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Strange Expectations in Affine Weyl Groups

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Abstract. We previously generalized P. Johnson's proof of D. Armstrong's conjecture for the expected number of boxes in a simultaneous core to simply-laced type. After recalling combinatorial core-like models for coroot lattices in the classical types, our main result is a generalization of this theorem to all affine Weyl groups.

Keywords: root system, affine Weyl group, integer partition, core, expected value

1 Introduction

1.1 Motivation

An *a-core* is an integer partition with no hook of length a. An *(a,b)-core* is a partition that is simultaneously an a-core and a b-core. For a and b relatively prime, it turns out that there are only finitely many (a,b)-cores:

$$\left|\mathsf{core}(a,b)\right| = \frac{1}{a+b}\binom{a+b}{b}.$$

For λ a partition, write $\lambda^{\intercal}$ for its conjugate and $\mathsf{size}(\lambda)$ for the number of its boxes. The starting point for a number of recent investigations has been Armstrong's conjecture on the average number of boxes in an (a,b)-core, and in a self-conjugate (a,b)-core [1, 2].

Theorem 1.1 ([5]). *For* $\gcd(a,b) = 1$,

$$\mathop{\mathbb{E}}_{\lambda\in\mathsf{core}(a,b)}(\mathsf{size}(\lambda)) = \frac{(a-1)(b-1)(a+b+1)}{24} = \mathop{\mathbb{E}}_{\substack{\lambda\in\mathsf{core}(a,b)\\ \lambda=\lambda^{\intercal}}}(\mathsf{size}(\lambda)).$$

Both equalities in Theorem 1.1 were proven by Johnson using weighted Ehrhart theory [5]. In [7], we generalized Armstrong's conjecture and Johnson's proof (of the first equality) to all *simply-laced* affine Weyl groups. In this extended abstract, we complete the generalization to all affine Weyl groups.

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1.2 Combinatorial Models of Coroot Lattices

The set of a-cores under the action of the affine symmetric group $\widetilde{\mathfrak{S}}_a$ is a well-studied combinatorial model for the coroot lattice $\mathcal{Q}^\vee_a$ of type A_{a-1}. Indeed, for all affine Weyl groups $\widetilde{W} = \widetilde{W}(X_n)$, there is a well-known $\widetilde{W}$-equivariant map from the group to the coroot lattice $\widetilde{W} = W \ltimes \mathcal{Q}^\vee_{X_n} \to \mathcal{Q}^\vee_{X_n}$

$$\widetilde{w} = w \cdot t_{-q} \mapsto \widetilde{w}^{-1}(0) = t_q \cdot w^{-1}(0) = t_q(0) = q,$$

which restricts to a $\widetilde{W}$-equivariant bijection on the cosets $W \backslash \widetilde{W}$.[1] Thus, combinatorial models for $\mathcal{Q}^\vee_{X_n}$ also give models for $W \backslash \widetilde{W}$, representatives usually taken to be dominant affine elements. In type A_{a-1}, these correspondences give $\widetilde{\mathfrak{S}}_a$-equivariant bijections

$$\begin{aligned} \mathsf{core}(a) &\leftrightarrow \mathcal{Q}^\vee_a \leftrightarrow \mathfrak{S}_a \backslash \widetilde{\mathfrak{S}}_a \\ \lambda &\leftrightarrow q_\lambda \ \leftrightarrow \widetilde{w}_\lambda. \end{aligned} \tag{1.1}$$

It is an easy exercise to produce similar combinatorial models for the quotients $W \backslash \widetilde{W}$ of other classical types ($X_n \in \{A_n, B_n, C_n, D_n\}$) by embedding $\mathcal{Q}^\vee_{X_n}$ into an appropriate type A coroot lattice. These, as well as a model for $X_n = G_2$, are illustrated for rank two root systems in Figure 1.

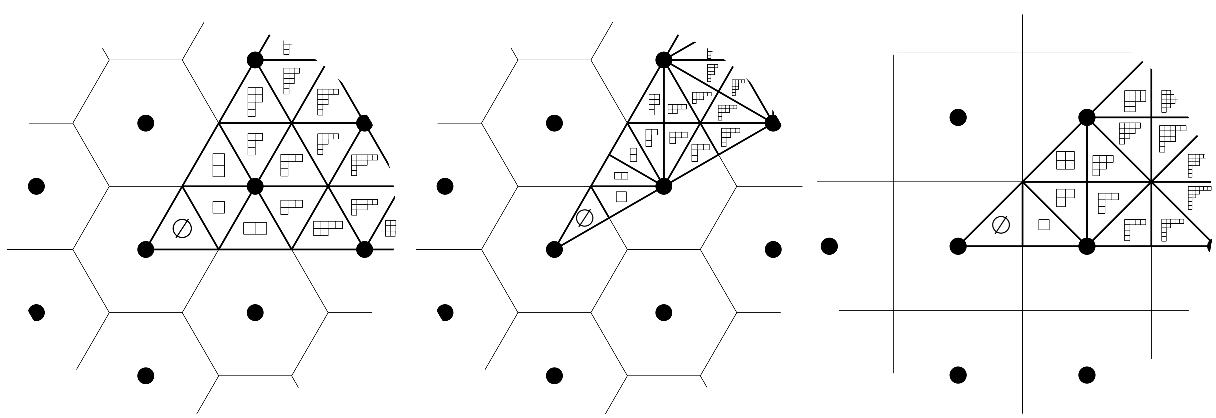


Figure 1: 3-cores in types A_2 and G_2, and self-conjugate 4-cores in type C_2.

