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Decompositions of Ehrhart h^*-polynomials for rational polytopes

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Abstract. The Ehrhart quasipolynomial of a rational polytope P encodes the number of integer lattice points in dilates of P, and the h^*-polynomial of P is the numerator of the accompanying generating function. We provide two decomposition formulas for the h^*-polynomial of a rational polytope. The first decomposition generalizes a theorem of Betke and McMullen for lattice polytopes. We use our rational Betke–McMullen formula to provide a novel proof of Stanley's Monotonicity Theorem for the h^*-polynomial of a rational polytope. The second decomposition generalizes a result of Stapledon, which we use to provide rational extensions of the Stanley and Hibi inequalities satisfied by the coefficients of the h^*-polynomial for lattice polytopes. Lastly, we apply our results to rational polytopes containing the origin whose duals are lattice polytopes.

Keywords: Ehrhart quasipolynomial, Ehrhart series, generating function, h^*-polynomial

1 Introduction

For a d-dimensional rational polytope $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Q}^d$) and a positive integer t, let $L_P(t)$ denote the number of integer lattice points in tP. Ehrhart's theorem [3] tells us that $L_P(t)$ is of the form $\operatorname{vol}(P)\, t^d + k_{d-1}(t)\, t^{d-1} + \cdots + k_1(t)\, t + k_0(t)$, where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in t. We call $L_P(t)$ the *Ehrhart quasipolynomial* of P, and Ehrhart proved that each period of $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ divides the *denominator* q of P, which is the least common multiple of all its vertex coordinate denominators. The *Ehrhart series* is the rational generating function

$$\operatorname{Ehr}(P; z) := \sum_{t \geq 0} L(P; t)\, z^t = \frac{h^*(P; z)}{(1 - z^q)^{d+1}},$$

where $h^*(P; z)$ is a polynomial of degree less than $q(d+1)$, the h^*-*polynomial* of P.[1]

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[1]Note that the h^*-polynomial depends not only on q (though that is implicitly determined by P), but also on our choice of representing the rational function $\operatorname{Ehr}(P; z)$, which in our form will not be in lowest terms.

Our first main contributions are generalizations of two well-known decomposition formulas of the h^*-polynomial for lattice polytopes due to Betke–McMullen [2] and Stapledon [11]. (All undefined terms are specified in the sections below.)

Theorem 3.2. For a triangulation T with denominator q of a rational d-polytope P,

$$\operatorname{Ehr}(P;z) = \frac{\sum_{\Omega \in T} B(\Omega;z)\, h(\Omega;z^q)}{(1-z^q)^{d+1}}\,.$$

Theorem 4.4. Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q. Then

$$h^*(P;z) = \frac{1-z^q}{1-z^\ell} \sum_{\Omega \in T} \left(B(\Omega;z) + B(\Omega';z)\right) h(\Omega;z^q)\,.$$

Our second main result is a generalization of inequalities provided by Hibi [5] and Stanley [9] that are satisfied by the coefficients of the h^*-polynomial for lattice polytopes.

Theorem 4.7. Let P be a rational d-polytope with denominator q, let $s := \deg h^*(P;z)$. The h^*-vector $(h_0^*, \ldots, h_{q(d+1)-1}^*)$ of P satisfies the following inequalities:

$$h_0^* + \cdots + h_{i+1}^* \geq h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-i}^*\,, \qquad i = 0, \ldots, \left\lfloor \frac{q(d+1)-1}{2} \right\rfloor - 1\,, \quad (1.1)$$

$$h_s^* + \cdots + h_{s-i}^* \geq h_0^* + \cdots + h_i^*\,, \qquad i = 0, \ldots, q(d+1)-1\,. \quad (1.2)$$

Inequality (1.1) is a generalization of a theorem by Hibi [5] for lattice polytopes, and (1.2) generalizes an inequality given by Stanley [9] for lattice polytopes, namely the case when $q = 1$. Both inequalities follow from the *a*/*b-decomposition* of the $\overline{h^*}$-polynomial for rational polytopes given in Theorem 4.6 in Section 4, which in turn generalizes results (and uses rational analogues of techniques) by Stapledon [11].

