orientable.

In particular, the coefficients of the LHS of (1.3) are polynomials in b, and they have nonnegative integer coefficients.

Branched covers and other developments. The full paper [1], of which we have chosen to keep the title, addresses several subjects which we omit in this extended abstract. The main one is the introduction of generalized branched coverings of the sphere by a non-orientable surface, which are in bijection with our generalized constellations, thus endowing the b-deformed tau function $\tau_b^{(k)}$ with a geometric meaning. In [1] we also construct the projective limit ($k \to \infty$) of $\tau_b^{(k)}$ and use it to introduce a non-orientable, b-weighted, analogue of the (classical or weighted) Hurwitz numbers (see e.g. [4]). In particular all results concerning constellations are extended to this setting, including b-weights, decomposition equations, and b-polynomiality, and some classical results such as the Cut-And-Join equation and piecewise polynomiality for classsical Hurwitz numbers are generalized. A special case of the b-weighted Hurwitz numbers enable us to introduce a b-deformation of monotone Hurwitz numbers and the HCIZ integral. Finally, see [1] for a more detailed discussion of the link of our results with the b- and Matching-Jack conjectures of Goulden and Jackson.

Method of proof and plan of the paper Our method of proof goes by showing that both sides of Equation (1.3) satisfy the same PDEs. The differential operators defining these PDEs take two different forms. For constellations, they follow from a combinatorial decomposition which is a vast generalization of the classical Tutte/Lehman-Walsh decomposition, in which the parameter "b" is appropriately inserted. For the "Jack polynomial" side, these equations are defined by two companion *Lax equations*, and they follow from algebro-combinatorial manipulations on Jack polynomials. Proving that the "Lax" and "combinatorial" forms are in fact equal is one of the hardest tasks of [1]. The proof relies on a combinatorial heuristic that can be made precise for $b \in \{0,1\}$. This heuristic is "lifted" towards an algebraic proof by developing a technical operator paradigm which occupies much of the paper [1].

This extended abstract is organized as follows. In Section 2, we introduce the (new) notion of constellations on general surfaces. In Section 3 we present the notion of MON, the associated b-weights, and the combinatorial decomposition equation. Finally in Section 4, we present the Lax version of the differential operators, the main technical result, we discuss Jack polynomials and give some ideas of the proof structure.

2 Maps and constellations

For us a *surface* is a compact, two dimensional, real manifold. By the classification theorem a connected surface $\mathcal{S}$ is uniquely determined by its Euler characteristic $\chi_{\mathcal{S}} \leq 2$ (or, equivalently, its *genus* $g_{\mathcal{S}} \in \frac{1}{2}\mathbb{N}$ given by $\chi_{\mathcal{S}} = 2 - 2g_{\mathcal{S}}$) together with the information whether $\mathcal{S}$ is orientable or not.

An embedding of a graph (possibly with multiple edges) into a surface which cuts it into simply connected pieces (called *faces*) is called *a map*. We consider maps up to homeomorphisms of surfaces. A small neighborhood of an edge around a vertex is called a *half-edge* and a small neighborhood of a vertex delimited by two consecutive half-edges is called *a corner*. It is convenient to represent a map by its *ribbon graph*, which is the surface with boundary made by a small neighbourhood of the graph on the surface it embeds in (see Figure 1–Center). Our first contribution is a notion of constellation that generalizes the classical notion from the orientable case [8].