Under the correspondence between a-cores and $\mathcal{Q}^\vee_a$ of Equation (1.1), the set of (a, b)-cores are exactly those coroot points that sit inside of a certain b-fold dilation of the fundamental alcove called the b-Sommers region (see Definition 4.1). The natural generalization of $\mathsf{core}(a, b)$ to any affine Weyl group is the intersection of the coroot lattice $\mathcal{Q}^\vee_{X_n}$ with the b-Sommers region, so that $\mathsf{core}(a, b) = \mathsf{core}(A_{a-1}, b)$, where

$$\mathsf{core}(X_n, b) := \mathcal{Q}^\vee_{X_n} \cap \mathcal{S}(b). \tag{1.2}$$

[1] Where $\widetilde{v} \in \widetilde{W}$ acts on the right for $\widetilde{w} \in W \backslash \widetilde{W}$ by $\widetilde{w} \cdot \widetilde{v}$, and on the left for $q \in \mathcal{Q}^\vee_{X_n}$ by $\widetilde{v}^{-1}(q)$.

1.3 Size Statistics

Under the bijections of Equation (1.1), we noticed in [7] that the number of boxes in λ could be computed from the coroot q_λ as described above, or the inversion set of $\widetilde{w}_\lambda^{-1}$, where $\mathsf{inv}(\widetilde{w}) = \widetilde{\Phi}^+ \cap \widetilde{w}(-\widetilde{\Phi}^+)$. More precisely:

Proposition 1.2 ([7, Proposition 6.4 & Corollary 6.7])**.** *Let λ be an a-core and $\rho^\vee$ be the sum of the fundamental coweights in type A_{a-1}. Then*

$$\mathsf{size}(\lambda) = \sum_{\alpha+k\delta\in\mathsf{inv}(\widetilde{w}_\lambda^{-1})} k = \left\langle \frac{a}{2}q_\lambda - \rho^\vee, q_\lambda \right\rangle.$$

It was natural to consider the corresponding statistic in *any* affine Weyl group $\widetilde{W}(X_n)$ acting on V, restricting to a certain finite set of coroots $\mathsf{core}(X_n, b)$ (defined below in Equation (1.2), in analogy with simultaneous (a, b)-cores) . For simply-laced Weyl groups, our result mirrored Theorem 1.1.

Theorem 1.3 ([7, Theorem 1.10])**.** *Let X_n be a simply-laced Cartan type with Coxeter number h, and let b be coprime to h. Then*

$$\mathop{\mathbb{E}}_{q\in\mathsf{core}(X_n,b)}(\mathsf{size}(q)) = \frac{n(b-1)(h+b+1)}{24}.$$

When applied to $X_n = A_{a-1}$ (so that $n = a-1$ and $h = a$), our result gives a proof of the left equality of Theorem 1.1 for the expected size of simultaneous (a, b)-cores. But since self-conjugate cores are a combinatorial model for coroots in the *non-simply-laced* type C_n, we were unable to similarly specialize Theorem 1.3 to conclude the right equality of Theorem 1.1 for the expected size of a self-conjugate simultaneous core.

Our mistake was to take Theorem 1.3 as evidence that we had determined the "correct" generalization of the number of boxes of an a-core to all affine Weyl groups—the trouble is that we were unable to apply the Ehrhart-theoretic techniques of Section 4.2 using this definition outside of simply-laced type. To be able to apply these techniques, it turns out that we must modify the above definitions to incorporate different root lengths.

1.4 Expected Size

Normalize root systems so that the highest root has length 2, and write r for the ratio of the length of a long to a short root. We define a new statistic on coroots that recovers our old definition of size in simply-laced type, but disagrees in non-simply-laced type. For $\widetilde{w} = w \cdot t_{-q} \in W \backslash \widetilde{W}$, define (the second equality is proven in Theorem 3.9)

$$\mathsf{size}^\vee(\widetilde{w}) := \left(\sum_{\substack{\alpha+k\delta\in\mathsf{inv}(\widetilde{w}^{-1}) \\ \alpha \text{ long}}} k \right) + r \left(\sum_{\substack{\alpha+k\delta\in\mathsf{inv}(\widetilde{w}^{-1}) \\ \alpha \text{ short}}} k \right) = \left\langle \frac{h}{2}q - \rho^\vee, q \right\rangle. \tag{1.3}$$

We interpret $\mathsf{size}^\vee$ as statistics on the combinatorial models of Section 3, showing (for example) that $\mathsf{size}^\vee$ in type C_n corresponds to the number of boxes in the corresponding self-conjugate $2n$-core (see Figure 1).

Following the same strategy as in [7], we find an affine Weyl group element that maps $\mathcal{S}(b)$ to a b-fold dilation of the fundamental alcove (correctly modifying the $\mathsf{size}^\vee$ statistic), and then apply Ehrhart theory to compute the expected value of $\mathsf{size}^\vee$ on $\mathsf{core}(X_n, b)$.

Theorem 1.4. *For X_n an irreducible rank n Cartan type with root system Φ,*

$$\mathop{\mathbb{E}}_{\lambda \in \mathsf{core}(X_n,b)} (\mathsf{size}^\vee(\lambda)) = \frac{rg^\vee}{h} \frac{n(b-1)(h+b+1)}{24},$$

where h is the Coxeter number of X_n, $g^\vee$ is the dual Coxeter number for $\Phi^\vee$, and r is the ratio of the length of a long root to the length of a short root in Φ.