This paper is an extended abstract of [1] and some proofs are omitted. The paper is structured as follows. In Section 2 we provide notation and background. In Section 3 we prove Theorem 3.2 and use this to give a novel proof of Stanley's Monotonicity Theorem. In Section 4 we prove Theorems 4.4 and 4.7. We conclude in Section 5 with some applications.

2 Set-Up and Notation

A *pointed simplicial cone* is a set of the form

$$K(\mathbf{W}) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : \lambda_i \geq 0 \right\},$$

where $\mathbf{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of n linearly independent vectors in $\mathbb{R}^d$. If we can choose $\mathbf{w}_i \in \mathbb{Z}^d$ then $K(\mathbf{W})$ is a *rational* cone and we assume this throughout this paper. Define the *open parallelepiped* associated with $K(\mathbf{W})$ as

$$\mathrm{Box}\,(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 < \lambda_i < 1 \right\}. \tag{2.1}$$

Observe that we have the natural involution $\iota : \mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d \to \mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d$ given by

$$\iota \left(\sum_i \lambda_i \mathbf{w}_i \right) := \sum_i (1 - \lambda_i) \mathbf{w}_i \,. \tag{2.2}$$

We set $\mathrm{Box}\,(\{0\}) := \{0\}$.

Let $u : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the last coordinate. We then define the *box polynomial* as

$$B(\mathbf{W}; z) := \sum_{\mathbf{v} \in \mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^d} z^{u(\mathbf{v})}. \tag{2.3}$$

If $\mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d = \emptyset$, then we set $B(\mathbf{W}; z) = 0$. We also define $B(\emptyset; z) = 1$.

Lemma 2.1. $B(\mathbf{W}; z) = z^{\sum_i u(\mathbf{w}_i)} B\left(\mathbf{W}; \frac{1}{z}\right)$.

Next, we define the *fundamental parallelepiped* $\Pi(\mathbf{W})$ to be a half-open variant of $\mathrm{Box}\,(\mathbf{W})$, namely,

$$\Pi(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 \le \lambda_i < 1 \right\}.$$

We also want to cone over a polytope P. If $P \subset \mathbb{R}^d$ is a rational polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate. Then

$$\mathrm{cone}\,(P) = \left\{ \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i, 1) : \lambda_i \ge 0 \right\} \subset \mathbb{R}^{d+1}. \tag{2.4}$$

We say a point is at *height* k in the cone if the point lies on $\mathrm{cone}\,(P) \cap \{\mathbf{x} : x_{d+1} = k\}$. Note that qP is embedded in $\mathrm{cone}\,(P)$ as $\mathrm{cone}\,(P) \cap \{\mathbf{x} : x_{d+1} = q\}$.

A *triangulation* T of a d-polytope P is a subdivision of P into simplices (of all dimensions). If all the vertices of T are rational points, define the *denominator* of T to be the least common multiple of all the vertex coordinate denominators of the faces of T. For each $\Delta \in T$, we define the *h-polynomial* of Δ with respect to T as

$$h_T(\Delta; z) := (1 - z)^{d - \dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left(\frac{z}{1 - z} \right)^{\dim(\Phi) - \dim(\Delta)}, \tag{2.5}$$

where the sum is over all simplices $\Phi \in T$ containing Δ. When T is clear from context, we omit the subscript. Note that when T is a boundary triangulation of P, the definition of the h-vector will be adjusted according to dimension, that is, d should be replaced by $d-1$ in (2.5).

For a d-simplex Δ with denominator p, let $\mathbf{W}$ be the set of ray generators of cone$\,(\Delta)$ at height p, which are all integral. We then define the *h^*-polynomial of* Δ as the generating function of the last coordinate of integer points in $\Pi(\mathbf{W}) =: \Pi(\Delta)$, that is, $h^*(\Delta;z) = \sum_{\mathbf{v}\in\Pi(\Delta)\cap\mathbb{Z}^{d+1}} z^{u(\mathbf{v})}$. With this consideration, the Ehrhart series of Δ can be expressed as $\mathrm{Ehr}(\Delta;z) = \frac{h^*(\Delta;z)}{(1-z^p)^{d+1}}$. We adjust this definition when Δ is a rational m-simplex of a triangulation T with denominator q. Namely, we let $\mathbf{W} = \{(\mathbf{r}_1,q),\ldots,(\mathbf{r}_{m+1},q)\}$, where the $(\mathbf{r}_i,q)$ are integral ray generators of cone$\,(\Delta)$ at height q. The corresponding h^*-polynomial of Δ is a function of q and the Ehrhart series of Δ can be expressed as