Definition 2.1 (Constellation, see Figure 1)**.** *Let $k \geq 1$ be an integer. A k-*constellation *is a map, equipped with a coloring of its vertices with colors in $\{0, 1, 2, \dots, k\}$, such that*

1. *each vertex colored by 0 (k, respectively) has only neighbours of color 1 ($k-1$ respectively),*
2. *for any $0 < i < k$ and for any vertex v colored by i, each corner of v separates vertices colored by $i-1$ and $i+1$.*

The *degree* of a face in a k-constellation is the number of corners of colour 0 it contains, which is the same as the number of corners of colour k, and as half the number of corners of any other colour. The *size* of a constellation $\mathbf{M}$ is its number of corners of colour 0 and is denoted by $|\mathbf{M}|$. A constellation of size n is *labelled* if its corners of colour 0 are labelled with the integers from 1 to n, and if each such corner carries an (arbitrary) orientation. A constellation is *rooted* if it is equipped with a distinguished oriented corner of colour 0, called the *root* (if the constellation is already labelled, the orientation of the root corner is already given, but for unlabelled maps, this orientation is part of the information given by the rooting). The *root vertex (or face, respectively)* is the vertex (or face) incident to the root corner.

The 2-profile of a k-constellation is the $k+2$-tuple $(\lambda, \mu, v_1(\mathbf{M}), \dots, v_k(\mathbf{M}))$, where λ is the partition encoding face degrees, μ is the partition encoding degrees of vertices of colour 0 and $v_i(\mathbf{M})$ is the number of vertices of colour i in $\mathbf{M}$ for $i \geq 1$. The Euler

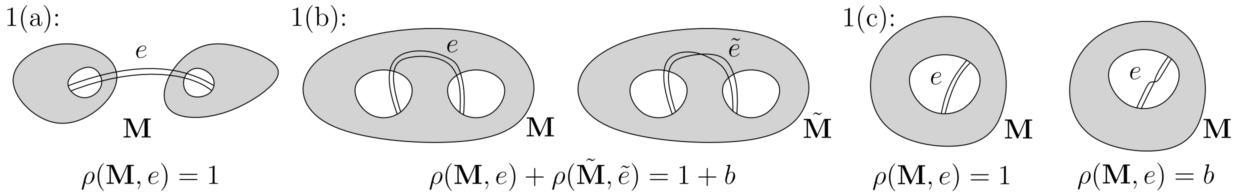

Figure 2: The main axioms of MONs.

characteristic $\chi(\mathbf{M})$ of a constellation (and its genus if it is connected) can be recovered from the Riemann-Hurwitz/Euler formula:

$$\chi(\mathbf{M}) = \ell(\lambda) + \ell(\mu) - \sum_{i=1}^{k} (n - v_i(\mathbf{M})). \tag{2.1}$$

In this paper we will be able to enumerate connected k-constellations fully controlling their 2-profiles and orientability, thus in particular we will always determine their underlying surface.

3 MON's and the b-deformed decomposition equation

3.1 MON's and weights

Our way to assign a b-dependent weight to a map proceeds by repeated edge-deletions. The weight attached to each deletion depends on a number of arbitrary choices subject to suitable axioms, encompassed by the concept of *measure of non-orientability*.

Definition 3.1 (MON; see Figure 2)**.** *A* measure of non-orientability (MON) *is a function $\rho(\cdot,\cdot)$ with value in $\mathbb{Q}[b]$ that associates to a vertex-colored map $\mathbf{M}$ and an edge e in $\mathbf{M}$, some value $\rho(\mathbf{M}, e)$ and that satisfies the following properties.*

1. *Let $\mathbf{N} := \mathbf{M} \setminus \{e\}$ and let c_1, c_2 be the two corners delimited by the endpoints of e in $\mathbf{N}$.*
   - *(a) If c_1, c_2 belong to two distinct connected components of $\mathbf{N}$, then $\rho(\mathbf{M}, e) = 1$.*
   - *(b) If c_1, c_2 belong to the same connected component of $\mathbf{N}$ but to two different faces, then let $\tilde{e}$ be the other edge that could be added to $\mathbf{N}$ between these corners to form a new map $\tilde{\mathbf{M}}$. Then $\rho(\mathbf{M}, e) + \rho(\tilde{\mathbf{M}}, \tilde{e}) = 1 + b$.*
   - *(c) If c_1, c_2 belong to the same face of $\mathbf{N}$, then $\rho(\mathbf{M}, e) = 1$ if e splits this face into two faces ("untwisted diagonal)" and $\rho(\mathbf{M}, e) = b$ otherwise ("twisted diagonal").*
2. *the value of $\rho(\mathbf{M}, e)$ depends only on the connected component of $\mathbf{M}$ containing e.*