The extra factor of $\frac{rg^\vee}{h}$ is *invisible* in the simply-laced case, where $\Phi^\vee = \Phi$, $g^\vee = h$, and $r = 1$. As an immediate application of Theorem 1.4, we conclude both equalities in Theorem 1.1 by specializing to these types. Interestingly, although the expected number of boxes in a simultaneous core and in a self-conjugate simultaneous core happen to be the same, the formulas have quite different interpretations: the factor of $a-1$ corresponds to the dimension n for ordinary simultaneous cores, but to $g^\vee$ in the self-conjugate case.

2 Affine Weyl Groups

Let $\Phi = \Phi^+ \sqcup \Phi^-$ be an irreducible crystallographic root system of Cartan type X_n with ambient space V. Let n be its rank, h its Coxeter number, Δ be its set of simple roots, and $\mathcal{Q}^\vee$ be its coroot lattice. Write $\omega_i^\vee$ for the fundamental coweights, and also set $\omega_0^\vee := 0$. Normalize the inner product on V so that $\langle \beta, \beta \rangle = 2$ for β a long root and define $r := \frac{\langle \beta,\beta \rangle}{\langle \alpha,\alpha \rangle}$ for β a long root and α a short root.

Recall that the corresponding Weyl group $W = W(X_n)$ is generated by the reflections $s_\alpha(x) := x - 2\frac{\langle \alpha,x \rangle}{\langle \alpha,\alpha \rangle}\alpha$ for $\alpha \in \Delta$. There is a unique *highest root* $\widetilde{\alpha} \in \Phi^+$ for which $\widetilde{\alpha} + \alpha \notin \Phi^+$ for any $\alpha \in \Delta$. The corresponding affine Weyl group $\widetilde{W} = W \ltimes \mathcal{Q}^\vee$ is generated by the s_α for $\alpha \in \Delta$ along with the additional affine simple reflection $s_{\widetilde{\alpha},1} := x - (\langle \widetilde{\alpha}, x \rangle - 1)\widetilde{\alpha}$. More generally, reflections act by $s_{\alpha,k}(x) = x - (\langle \alpha, x \rangle - k)\alpha^\vee = s_\alpha(x) + k\alpha^\vee$.

Recall that $\widetilde{W}$ acts on the affine roots $\widetilde{\Phi} = \{\alpha + k\delta : \alpha \in \Phi, k \in \mathbb{Z}\}$ by $\widetilde{w} \cdot (\alpha + k\delta) = w(\alpha) + (k - \langle \alpha, q \rangle)\delta$ when $\widetilde{w} = w \cdot t_q$ for $w \in W$ and $q \in \mathcal{Q}^\vee$ Given a reduced word $\widetilde{\mathsf{w}} = a_1 a_2 \cdots a_\ell$ for $\widetilde{w} \in \widetilde{W}$ (with the a_i simple reflections), we associate the affine roots in the inversion set of $\widetilde{w}$ to the letters a_i in $\widetilde{\mathsf{w}}$ by $(a_1 \cdots a_{i-1})(\alpha_{a_i}) = \beta_i + k_i\delta$. We call the ordered sequence $\mathsf{inv}(\widetilde{\mathsf{w}}) = \beta_1 + k_1\delta, \beta_2 + k_2\delta, \ldots, \beta_\ell + k_\ell\delta$ the *inversion sequence* corresponding to the word $\widetilde{\mathsf{w}}$. (These record the affine hyperplanes that separate $w(\mathcal{A})$ from the fundamental alcove $\mathcal{A}$.)

Example 2.1. We compute the inversion sequence for the reduced word $s_0s_1s_2s_1s_0s_1$ for the element $\widetilde{w} = s_1s_2t_{-2\alpha_2^\vee} \in \widetilde{A}_2$ (see also Figure 2):

$$\mathsf{inv}(s_0s_1s_2s_1s_0s_1) = -\widetilde{\alpha}+\delta, -\alpha_2+\delta, -\widetilde{\alpha}+2\delta, -\alpha_1+\delta, -\widetilde{\alpha}+3\delta, -\alpha_2+2\delta.$$

Theorem 2.2. *The $\widetilde{W}$-equivariant map $\widetilde{W} = W \ltimes \mathcal{Q}^\vee \to \mathcal{Q}^\vee$ defined by*

$$\widetilde{w} = w \cdot t_{-q} \mapsto \widetilde{w}^{-1}(0) = t_q \cdot w^{-1}(0) = t_q(0) = q$$

restricts to a $\widetilde{W}$-equivariant bijection on the cosets $W\backslash\widetilde{W}$.

3 Combinatorial Models for Coroot Lattices

We explain the a-core model for the type A_{a-1} coroot lattice $\mathcal{Q}_a^\vee$. Similar combinatorial models for affine Weyl groups of classical type can be produced using coroot lattice embeddings, as in [4]. We illustrate this is type G_2, interpreting the statistics $\mathsf{size}_i^\vee$ and $\mathsf{size}^\vee$ on the model.

3.1 Partitions, Abaci, and the Coroot Lattice in Type A

An integer partition λ (in English notation) can be characterized by its *boundary word*—the bi-infinite sequence of •s and ∘s (with •s representing steps up and ∘s representing steps right) that begins with an infinite sequence of only •s and ends with an infinite sequence of only ∘s, encoding the boundary of λ when read from bottom left to top right. Partitioning this sequence into consecutive subsequences of length a and stacking them vertically gives an *a-abacus* representation of λ. Finally, an a-abacus is called *balanced* if we can draw a horizontal line between two rows with as many ∘s above the line as •s below; every partition has a unique representation as a balanced a-abacus.