$$\mathrm{Ehr}(\Delta;z) = \frac{h^*(\Delta;z)}{(1-z^q)^{m+1}}.$$

We may think of $h^*(\Delta;z)$ as computed via $\sum_{\mathbf{v}\in\Pi(\mathbf{W})\cap\mathbb{Z}^{d+1}} z^{u(\mathbf{v})}$.

3 Rational Betke–McMullen Decomposition

3.1 Decomposition à la Betke–McMullen

Let P be a rational d-polytope and T be a triangulation of P with denominator q. For an m-simplex $\Delta \in T$, let $\mathbf{W} = \{(\mathbf{r}_1,q),\ldots,(\mathbf{r}_{m+1},q)\}$, where the $(\mathbf{r}_i,q)$ are the integral ray generators of cone$\,(\Delta)$ at height q as above. Further, set $B(\mathbf{W};z) =: B(\Delta;z)$ and similarly $\mathrm{Box}\,(\mathbf{W}) =: \mathrm{Box}\,(\Delta)$. We emphasize that the h^*-polynomial, fundamental parallelepiped, and box polynomial of Δ depend on the denominator q of T.

A point $\mathbf{v} \in$ cone$\,(\Delta)$ can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i,q)$ for $\lambda_i \geq 0$. Define

$$I(\mathbf{v}) := \{i \in [m+1] : \lambda_i \in \mathbb{Z}\} \qquad \text{and} \qquad \overline{I(\mathbf{v})} := [m+1] \setminus I, \tag{3.1}$$

where $[m+1] := \{1,\cdots,m+1\}$.

Lemma 3.1. *Fix a triangulation T with denominator q of a rational d-polytope P and let $\Delta \in T$. Then for $\Omega := \mathrm{conv}\left\{\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v})}\right\} \subseteq \Delta$, $h^*(\Delta;z) = \sum_{\Omega\subseteq\Delta} B(\Omega;z)$.*

Theorem 3.2. *For a triangulation T with denominator q of a rational d-polytope P,*

$$\mathrm{Ehr}(P;z) = \frac{\sum_{\Omega\in T} B(\Omega;z)\, h(\Omega;z^q)}{(1-z^q)^{d+1}}\,.$$

Proof. We write P as the disjoint union of all open nonempty simplices in T and use Ehrhart–Macdonald reciprocity [3, 8]:

$$\begin{aligned}
\mathrm{Ehr}(P;z) &= 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} \mathrm{Ehr}(\Delta^\circ; z) = 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} (-1)^{\dim(\Delta)+1} \mathrm{Ehr}\left(\Delta; \frac{1}{z}\right) \\
&= 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} (-1)^{\dim(\Delta)+1} \frac{h^*\left(\Delta; \frac{1}{z}\right)}{\left(1 - \frac{1}{z^q}\right)^{\dim(\Delta)+1}} \\
&= 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} \frac{(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)}\, h^*\left(\Delta; \frac{1}{z}\right)}{(1 - z^q)^{d+1}}.
\end{aligned}$$

Note that the Ehrhart series of each Δ is being written as a rational function with denominator $(1 - z^q)^{d+1}$. Using Lemma 3.1,

$$\begin{aligned}
\mathrm{Ehr}(P;z) &= 1 + \sum_{\Delta \in T \setminus \emptyset} \frac{(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} B\left(\Omega; \frac{1}{z}\right)}{(1 - z^q)^{d+1}} \\
&= \frac{\sum_{\Delta \in T} \left[(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} B\left(\Omega; \frac{1}{z}\right)\right]}{(1 - z^q)^{d+1}}.
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
h^*(P;z) &= \sum_{\Delta \in T} \left[(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} B\left(\Omega; \frac{1}{z}\right) \right] \\
&= \sum_{\Delta \in T} \left[(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} (z^q)^{-\dim(\Omega)-1} B(\Omega; z) \right] \\
&= \sum_{\Omega \in T} \sum_{\Omega \subseteq \Delta} (z^q)^{\dim(\Delta) - \dim(\Omega)} (1 - z^q)^{d - \dim(\Delta)} B(\Omega; z) \\
&= \sum_{\Omega \in T} \left[B(\Omega; z) (1 - z^q)^{d - \dim(\Omega)} \sum_{\Omega \subseteq \Delta} \left(\frac{z^q}{1 - z^q} \right)^{\dim(\Delta) - \dim(\Omega)} \right].
\end{aligned}$$