A MON ρ is integral *if $\rho(\mathbf{M}, e)$ belongs to $\{1, b\}$ for any $\mathbf{M}$ and e, and if the following is true: for every pair $(\mathbf{M}, e)$ which is in case (b) above, and such that $\mathbf{M}$ is orientable, we have $\rho(\mathbf{M}, e) = 1$ and $\rho(\tilde{\mathbf{M}}, \tilde{e}) = b$.*

The idea of using MON's or their variants already appeared in previous works on the b-conjecture starting from [7]. Here we have added Axiom (2) which is necessary for some of our arguments. We also allow MON's to take values in $\mathbb{Q}[b]$ (previous authors only consider what we call here *integral* MON's), which we believe is natural [1, Rem. 3].

In [1], we also introduce the notion of *coherent MON*, which is too technical to be stated here. A coherent MON satisfies an additional restriction that implies the following property, which we can take as an alternative definition for the purposes of this extended abstract: if c is a corner of colour j in $\mathbf{M}$ and f is a face in $\mathbf{M}$, then the average value, among all possible additions of an edge e between c and some corner of colour $(j+1)$ in f, of the value $\rho(\mathbf{M} \cup \{e\}, e)$, is equal to $\frac{1+b}{2}$. It is crucial for us (but easy to see, [1, Lem. 3.3]) that there exist MONs which are both coherent and integral.

3.2 The combinatorial decomposition

We now present an algorithm that enables one to "exhaust" any rooted constellation by repeated edge deletions. It is a generalization of the classical "root-edge decompositions" going back to Tutte for the planar case and to Lehman and Walsh for higher genera.

Define the *right-path* of the corner c in the map $\mathbf{M}$ as the sequence of k edges following the corner c along the face it belongs to in $\mathbf{M}$ (Figure 1-Center). It is easy to see that removing a right-path from a k-constellation, one obtains again a k-constellation. The *combinatorial decomposition* of a connected rooted k-constellation $(\mathbf{M}, c)$ is defined as follows: remove the right-path of the root, and iterate the operation on each corner incident to the root vertex until the root vertex is isolated; delete the root vertex; then, iterate the construction on each remaining connected component, until no edge remains[1].

Assuming that an underlying MON ρ has been fixed, every time an edge e is deleted by the algorithm, we say that the weight $\rho(\mathbf{N}, e)$ is *collected* by the algorithm (where $\mathbf{N}$ is the current map at the time e is deleted). The following crucial property follows directly from the axioms of MON's.

Definition-Lemma 3.2. *Let ρ be a MON, and let $(\mathbf{M}, c)$ be a connected rooted k-constellation. We define the* weight $\vec{\rho}(\mathbf{M}, c)$ *of $(\mathbf{M}, c)$ as the product of all the weights collected during the combinatorial decomposition of $(\mathbf{M}, c)$. If the MON ρ is integral then we have*

$$\vec{\rho}(\mathbf{M}, c) = b^{\nu_\rho(\mathbf{M}, c)},$$

where $\nu_\rho(\mathbf{M}, c)$ is a nonnegative integer, which is zero if and only if $\mathbf{M}$ is orientable.

[1] We refer to the paper [1] for the detailed construction, in which each remaining component is canonically rooted. For technical reasons, an operation of *duality* has to be applied to each remaining connected component before iterating. This duality generalizes the classical map duality and it exchanges the roles of vertices of colour 0 and faces. We omit these subtle considerations here, see [1] for more.