An integer partition λ is an a-core if and only if its a-abacus representation is *flush*—that is, if each of the vertical "runners" of the abacus consists of an infinite sequence of only •s followed by an infinite sequence of only ∘s. A flush, balanced a-abacus can be encoded as the a-tuple of signed distances from the lowest • in each runner to the line witnessing the balanced condition—the balanced condition ensures that these distances sum to zero. See Figure 2 for an illustration.

In type A_{a-1}, the simple roots are $\alpha_i := e_i - e_{i+1}$ for $1 \leq i < a$, the highest root is $\widetilde{\alpha} := e_1 - e_a$, and the coroot lattice[2] is $\mathcal{Q}_a^\vee = \mathcal{Q}_{A_{a-1}}^\vee := \{q = (q_1, q_2 \ldots, q_a) \in \mathbb{Z}^a : \sum_{i=1}^a q_i = 0\}$. By the discussion above, $\mathcal{Q}_a^\vee$ is in bijection with the set of a-cores $\mathsf{core}(a)$. For $q \in \mathcal{Q}_a^\vee$, we write λ_q for the corresponding a-core; for $\lambda \in \mathsf{core}(a)$, we write q_λ for the corresponding

[2]For safety—even though roots and coroots can be identified in type A—we already throw in the distinguishing check.

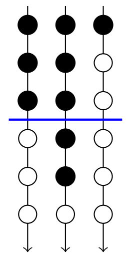
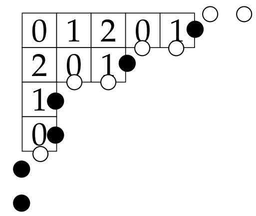
Coroot point $q = (0, 2, -2)$, abacus  and 3-core $\lambda =$ 

Figure 2: An example of the bijection between a-core, abaci, and $\mathcal{Q}^\vee_{A_{a-1}}$ (for $a = 3$).

coroot. The action of the affine symmetric group $\widetilde{\mathfrak{S}}_a = \widetilde{W}(A_{a-1})$ on $\mathcal{Q}^\vee_a$ is generated by the usual simple reflections along with an additional affine simple reflection:

$$s_i(q_1, \ldots, q_i, q_{i+1}, \ldots, q_a) = (q_1, \ldots, q_{i+1}, q_i, \ldots, q_a), \text{ and}$$
$$s_0(q_1, \ldots, q_a) = (q_a + 1, \ldots, q_1 - 1).$$

It is elegant to rephrase this action of $\widetilde{\mathfrak{S}}_a$ in the language of a-cores. We think of an a-core as an order ideal in $\mathbb{N} \times \mathbb{N}$, where each $(i, j) \in \mathbb{N} \times \mathbb{N}$ is indexed by its content $(i - j) \bmod a$. For $0 \leq i < a$, the action of a simple reflection s_i is then to add or remove all possible boxes indexed by i. The following theorem now follows immediately.

Theorem 3.1. *The map $\lambda \to q_\lambda$ is an $\widetilde{\mathfrak{S}}_a$-equivariant bijection between* $\mathsf{core}(a)$ *and $\mathcal{Q}^\vee_a$, with inverse given by $q \to \lambda_q$.*

3.2 The statistic size in type A

We interpret the size statistic for the number of boxes in an a-core in terms of the affine symmetric group $\widetilde{\mathfrak{S}}_a$). Recall that in type A_{a-1} (up to the usual normalization that the sum of the entries ought to be zero, which we will safely ignore), we have

$$\omega^\vee_i = \sum_{j=1}^{i} e_i = (\underbrace{1, 1, \ldots, 1}_{i \text{ ones}}, 0, 0, \ldots, 0) \text{ and } \rho^\vee = \sum_{i=1}^{a-1} \omega^\vee_i = (a-1, a-2, \ldots, 1, 0),$$

where we use the convention that $\omega_0 = 0$. For λ a partition, write $\lambda^\intercal$ for its conjugate, $\mathsf{size}_i(\lambda)$ for the the number of boxes in λ with content $i \bmod a$, and $\mathsf{size}(\lambda)$ for the total number of its boxes. For $q = (q_1, \ldots, q_a) \in \mathcal{Q}^\vee_a$, write $q^\intercal := (-q_a, \ldots, -q_1)$ and define

$$\mathsf{size}^\vee_i(q) := \left\langle \frac{1}{2} q - \omega^\vee_i, q \right\rangle \text{ and } \mathsf{size}^\vee(q) = \sum_{i=1}^{a-1} \mathsf{size}^\vee_i(q) := \left\langle \frac{a}{2} q - \rho^\vee, q \right\rangle.$$

Example 3.2. Continuing Example 2.1, the element $\widetilde{w} = s_1 s_2 t_{-2\alpha_2^\vee} \in \widetilde{A}_2$ corresponds to the coroot $q = 2\alpha_2^\vee = (0, 2, -2)$ and the 3-core $\lambda = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}$. This λ has four boxes with content $0 \bmod 3$, four with content $1 \bmod 3$, and two with content $2 \bmod 3$. We compute (again, safely ignoring normalization)

$$\begin{aligned}
\mathsf{size}_0^\vee(q) &= \left\langle \frac{1}{2}(0,2,-2), (0,2,-2) \right\rangle = 4 = \mathsf{size}_0(\lambda), \\
\mathsf{size}_1^\vee(q) &= \left\langle \frac{1}{2}(0,2,-2) - (1,0,0), (0,2,-2) \right\rangle = 4 = \mathsf{size}_1(\lambda), \\
\mathsf{size}_2^\vee(q) &= \left\langle \frac{1}{2}(0,2,-2) - (1,1,0), (0,2,-2) \right\rangle = 2 = \mathsf{size}_2(\lambda).
\end{aligned}$$