Using the definition of the h-polynomial, the theorem follows. □

3.2 Rational h^*-Monotonicity

We now show how the following theorem follows from our rational Betke–McMullen formula.

Theorem 3.3 (Stanley Monotonicity [10]). *Suppose that $P \subseteq Q$ are rational polytopes with qP and qQ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the h^*-polynomials via*

$$\operatorname{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{\dim(P)+1}} \qquad \textit{and} \qquad \operatorname{Ehr}(Q;z) = \frac{h^*(Q;z)}{(1-z^q)^{\dim(Q)+1}}.$$

Then $h_i^*(P;z) \le h_i^*(Q;z)$ coefficient-wise.

The following lemma assumes familiarity with Cohen–Macaulay complexes and related theory.

Lemma 3.4. *Suppose that P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that (1) the join $T * \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subcomplex of T' and (2) $\dim(P * \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) = \dim(Q)$. For every face $\Omega \in T$, the coefficient-wise inequality $h_T(\Omega;z) \le h_{T'}(\Omega,z)$ holds.*

Proof of Theorem 3.3. Let P be a polytope contained in Q. Let T be a triangulation of P and let T' be a triangulation of Q such that T' restricted to P is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions given in Lemma 3.4. (Note that such a triangulation T' can always be obtained from T, e.g., by extending T using a placing triangulation.) By Theorem 3.2, $h^*(P;z) = \sum_{\Omega \in T} B(\Omega;z)\, h_T(\Omega;z^q)$. Since P is contained in Q,

$$h^*(Q;z) = \sum_{\Omega \in T} B(\Omega;z)\, h_{T'|_P}(\Omega;z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega;z)\, h_{T'}(\Omega;z^q).$$

By Lemma 3.4, the coefficients of $\sum_{\Omega \in T} B(\Omega;z) h_{T'}(\Omega;z^q)$ dominate the coefficients of $\sum_{\Omega \in T} B(\Omega;z) h_T(\Omega;z^q)$. This further implies that the coefficients of $h^*(Q;z)$ dominate the coefficients of $h^*(P;z)$ since

$$\begin{aligned}\sum_{\Omega \in T} B(\Omega;z)\, h_T(\Omega;z^q) &\le \sum_{\Omega \in T} B(\Omega;z)\, h_{T'}(\Omega;z^q) \\ &\le \sum_{\Omega \in T} B(\Omega;z)\, h_{T'|_P}(\Omega;z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega;z)\, h_{T'}(\Omega;z^q). \end{aligned} \qquad \square$$

4 h^*-Decompositions from Boundary Triangulations

4.1 Set-up

Throughout this section we will use the following set-up. Fix a boundary triangulation T with denominator q of a rational d-polytope P. Take $\ell \in \mathbb{Z}_{>0}$, such that ℓP contains a lattice point $\mathbf{a}$ in its interior. Thus $(\mathbf{a}, \ell) \in \operatorname{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$ is a lattice point in the interior of the cone of P at height ℓ, and $\operatorname{cone}((\mathbf{a},\ell))$ is the ray through the point $(\mathbf{a},\ell)$. We cone over each $\Delta \in T$ and define $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$ where the $(\mathbf{r}_i, q)$ are integral ray generators of $\operatorname{cone}(\Delta)$ at height q. As before, we have the associated box polynomial $B(\mathbf{W}; z) =: B(\Delta; z)$. Now, let $\mathbf{W}' = \mathbf{W} \cup \{(\mathbf{a},\ell)\}$ be the set of generators from $\mathbf{W}$ together with $(\mathbf{a},\ell)$ and we set $\operatorname{cone}(\Delta')$ to be the cone generated by $\mathbf{W}'$, with associated box polynomial $B(\mathbf{W}'; z) =: B(\Delta'; z)$.