3.3 Generating functions of maps and the decomposition equations.

If $\mathbf{M}$ is a constellation we denote by $cc(\mathbf{M})$ its number of connected components, and by $F(\mathbf{M})$ the set of its faces. For $i \geq 0$ we denote by $V_i(\mathbf{M})$ the set of vertices of color i and by $v_i(\mathbf{M})$ its cardinality. Recall also that its size $|\mathbf{M}|$ is its number of corners of colour 0.

For the rest of this paper we fix indeterminates b, $\mathbf{p} = (p_i)_{i\geq 1}$, $\mathbf{q} = (q_i)_{i\geq 1}$, $\boldsymbol{y} = (y_i)_{i\geq 0}$, $\boldsymbol{u} = (u_i)_{i\geq 1}$, which we will use to track combinatorial parameters of constellations in our generating functions. Let $(\mathbf{M}, c)$ be a rooted k-constellation, and f_c be its root face. We associate to $(\mathbf{M}, c)$ the monomial

$$\vec{\kappa}(\mathbf{M}, c) := y_{\deg(f_c)} \prod_{f \in F(\mathbf{M})\setminus\{f_c\}} p_{\deg(f)} \prod_{v \in V_0(\mathbf{M})} q_{\deg(v)} \prod_{i=1}^{k} u_i^{v_i(\mathbf{M})} = \frac{y_{\deg(f_c)}}{p_{\deg(f_c)}} \kappa(\mathbf{M}),$$

where we recall that the monomial $\kappa(\mathbf{M})$ is defined by (1.4).

Definition 3.3. *Let ρ be a MON. Let $\vec{H}_\rho(t; \mathbf{p}, \mathbf{q}, \boldsymbol{y}, u_1, \ldots, u_k) \in \mathbb{Q}(b)[\boldsymbol{y}, \mathbf{p}, \mathbf{q}, u_1, \ldots, u_k][[t]]$ be the multivariate generating function of rooted connected k-constellations given by*

$$\vec{H}_\rho(t; \mathbf{p}, \mathbf{q}, \boldsymbol{y}, u_1, \ldots, u_k) := \sum_{n\geq 1} \sum_{(\mathbf{M},c)} t^n \vec{\rho}(\mathbf{M}, c) \vec{\kappa}(\mathbf{M}, c), \tag{3.1}$$

where the second sum is taken over rooted connected (unlabelled) k-constellations of size n. We also consider the multivariate generating function of possibly disconnected labelled *k-constellations given by the formula*

$$F_\rho := 1 + \sum_{\mathbf{M}} \frac{t^{|\mathbf{M}|}}{2^{|\mathbf{M}|-cc(\mathbf{M})}|\mathbf{M}|!} \frac{\tilde{\rho}(\mathbf{M})\kappa(\mathbf{M})}{(1+b)^{cc(\mathbf{M})}}, \tag{3.2}$$

where the b-weight $\tilde{\rho}(\mathbf{M})$ is defined from $\vec{\rho}(\mathbf{M}, c)$, multiplicatively over connected components, and averaged over the choice of a random root in each component.

These functions are related by a version of the "Exp-Log" principle, namely

Lemma 3.4. *Introduce the operator $\Theta_Y := \sum_{i\geq 1} p_i \frac{\partial}{\partial y_i}$. Then we have*

$$(1+b)t\frac{\partial}{\partial t} \ln F_\rho = \Theta_Y \vec{H}_\rho. \tag{3.3}$$

The function F_ρ is characterized by the following "decomposition equations", which originates in the combinatorial decomposition of the previous section.