Proposition 3.3. *For $q \in \mathcal{Q}_a^\vee$, $\lambda_{q^\intercal} = \lambda_q^\intercal$ and $\mathsf{size}^\vee(q) = \mathsf{size}(\lambda_q))$. Furthermore, for any $0 \leq i \leq a-1$, $\mathsf{size}_i(\lambda) = \mathsf{size}_i^\vee(q_\lambda)$.*

Proof. We compute directly that the number of boxes in the a-core λ_q is given by

$$\sum_{\bullet \in \lambda_q} \#\{\circ \text{ to the left of } \bullet\} = \sum_{1 \leq i < j \leq n} \frac{(x_i - x_j - 1)(x_i - x_j)}{2} = \left\langle \frac{a}{2} q - \rho^\vee, q \right\rangle,$$

where the first equality comes from counting the number of inversions coming from $\bullet$s and $\circ$s between runner i and runner j, while the second equality comes from rearranging and using $\rho^\vee = (a, a-1, \ldots, 1) - \frac{a+1}{2}(1, 1, \ldots, 1)$. The statement about conjugation follows by observing that the boundary path of a partition and its conjugate are related by reversing and interchanging $\bullet \leftrightarrow \circ$. The remaining statement about size_i will follow from the more general Theorem 3.8 below. □

3.3 The statistic size in general type

We now turn to the general definition of the size statistic for affine Weyl groups. In fact, we generalize the refined statistic size_i (keeping track of the number of boxes with content $i \bmod a$ in type A) for both models of X_n-cores: first for $W \backslash \widetilde{W}$, and then for $\mathcal{Q}_{X_n}^\vee$.

Definition 3.4. Fix $\widetilde{w} \in \widetilde{W}$ and a reduced word $\widetilde{\mathsf{w}} = a_1 \cdots a_\ell$ for $\widetilde{w}$, with inversion sequence $\mathsf{inv}(\widetilde{\mathsf{w}}) = \beta_1 + k_1\delta, \beta_2 + k_2\delta, \ldots, \beta_\ell + k_\ell\delta$. For any $i \in \{0, 1, \ldots, n\}$ with corresponding simple reflection s_i and simple root α_i, define

$$\mathsf{size}_i^\vee(\widetilde{\mathsf{w}}) = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \sum_{\substack{1 \leq j \leq \ell \\ a_j = s_i}} k_j.$$

Example 3.5. Continuing Examples 2.1 and 3.2, we recall that the inversion sequence for $\widetilde{W} = \widetilde{A}_2$ and $\widetilde{\mathsf{w}} = s_0 s_1 s_2 s_1 s_0 s_1$ is

$$\mathsf{inv}(\widetilde{\mathsf{w}}) = -\widetilde{\alpha} + 1 \cdot \delta, -\alpha_2 + 1 \cdot \delta, -\widetilde{\alpha} + 2 \cdot \delta, -\alpha_1 + 1 \cdot \delta, -\widetilde{\alpha} + 3 \cdot \delta, -\alpha_2 + 2 \cdot \delta,$$

and we compute $\mathsf{size}_0(\widetilde{\mathsf{w}}) = 1 + 3 = 4$, $\mathsf{size}_1(\widetilde{\mathsf{w}}) = 1 + 1 + 2 = 4$, and $\mathsf{size}_2(\widetilde{\mathsf{w}}) = 2$.

Proposition 3.6. *For any two reduced words $\widetilde{\mathsf{w}}$ and $\widetilde{\mathsf{w}}'$ representing the same group element $\widetilde{w} \in \widetilde{W}$, we have $\mathsf{size}_i^\vee(\widetilde{\mathsf{w}}) = \mathsf{size}_i^\vee(\widetilde{\mathsf{w}}')$.*

Definition 3.7. For $q \in \mathcal{Q}^\vee_{X_n}$, define $\mathsf{size}_i^\vee(q) = \left\langle \frac{c_i}{2} q - \omega_i^\vee, q \right\rangle$, where the highest root $\widetilde{\alpha}$ is expressed in terms of the simple roots as $\widetilde{\alpha} = \sum_{i=1}^n c_i \alpha_i$ (so that $\langle \omega_i^\vee, \widetilde{\alpha} \rangle = c_i$), with $c_0 := 1$.

Theorem 3.8. *Under the bijection of Theorem 2.2 sending a dominant affine element $\widetilde{w} = w t_{-q} \in W \backslash \widetilde{W}$ to the coroot $q \in \mathcal{Q}^\vee_{X_n}$, we have $\mathsf{size}_i^\vee(\widetilde{w}) = \mathsf{size}_i^\vee(q)$.*

Proof. Let $j \neq 0$ and let $i \in \{0, 1, \ldots, n\}$. We compute $\mathsf{size}_i^\vee(s_j(q))$:

$$\begin{aligned}
\mathsf{size}_i^\vee(s_j(q)) &= \left\langle \frac{c_i}{2} s_j(q) - \omega_i^\vee, s_j(q) \right\rangle \\
&= \left\langle \frac{c_i}{2} \left[q - \langle \alpha_j^\vee, q \rangle \alpha_j \right] - \omega_i^\vee, \left[q - \langle \alpha_j^\vee, q \rangle \alpha_j \right] \right\rangle \\
&= \left\langle \frac{c_i}{2} q - \omega_i^\vee, q \right\rangle + \left\langle \alpha_j^\vee, q \right\rangle \cdot \langle \omega_i^\vee, \alpha_j \rangle \\
&= \mathsf{size}_i^\vee(q) + \begin{cases} \left\langle \alpha_j^\vee, q \right\rangle & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.
\end{aligned}$$

Similarly, we compute $\mathsf{size}_i^\vee(s_0(q)) = \mathsf{size}_i^\vee(q) + \begin{cases} 1 - \langle \widetilde{\alpha}, q \rangle & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$.