Corollary 4.1. *For each face Δ of T,*

$$B(\Delta; z) = z^{q(\dim(\Delta)+1)} B\left(\Delta; \tfrac{1}{z}\right) \qquad \text{and} \qquad B(\Delta'; z) = z^{q(\dim(\Delta)+1)+\ell} B\left(\Delta'; \tfrac{1}{z}\right).$$

Observe that when $\Delta = \emptyset$ is the empty face, $B(\emptyset; z) = 1$, but $B(\emptyset'; z) = B((\mathbf{a},\ell); z)$. This differs from the scenario in [11] where Stapledon's set-up determined that $B(\emptyset', z) = 0$. For a real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. Additionally, define the *fractional part* of x to be $\{x\} = x - \lfloor x \rfloor$.

4.2 Boundary Triangulations

For each $\mathbf{v} \in \operatorname{cone}(P)$ we associate two faces $\Delta(\mathbf{v})$ and $\Omega(\mathbf{v})$ of T, as follows. The face $\Delta(\mathbf{v})$ is chosen to be the minimal face of T such that $\mathbf{v} \in \operatorname{cone}(\Delta'(\mathbf{v}))$, and we define

$$\Omega(\mathbf{v}) := \operatorname{conv}\left\{\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v})}\right\} \subseteq \Delta(\mathbf{v}),$$

where $\overline{I(\mathbf{v})}$ is defined as in (3.1) and the $(\mathbf{r}_i, q)$ are ray generators of $\operatorname{cone}(\Delta)$. In an effort to make our statements and proofs less notation heavy, for the rest of this section we write $\Delta(\mathbf{v}) = \Delta$ and $\Omega(\mathbf{v}) = \Omega$ with the understanding that both depend on $\mathbf{v}$. Furthermore, for $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) + \lambda(\mathbf{a},\ell)$ where $\lambda, \lambda_i \geq 0$, define $\{\mathbf{v}\} := \sum_{i \in \overline{I(\mathbf{v})}} \{\lambda_i\}(\mathbf{r}_i, q) + \{\lambda\}(\mathbf{a},\ell)$.

Lemma 4.2. *Given $\mathbf{v} \in \operatorname{cone}(P)$, construct $\Delta = \Delta(\mathbf{v})$ as described above, with $\operatorname{cone}(\Delta)$ generated by $(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)$. Then $\mathbf{v}$ can be written uniquely as*

$$\{\mathbf{v}\} + \sum_{i \in I(\mathbf{v})} (\mathbf{r}_i, q) + \sum_{i=1}^{m+1} \mu_i(\mathbf{r}_i, q) + \mu(\mathbf{a},\ell), \tag{4.1}$$

where $\mu, \mu_i \in \mathbb{Z}_{\geq 0}$.

Corollary 4.3. *Continuing the notation above,*

$$u(\mathbf{v}) = u(\{\mathbf{v}\}) + q(\dim \Delta(\mathbf{v}) - \dim \Omega(\mathbf{v})) + \sum_{i=1}^{m+1} q\, \mu_i(\mathbf{v}) + \mu(\mathbf{v})\, \ell\,. \tag{4.2}$$

The following theorem provides a decomposition of the h^*-polynomial of a rational polytope in terms of box and h-polynomials. It is important to note again that the h^*-polynomial depends on the denominator of the boundary triangulation.

Theorem 4.4. *Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q. Then*

$$h^*(P;z) = \frac{1-z^q}{1-z^\ell} \sum_{\Omega \in T} \left(B(\Omega;z) + B(\Omega';z)\right) h(\Omega;z^q).$$