Theorem 3.5 (Main PDE's – combinatorial form)**.** *Let ρ be any coherent MON. Then the generating series $F_\rho \equiv F_\rho(t; \mathbf{p}, \mathbf{q}, \boldsymbol{y}, u_1, \ldots, u_k)$ satisfies the following set of equations, for $m \geq 1$:*

$$m\frac{q_m \partial}{\partial q_m} F_\rho = \Theta_Y t^m \cdot q_m \cdot \left(Y_+ \prod_{l=1}^{k} (\Lambda_Y + u_l)\right)^m \frac{y_0}{1+b} F_\rho, \tag{3.4}$$

where

$$\Lambda_Y := (1+b)\sum_{i,j\geq 1} y_{i+j-1}\frac{i\partial^2}{\partial p_i \partial y_{j-1}} + \sum_{i,j\geq 1} y_{i-1}p_j\frac{\partial}{\partial y_{i+j-1}} + b\cdot\sum_{i\geq 0} y_i\frac{i\partial}{\partial y_i},$$
$$Y_+ := \sum_{i\geq 0} y_{i+1}\frac{\partial}{\partial y_i}.$$

The proof of Theorem 3.5 closely follows the combinatorial decomposition: roughly speaking, each term in the operator Λ_Y corresponds to a way to add an edge from the root corner of a map in a k-constellation. The product over $l = 1..k$ in (3.4) corresponds, very roughly, to the k edges to be added to construct the right-path of the root, while the parameter m controls the degree of the root vertex (so the operation of constructing a right-path has to be iterated m times). That being said, there are considerable difficulties to overcome to make such an interpretation valid in the presence of b-weights.

The main one is that the b-weight $\vec{\rho}(\mathbf{M}, c)$ is defined for rooted objects, while (3.4) suggests a (randomized) decomposition applied to unrooted objects. For $b \in \{0, 1\}$ this would make no big difference, but since the b-weight and root-degree could be complicatedly correlated, here this introduces a difficulty. Note moreover that working with rooted objects (and with a deterministic, rather than random, decomposition) is necessary for applications to the integrality of coefficients, as averaging over the choice of a root may otherwise introduce rational numbers. In fact, we give in [1] a "connected" version of (3.4), which is non-linear and on which integrality of coefficients is apparent. The subtle problem of correlations between degrees and b-weight is also the reason why we need to introduce duality as mentioned in the footnote page 7.

4 The Lax equations, Jack polynomials, and the proof

The theorem below shows that the operators that appear in the decomposition equations can be alternatively defined inductively by certain recurrence relations involving commutators, which is the crucial link between Jack polynomials and constellations. Their proof is the hardest part of the paper [1].

Definition 4.1. *The Laplace–Beltrami operator D_{1+b} is the differential operator defined by*

$$D_{1+b} = \frac{1}{2}\left((1+b)\sum_{i,j\geq 1} p_{i+j}\frac{ij\partial^2}{\partial p_i \partial p_j} + \sum_{i,j\geq 1} p_i p_j \frac{(i+j)\partial}{\partial p_{i+j}} + b\cdot\sum_{i\geq 1} p_i\frac{i(i-1)\partial}{\partial p_i}\right). \quad (4.1)$$

Here and below $[\cdot,\cdot]$ denotes the algebra commutator, $[A, B] = AB - BA$.

Definition 4.2. *Define the three differential operators* $(A_j)_{j\geq 1}$, $(B^{(k)}_m)_{m\geq 1}$, *and* $\Omega^{(k)}_Y$ *on* $\mathcal{P} := \mathbb{Q}(b)[p_1, p_2, \dots]$ *by the fact that the formal power series of operators*

$$A(s) := \sum_{j\geq 0} \frac{s^j}{j!} A_{j+1}$$

$$B^{(k)}(s) := \sum_{m\geq 0} \frac{s^m}{m!} B^{(k)}_{m+1}$$

each satisfies a Lax equation with respective Lax pairs $(A(s), D_{1+b})$*and*$(B^{(k)}(s), \Omega^{(k)}_Y)$, *namely*

$$\frac{d}{ds}A(s) = [D_{1+b}, A(s)] \quad \text{and} \quad \frac{d}{ds}B^{(k)}(s) = [\Omega^{(k)}_Y, B^{(k)}(s)],$$