We now argue by induction on the length of the dominant affine element $\widetilde{w}$, with base case coming from the identity $e \leftrightarrow 0$ giving $\mathsf{size}_i^\vee(e) = \mathsf{size}_i^\vee(0) = 0$. Consider now $\widetilde{w} = a_1 a_2 \cdots a_{\ell-1} = w \cdot t_{-q}$ with (right) ascent s_j so that $\widetilde{w} s_j$ is still dominant. We can compare the root $-(\widetilde{w} s_j)(\alpha_j) = -\widetilde{w}(-\alpha_j) = \widetilde{w}(\alpha_j)$ with the previous computations to conclude the result by induction: for $j \neq 0$, we have $\widetilde{w}(\alpha_j) = w(\alpha_j) + \langle \alpha_j, q \rangle \delta$, while if $j = 0$, then $\widetilde{w}(-\widetilde{\alpha} + \delta) = w(-\widetilde{\alpha}) + (1 - \langle \widetilde{\alpha}, q \rangle)\delta$. The result follows by observing that $\mathsf{size}_i^\vee(\widetilde{w})$ picks off the coefficient of δ for this last inversion, recording the correction factor of r if α_j was a short root. □

Because $\frac{2}{\langle \alpha_i, \alpha_i \rangle} = 1$ if α_i is long, and is r if α_i is short, and because $\sum_{i=0}^n c_i = h$ and $\sum_{i=0}^n \omega_i^\vee = \rho^\vee$, we conclude the following theorem.

Theorem 3.9. *For $\widetilde{w} \in W \backslash \widetilde{W}$, define*

$$\mathsf{size}^\vee(\widetilde{w}) = \sum_{i=0}^{n} \mathsf{size}_i^\vee(\widetilde{w}) = \left(\sum_{\substack{\alpha+k\delta \in \mathsf{inv}(\widetilde{w}^{-1}) \\ \alpha \text{ long}}} k \right) + r \left(\sum_{\substack{\alpha+k\delta \in \mathsf{inv}(\widetilde{w}^{-1}) \\ \alpha \text{ short}}} k \right).$$

For $q \in \mathcal{Q}^\vee_{X_n}$, define $\mathsf{size}^\vee(q) = \sum_{i=0}^{n} \mathsf{size}_i^\vee(q) = \left\langle \frac{h}{2}q - \rho^\vee, q \right\rangle$. Then if $\widetilde{w} = wt_{-q} \in W \backslash \widetilde{W}$ corresponds to the coroot $q \in \mathcal{Q}^\vee_{X_n}$, we have $\mathsf{size}^\vee(\widetilde{w}) = \mathsf{size}^\vee(q)$.

3.4 Other Types

It is easy to embed coroot lattices $\mathcal{Q}_{X_n}$ for the classical types into appropriate coroot lattices of type A. In so doing we find combinatorial models for their coroot lattices. and in so doing give combinatorial interpretations of $\mathsf{size}^\vee$. We illustrate this method here for the exceptional type G_2—we do not know of a similar method for finding reasonable combinatorial models for the remaining types F_4, E_6, E_7, or E_8.

As usual, we wish to think of G_2 as acting on the orthogonal complement of $\mathbb{R}(1,1,1)$ in $\mathbb{R}^3$. The simple roots for G_2 can be taken to be $\alpha_1 := (1,-1,0)$ and $\alpha_2 := \frac{1}{3}(-1,2,-1)$. With these conventions, the type G_2 coroot lattice $\mathcal{Q}^\vee_{G_2}$ coincides with the coroot lattice for $\widetilde{\mathfrak{S}}_3$. The map $q \mapsto \lambda_q$ therefore gives a bijection between $\mathcal{Q}^\vee_{G_2}$ and 3-cores. It remains to determine the action of $\widetilde{G}_2$ on 3-cores. The action of the affine Weyl group $\widetilde{G}_2 = \langle s_0^G, s_1^G, s_2^G : (s_0^G s_1^G)^3 = (s_1^G s_2^G)^6 = (s_0 s_2)^2 = e \rangle$ on $\mathcal{Q}^\vee_{G_2}$ is given explicitly by

$$\begin{aligned} s_1^G(q_1,q_2,q_3) &= (q_2,q_1,q_3), \\ s_2^G(q_1,q_2,q_3) &= (-q_3,-q_2,-q_1), \text{ and} \\ s_0^G(q_1,q_2,q_3) &= (q_3+1,q_2,q_1-1). \end{aligned}$$

We emulate the action of $\widetilde{G}_2$ by $s_1^G(\lambda_q) = s_1(\lambda_q), s_2^G(\lambda_q) = \lambda_q^\intercal$, and $s_0^G(\lambda_q) = s_0(\lambda_q)$. For λ a 3-core, define $\mathsf{size}_G^\vee(\lambda) := \lambda_0 + \lambda_1 + 4\lambda_2$. We obtain a combinatorial model for $\mathcal{Q}^\vee_{G_2}$.