Proof. By Corollary 4.3,

$$\begin{aligned}
\frac{h^*(P;z)}{(1-z^q)^{d+1}} &= \sum_{\mathbf{v} \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} \\
&= \sum_{\mathbf{v} \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{u(\{\mathbf{v}\}) + q(\dim\Delta(\mathbf{v}) - \dim\Omega(\mathbf{v})) + \sum_{i=1}^{\dim(\Delta)+1} q\mu_i(\mathbf{v}) + \mu(\mathbf{v})\ell} \\
&= \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} z^{q(\dim\Delta - \dim\Omega)} \sum_{\mathbf{v} \in (\mathrm{Box}(\Omega) \cup \mathrm{Box}(\Omega')) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} \sum_{\mu_i, \mu \geq 0} z^{\sum_{i=1}^{\dim(\Delta)+1} q\mu_i + \mu\ell} \\
&= \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} \frac{\left(B(\Omega;z) + B(\Omega';z)\right) z^{q(\dim\Delta - \dim\Omega)}}{(1-z^q)^{\dim(\Delta)+1}(1-z^\ell)} \\
&= \frac{1}{1-z^\ell} \sum_{\Omega \in T} \left(B(\Omega;z) + B(\Omega';z)\right) \sum_{\Omega \subseteq \Delta} \frac{(z^q)^{\dim(\Delta) - \dim(\Omega)}}{(1-z^q)^{\dim(\Delta)+1}} \\
&= \frac{1}{(1-z^\ell)(1-z^q)^d} \sum_{\Omega \in T} \left(B(\Omega;z) + B(\Omega';z)\right) h(\Omega;z^q)\,.
\end{aligned}$$

□

4.3 Rational Stapledon Decomposition and Inequalities

Using Theorem 4.4, we can rewrite the h^*-polynomial of a rational polytope P as

$$h^*(P;z) = \frac{1+z+\cdots+z^{q-1}}{1+z+\cdots+z^{\ell-1}} \sum_{\Omega \in T} \left(B(\Omega;z) + B(\Omega';z)\right) h(\Omega;z^q)\,.$$

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1+z+\cdots+z^{\ell-1}\right) h^*(P;z)\,. \tag{4.3}$$

We know that $h^*(P;z)$ is a polynomial of degree at most $q(d+1)-1$, thus $\overline{h^*}(P;z)$ has degree at most $q(d+1)+\ell-2$. We set f to be the degree of $\overline{h^*}(P;z)$ and s to be the degree of $h^*(P;z)$. We can recover $h^*(P;z)$ from $\overline{h^*}(P;z)$ for a chosen value of ℓ; if we write

$$\overline{h^*}(P;z) = \overline{h^*_0} + \overline{h^*_1}z + \cdots + \overline{h^*_f}z^f,$$

then

$$\overline{h_i^*} = h_i^* + h_{i-1}^* + \cdots + h_{i-l+1}^* \qquad i = 0, \ldots, f, \tag{4.4}$$

and we set $h_i^* = 0$ when $i > s$ or $i < 0$.

Proposition 4.5. *Let P be a rational d-polytope with denominator q and Ehrhart series*

$$\mathrm{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}.$$

Then $\deg h^*(P;z) = s$ if and only if $(q(d+1)-s)P$ is the smallest integer dilate of P that contains an interior lattice point.

The following result provides a decomposition of the $\overline{h^*}$-polynomial which we refer to as an *a/b-decomposition*. It generalizes [11, Theorem 2.14] to the rational case.

Theorem 4.6. *Let P be a rational d-polytope with denominator q, and let $s := \deg h^*(P;z)$. Then $\overline{h^*}(P;z)$ has a unique decomposition*

$$\overline{h^*}(P;z) = a(z) + z^\ell b(z),$$

where $\ell = q(d+1) - s$ and $a(z)$ and $b(z)$ are polynomials with integer coefficients satisfying $a(z) = z^{q(d+1)-1} a\left(\frac{1}{z}\right)$ and $b(z) = z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right)$. Moreover, the coefficients of $a(z)$ and $b(z)$ are nonnegative.

Proof. Let a_i and b_i denote the coefficients of z^i in $a(z)$ and $b(z)$, respectively. Set

$$a_{i+1} = h_0^* + \cdots + h_{i+1}^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-1-i}^*, \tag{4.5}$$