with initial conditions $A_1 = p_1/(1+b)$ *and* $B^{(k)}_1 = \sum_{j=1}^{k+1} e_{k+1-j}(u_1, \dots, u_k)A_j$, *and with*

$$\Omega^{(k)}_Y := \sum_{j=1}^{k+1} e_{k+1-j}(u_1, \dots, u_k)A_{j+1}.$$

The following theorem shows that the operators A_j and $B^{(k)}_m$ defined by the Lax equation are in fact the "building blocks" which appear in the decomposition equation (3.4).

Theorem 4.3 (Combinatorial operators solve the Lax equations)**.** *The operators* $A_j, B^{(k)}_m$ *defined above admit the explicit expression, on* $\mathcal{P}$,

$$A_j = \Theta_Y Y_+ \Lambda_Y^{j-1} \frac{y_0}{1+b}, \qquad B^{(k)}_m = (m-1)!\Theta_Y \big(Y_+ \prod_{i=1}^{k} (\Lambda_Y + u_i)\big)^m \frac{y_0}{1+b}. \tag{4.2}$$

The proof of Theorem 4.3 is the crucial part of [1] occupying most of its content. The technical algebraic proof has its origin in the following combinatorial ideas.

Sketch of the proof of Theorem 4.3 for $b \in \{0, 1\}$. The Lax equation for $A(s)$ is equivalent to the recurrence relation

$$A_1 = p_1/(1+b) \quad , \quad A_{j+1} = [D_{1+b}, A_j], \text{ , for } j \geq 1. \tag{4.3}$$

By induction, assume (4.2) for some $j \geq 1$. By analyzing carefully the root-edge decomposition, one can see that the operator A_j has the following combinatorial interpretation: its creates an isolated root vertex (operator $y_0/(1+b)$), then a path of length $j-1$ from this root (iteration of the operator Λ_Y, where the y-variables keep track of the root face degree), and it "anonymizes" the root face (operator Θ_Y). Similarly, it can be shown that the operator D_{1+b} can be interpreted as adding an edge of given color, at an arbitrary position in the map.

The commutator $[D_{1+b}, A_j] = D_{1+b}A_j - A_jD_{1+b}$ therefore has the following interpretation: create a path of length $j-1$, then an edge, or do it in the converse order with a minus sign. The two are almost the same, except that the second way to proceed does not include the case when the edge is added from the very last corner of the newly created path (note the operator Y_+ in (4.2)), or equivalently when this creates a path of length j. Thus by (4.3) the effect of A_{j+1} is to create a path of length j, which concludes the induction.

For $b \in \{0,1\}$, the proof of the explicit form (4.2) of $B_m^{(k)}$ can be done in a similar way (where now $\frac{1}{(m-1)!}B_m^{(k)}$ and $\Omega_Y^{(k)}$ have the effect of creating a new vertex of colour 0 and degree m, and a path of length k, respectively). See [1, Sec. 4.3]. □

How to fix the proof for general b? We insist that the last sketch of proof does *not* work for a generic b. Indeed, it is based on the idea of constructing the same map by adding the same edges in different orders, but in the general case there is no reason *a priori* that different orders give the same contribution to the b-weight. This is a *major* problem, and the whole strategy of [1] is designed to overcome this difficulty.

A natural idea, to make the same approach work in general, would be to look for some sort of combinatorial operation, such as an involution, which would compare the different possible ways to add two edges (or two k-paths) to a given map in different orders, and that would ensure that the overall contributions to the b-weight are the same. This turns out to be (very) tricky but possible for the first Lax equation (for $A(s)$), but becomes intractable for the second one, in particular because such an operation would have to be nonlocal (since k is not a priori bounded).