Theorem 3.10. *The map $q \mapsto \lambda_q$ is a $\widetilde{G}_2$-equivariant bijection between the G_2 coroot lattice and 3-cores: s_1^G acts on a 3-core by adding or removing all boxes of content 1, s_0^G acts similarly on boxes of content 0, and s_2^G acts by conjugation. For $q \in \mathcal{Q}^\vee_{G_2}$, $\mathsf{size}_G^\vee(\lambda_q) = \mathsf{size}^\vee(q)$.*

4 Expected Size of Simultaneous Cores

Johnson [5] showed that (a,b)-cores satisfy certain inequalities; when considering them as elements of $\mathcal{Q}^\vee_{A_{a-1}}$, these place them inside a particular simplex that had previously been considered by Sommers [6]. Recall that Φ is a root system with an irreducible Cartan type X_n, Coxeter number h. For $1 \leq i < h$, write Φ_i to denote the set of positive roots of height i.

Definition 4.1. For b coprime to h, write $b = q_b h + r_b$ with $t, r \in \mathbb{Z}_{\geq 0}$ and $0 < r_b < h$. We define the *b-Sommers region*

$$\mathcal{S}_{X_n}(b) := \left\{ x \in V : \begin{array}{rl} \langle x, \alpha \rangle \geq -q_b & \text{for all } \alpha \in \Phi_{r_b} \text{ and} \\ \langle x, \alpha \rangle \leq q_b + 1 & \text{for all } \alpha \in \Phi_{h-r_b} \end{array} \right\}.$$

As in Equation (1.2), a natural generalization of $\mathsf{core}(a,b)$ to any affine Weyl group is the intersection of the coroot lattice $\mathcal{Q}^\vee_{X_n}$ with $\mathcal{S}_{X_n}(b)$, so that $\mathsf{core}(a,b) = \mathsf{core}(A_{a-1}, b)$.

4.1 The Sommers Region and the Fundamental Alcove

We would like to perform the $\mathsf{size}^\vee$-weighted enumeration of $\mathsf{core}(X_n, b)$ using Ehrhart theory. Unfortunately, the family $\{\mathcal{S}_{X_n}(b) : \gcd(b,h) = 1\}$ does not consist of dilations of a fixed polytope—but this difficulty can be circumvented. Define, for any $x \in V$, the statistic

$$\mathsf{size}^{(b)}(x) := \frac{h}{2}\left(\left\| x - \frac{b\rho^\vee}{h} \right\|^2 - \left\| \frac{\rho^\vee}{h} \right\|^2 \right). \tag{4.1}$$

Notice that when $q \in \mathcal{Q}^\vee_{X_n}$ and $b = 1$, we have $\mathsf{size}^{(1)}(q) = \mathsf{size}^\vee(q)$ (cf. Theorem 3.9).

We recall from [7, §4] that there is a unique element $\widetilde{w}_b \in \widetilde{W}$ such that $\frac{b}{h}\rho^\vee = \widetilde{w}_b(\frac{\rho^\vee}{h})$, and that left-multiplication by this element maps $\mathcal{S}_{X_n}(b)$ onto the b-fold dilation of the fundamental alcove $\mathcal{A}$.

Theorem 4.2. *For b coprime to h, the following equality of multisets holds:*

$$\left\{ \mathsf{size}^\vee(q) : q \in \mathsf{core}(X_n, b) \right\} = \left\{ \mathsf{size}^{(b)}(q) : q \in b\mathcal{A} \cap \mathcal{Q}^\vee_{X_n} \right\}.$$

Proof. We first note that since $\widetilde{w}_b$ maps $\mathcal{S}_{X_n}(b)$ onto $b\mathcal{A}$, and also $\widetilde{w}_b \in \widetilde{W}$ and thus is a $\mathcal{Q}^\vee_{X_n}$-preserving bijection, it restricts to a bijection $\mathsf{core}(\widetilde{W}, b) \to b\mathcal{A} \cap \mathcal{Q}^\vee$. Write $\widetilde{w}_b = t_{q_0} w$; then $\widetilde{w}_b = t_{\frac{b\rho^\vee}{h}} w t_{-\frac{\rho^\vee}{h}}$. Since $\|\cdot\|$ is W-invariant, for $q \in \mathsf{core}(\widetilde{W}, b)$:

$$\begin{aligned} \mathsf{size}^{(b)}(\widetilde{w}_b(q)) &= \frac{h}{2}\left(\left\| t_{\frac{b}{h}\rho^\vee} w t_{\frac{1}{h}\rho^\vee}(q) - \frac{b\rho^\vee}{h} \right\|^2 - \left\| -\frac{\rho^\vee}{h} \right\|^2 \right) \\ &= \frac{h}{2}\left(\left\| w\left(q - \frac{\rho^\vee}{h}\right) \right\|^2 - \left\| \frac{\rho^\vee}{h} \right\|^2 \right) = \mathsf{size}^\vee(q). \end{aligned}$$

□

4.2 Proof Sketch

We are now ready to sketch a proof of Theorem 1.4:

Theorem 1.4. *For X_n an irreducible rank n Cartan type with root system Φ,*

$$\mathop{\mathbb{E}}_{\lambda \in \mathsf{core}(X_n,b)}(\mathsf{size}^\vee(\lambda)) = \frac{rg^\vee}{h}\frac{n(b-1)(h+b+1)}{24},$$

where h is the Coxeter number of X_n, $g^\vee$ is the dual Coxeter number for $\Phi^\vee$, and r is the ratio of the length of a long root to the length of a short root in Φ.