and

$$b_i = -h_0^* - \cdots - h_i^* + h_s^* + \cdots + h_{s-i}^*. \tag{4.6}$$

Using (4.4) and the fact that $\ell = q(d+1) - s$, we compute that

$$\begin{aligned} a_i + b_{i-\ell} &= h_0^* + \cdots + h_i^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-i}^* - h_0^* - \cdots - h_{i-\ell}^* + h_s^* \\ &\quad + \cdots + h_{s-i+\ell}^* = h_{i-\ell+1}^* + \cdots + h_i^* = \overline{h_i^*}, \\ a_i - a_{q(d+1)-1-i} &= h_0^* + \cdots + h_i^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-i}^* - h_0^* - \cdots - h_{q(d+1)-1-i}^* \\ &\quad + h_{q(d+1)-1}^* + \cdots + h_{i+1}^* = 0, \\ b_i - b_{q(d+1)-1-\ell-i} &= -h_0^* - \cdots - h_i^* + h_s^* + \cdots + h_{s-i}^* + h_0^* + \cdots + h_i^* \\ &\quad - h_s^* - \cdots - h_{s-i-1}^* - h_s^* - \cdots - h_{i+1}^* = 0, \end{aligned}$$

for $i = 0, \ldots, q(d+1)-1$. Thus, we obtain the decomposition desired. The uniqueness property follows (4.5) and (4.6).

Let T be a regular boundary triangulation of P. By Theorem 4.4 and (4.3), we can set

$$a(z) = (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega; z)\, h(\Omega; z^q)\,, \tag{4.7}$$

and

$$b(z) = z^{-\ell}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z)\, h(\Omega; z^q)\,, \tag{4.8}$$

so that $\overline{h^*}(P; z) = a(z) + z^\ell b(z)$. By Proposition 4.5, the dilate kP contains no interior lattice points for $k = 1, \ldots, \ell - 1$, so if $\mathbf{v} \in \operatorname{Box}(\Omega') \cap \mathbb{Z}^{d+1}$ for $\Omega \in T$, then $u(\mathbf{v}) \geq \ell$. Hence, $b(z)$ is a polynomial. We now need to verify that

$$a(z) = a^{q(d+1)-1} a\left(\tfrac{1}{z}\right) \qquad \text{and} \qquad b(z) = z^{q(d+1)-1-\ell} b\left(\tfrac{1}{z}\right).$$

It is a well-known property of the h-vector in (2.5) that $h(\Omega, z^q) = z^{q(d-1-\dim(\Omega))} h(\Omega; z^{-q})$.

Using the aforementioned and Corollary 4.1, we determine that

$$\begin{aligned} z^{q(d+1)-1} a\left(\frac{1}{z}\right) &= z^{q(d+1)-1}\left(1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}}\right) \sum_{\Omega \in T} B\left(\Omega; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\ &= z^{qd}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(\dim(\Omega)+1)} B(\Omega, z)\, z^{-q(d-1-\dim \Omega)} h(\Omega; z^q) \\ &= (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega, z)\, h(\Omega; z^q) = a(z) \end{aligned}$$

and

$$\begin{aligned} z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right) &= z^{q(d+1)-1-\ell} z^\ell \left(1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}}\right) \sum_{\Omega \in T} B\left(\Omega'; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\ &= z^{qd}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(\dim \Omega+1)-\ell} B(\Omega'; z)\, z^{-q(d-1-\dim \Omega)} h(\Omega; z^q) \\ &= z^{-\ell}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z)\, h(\Omega; z^q) = b(z)\,. \end{aligned}$$

Lastly, recall that the box polynomials and the h-polynomials are nonnegative, so a sum of products of box polynomials and h-polynomials will also be nonnegative. Thus, the result holds. □

The next theorem follows as a corollary to Theorem 4.6 and gives inequalities satisfied by the coefficients of the h^*-polynomial for rational polytopes.

Theorem 4.7. *Let P be a rational d-polytope with denominator q, let $s := \deg h^*(P; z)$ and $\ell := q(d+1) - s$. The h^*-vector $(h_0^*, \ldots, h_{q(d+1)-1}^*)$ of P satisfies the following inequalities:*

$$h_0^* + \cdots + h_{i+1}^* \geq h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-i}^*\,, \qquad i = 0, \ldots, \left\lfloor \frac{q(d+1)-1}{2} \right\rfloor - 1\,, \tag{4.9}$$

$$h_s^* + \cdots + h_{s-i}^* \geq h_0^* + \cdots + h_i^*\,, \qquad i = 0, \ldots, q(d+1) - 1\,. \tag{4.10}$$

Proof. By (4.5) and (4.6) if follows that (4.9) and (4.10) hold if and only if $a(z)$ and $b(z)$ are nonnegative, respectively, which in turn follows from Theorem 4.6. □