Our strategy of proof consists in "lifting" these combinatorial heuristics to an algebraic framework of operators, on which involutions and cancellations arguments can be replaced by abstract algebraic relations. This requires in particular to introduce several other sets of variables in addition to $(p_i)_{i\geq 1}, (y_i)_{i\geq 1}$, that heuristically represent several root faces in "multi-rooted" objects. We then appropriately promote the operators D_{1+b}, Λ_Y to a much larger polynomial ring, on which several remarkable commutation relations can be established, which, in the end, lead us to Theorem 4.3. See [1, Sec. 4].

4.1 Jack polynomials and the Laplace–Beltrami operator

We let Sym denote the algebra of symmetric functions over the field $\mathbb{Q}(b)$ of rational functions in b with rational coefficients, and we use the standard notation $p_\lambda, m_\lambda, e_\lambda, \dots$ for standard bases [9]. Since Sym $= \mathbb{Q}(b)[p_1, p_2, \dots]$ then clearly the Laplace–Beltrami operator D_{1+b} given by (4.1) acts on the symmetric function algebra. Its importance is reflected in the following characterization of Jack symmetric functions:

Definition-Proposition 4.4. *Let $\alpha = 1 + b$. There is a unique family of symmetric functions $\{J_\lambda^{(\alpha)}\}$ called* Jack symmetric functions *such that for each partition λ,*

- $D_{1+b} J_\lambda^{(\alpha)} = \left(\sum_{\square\in\lambda} c_\alpha(\square)\right) J_\lambda^{(\alpha)}$;
- $J_\lambda^{(\alpha)} = \mathrm{hook}_\alpha(\lambda) m_\lambda + \sum_{\nu<\lambda} a_\nu^\lambda m_\nu$, *where* $a_\nu^\lambda \in \mathbb{Q}(b)$.

Here $c_\alpha(i,j) = \alpha i - j - b$ *is the "*(α)*-content" of the box* $\square = (i,j)$ *and*

$$\mathrm{hook}_\alpha(\lambda) = \prod_{\square\in\lambda} \left(\alpha a_\lambda(\square) + \ell_\lambda(\square) + 1\right),\ \ \mathrm{hook}'_\alpha(\lambda) = \prod_{\square\in\lambda} \left(\alpha a_\lambda(\square) + \ell_\lambda(\square) + \alpha\right)$$

are (α)*-deformations of the hook-length product, with* $a_\lambda(\square)$ *and* $\ell_\lambda(\square)$ *denoting the "arm" and "leg" lengths of* $\square$ *in* λ*, [9].*

The following properties are classical results of Stanley [11]:

- the coefficient $[J_\mu^{(\alpha)}] p_1 J_\lambda^{(\alpha)}$ is non-zero only if $\lambda \subset \mu$,
- $J_\lambda^{(\alpha)}(\underline{u}) = \prod_{\square\in\lambda\setminus(1,1)} (u + c_\alpha(\square))$, where we recall that $\underline{u}$ is the specialization $p_i \equiv u$.

Using Definition-Proposition 4.4, these two properties, and a number of algebraic manipulations, one can construct inductively ([1, Sec.5]) a family of differential operators that cancel the function $\tau_b^{(k)}$ defined by (1.2). The relations defining these operators are equivalent to the Lax equations of Definition 4.2, and we obtain with some work:

Theorem 4.5 (Main PDE – Lax form)**.** *The function* $\tau_b^{(k)}$ *satisfies the equation, for* $m \geq 1$*:*

$$\frac{m!\partial}{\partial q_m} \tau_b^{(k)} = t^m B_m^{(k)} \tau_b^{(k)}. \tag{4.4}$$

By Theorem 4.3, the differential equation (4.4) solved by $\tau_b^{(k)}$ is in fact the same as the combinatorial equation (3.4) solved by F_ρ (Theorem 3.5, for ρ a coherent MON). It follows easily that these two series are equal, which given Lemma 3.4, is precisely the content of Theorem 1.1, provided we choose a MON ρ which is both coherent and integral.

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