We do this by computing the left-hand side explicitly for each type X_n, which by Theorem 4.2 is

$$\mathop{\mathbb{E}}_{q \in \mathsf{core}(X_n,b)}(\mathsf{size}^\vee(q)) = \frac{1}{|\mathsf{core}(X_n,b)|}\sum_{q\in\mathsf{core}(X_n,b)}\mathsf{size}^\vee(q) = \frac{1}{|b\mathcal{A}\cap\mathcal{Q}^\vee_{X_n}|}\sum_{q\in b\mathcal{A}\cap\mathcal{Q}^\vee_{X_n}}\mathsf{size}^{(b)}(q).$$

The denominator was explicitly (and uniformly) calculated by Haiman [3]. To compute the sum, we first record the vertices of the fundamental alcove $\mathcal{A}$: they are $\Gamma := \{0\} \cup \left\{\frac{\omega_i^\vee}{c_i} : 1 \le i \le n\right\}$, where and c_i are defined by $\widetilde{\alpha} = \sum_{i=1}^n c_i\alpha_i$. As in [7], we proceed by translating the problem to the coweight lattice. Define the *extended affine Weyl group* of type X_n by $\widetilde{W}_{\mathrm{ex}} := W \ltimes \Lambda^\vee_{X_n}$, and write the group of automorphisms for $b\mathcal{A}$ as $b\Omega := \{\widetilde{w} \in \widetilde{W}_{\mathrm{ex}} : \widetilde{w}(b\mathcal{A}) = b\mathcal{A}\}$. These groups are isomorphic for all b, and in particular have constant order f.

Proposition 4.3 ([7, Theorem 2.5 & Lemma 6.11])**.** *For any b coprime to h:*

(a) The action of $b\Omega$ on $b\mathcal{A}\cap\Lambda^\vee_{X_n}$ is free.

(b) Each $b\Omega$ orbit of $b\mathcal{A}\cap\Lambda^\vee_{X_n}$ contains exactly one element of $b\mathcal{A}\cap\mathcal{Q}^\vee_{X_n} = \mathsf{core}(X_n,b)$.

(c) For any $\widetilde{w} \in b\Omega$ any any $\omega \in \Lambda^\vee_{X_n}$, $\mathsf{size}^\vee_b(\omega) = \mathsf{size}^\vee_b(\widetilde{w}\cdot\omega)$

Using this, we finish the translation from $\mathcal{Q}^\vee_{X_n}$ to $\Lambda^\vee_{X_n}$:

$$\mathop{\mathbb{E}}_{q \in \mathsf{core}(X_n,b)}(\mathsf{size}^\vee(q)) = \frac{1}{|b\mathcal{A}\cap\mathcal{Q}^\vee_{X_n}|}\cdot\frac{1}{f}\sum_{q\in b\mathcal{A}\cap\Lambda^\vee_{X_n}}\mathsf{size}^{(b)}(q).$$

Let us now recall the relevant Ehrhart-theoretic tools. For any degree-r polynomial $F : \mathbb{R}^n \to \mathbb{R}$, its *weighted lattice point enumerator* is $\mathcal{A}_F(b) := \sum\limits_{q\in b\mathcal{A}\cap\Lambda^\vee_{X_n}} F(q)$ over $\Lambda^\vee_{X_n}$. This $\mathcal{A}_F(b)$ is a quasipolynomial in b, of degree $n+r$ and period $c := \mathsf{lcm}(c_1,\ldots,c_n)$, where the c_i are again the denominators of the vertices of $\mathcal{A}$. As $\mathsf{size}^{(b)}$ changes with b, Ehrhart theory appears to be inapplicable—however, a judicious rewriting shows that this is not the case.

Proposition 4.4. *The weighted lattice point enumerator $\mathcal{A}_{\mathsf{size}^{(b)}}(b)$ is a quasipolynomial in b of degree $n+2$ and period $c_{X_n} := \mathsf{lcm}(c_1, \ldots, c_n)$.*

Proof. Notice that $\mathsf{size}_b^\vee(x) = \frac{h}{2}\|x\|^2 - b\langle x, \rho^\vee\rangle + (b^2-1)\frac{\|\check{\rho}\|^2}{2h}$. Thus we find that

$$\mathcal{A}_{\mathsf{size}^{(b)}}(b) = \tfrac{h}{2}\mathcal{A}_{\|\cdot\|^2}(b) - b\mathcal{A}_{\langle\cdot,\rho^\vee\rangle}(b) + (b^2-1)\mathcal{A}_{\frac{\|\rho^\vee\|^2}{2h}}(b)$$

is a quasipolynomial in b of degree $n+2$ and period c_{X_n}. $\square$

Therefore, to complete the proof of Theorem 1.4, we may compute the quasipolynomial on for all components that contain a residue $b \bmod c_{X_n}$ that is coprime to h. For the exceptional types, this is already a finite and computationally feasible calculation. For classical types, we use Ehrhart reciprocity to see that $-e_i$ is a root of $\mathcal{A}_{\mathsf{size}^{(b)}}(b)$ for all exponents e_i; when working through the details, this reduces the problem to interpolating a factor that is at worst quadratic (in type D_n). Thus, by explicitly computing the lattice points contained in several dilates of $\mathcal{A}$, we complete the missing factor and recover the desired formula.

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