5 Applications

5.1 Rational Reflexive Polytopes

A lattice polytope is *reflexive* if its dual is also a lattice polytope. Hibi [6] proved that a lattice polytope P is the translate of a reflexive polytope if and only if $\mathrm{Ehr}\left(P;\frac{1}{z}\right) = (-1)^{d+1}z\,\mathrm{Ehr}(P;z)$ as rational functions, that is, $h^*(z)$ is palindromic. More generally, Fiset and Kaspryzk [4, Corollary 2.2] proved that a rational polytope P whose dual is a lattice polytope has a palindromic h^*-polynomial. The following proposition provides an alternate route to Fiset and Kaspryzk's result.

Theorem 5.1. *Let P be a rational polytope containing the origin. The dual of P is a lattice polytope if and only if $\overline{h^*}(P;z) = h^*(z) = a(z)$, that is, $b(z) = 0$ in the a/b-decomposition of $\overline{h^*}(P;z)$ from Theorem 4.4.*

5.2 Reflexive Polytopes of Higher Index

Kasprzyk and Nill [7] introduced the following class of polytopes .

Definition 5.2. A lattice polytope P is a *reflexive polytope of higher index $\mathcal{L}$* (also known as an *$\mathcal{L}$-reflexive polytope*), for some $\mathcal{L} \in \mathbb{Z}_{>0}$, if the following conditions hold:

- P contains the origin in its interior;
- The vertices of P are primitive, i.e., the line segment joining each vertex to $\mathbf{0}$ contains no other lattice points;
- For any facet F of P the local index $\mathcal{L}_F$ equals $\mathcal{L}$, i.e., the integral distance of $\mathbf{0}$ from the affine hyperplane spanned by F equals $\mathcal{L}$.

The 1-reflexive polytopes are the reflexive polytopes mentioned earlier in the section. Kaspryzk and Nill proved that if P is a lattice polytope with primitive vertices containing the origin in its interior then P is $\mathcal{L}$-reflexive if and only if $\mathcal{L}P^*$ is a lattice polytope having only primitive vertices. In this case, $\mathcal{L}P^*$ is also $\mathcal{L}$-reflexive. They investigated $\mathcal{L}$-reflexive polygons. In particular, they show that there is no $\mathcal{L}$-reflexive polygon of even index. Furthermore, they provide a family of $\mathcal{L}$-reflexive polygons arising for each odd index:

$$P_{\mathcal{L}} = \mathrm{conv}\left\{\pm(0,1), \pm(\mathcal{L},2), \pm(\mathcal{L},1)\right\}.$$

We are interested in the dual of $P_{\mathcal{L}}$:

$$P^*_{\mathcal{L}} = \text{conv}\left\{\pm\left(\frac{1}{\mathcal{L}},0\right), \pm(\frac{2}{\mathcal{L}},-1), \pm\left(\frac{1}{\mathcal{L}},-1\right)\right\}.$$

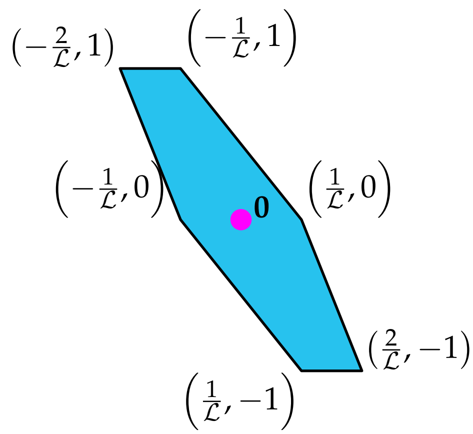

Figure 1: The rational hexagon $P^*_{\mathcal{L}}$.

Applying Theorems 4.4 and 5.1 we conclude to following.

Proposition 5.3. *For $\mathcal{L} = 2k+1$,*

$$h^*(P^*_{\mathcal{L}};z) = (1+z+\cdots+z^{\mathcal{L}})\left(1+4z^{\mathcal{L}}+z^{2\mathcal{L}}+4\left(\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i\right)+2\sum_{i=1}^{\mathcal{L}-1} z^{2i}\right).$$

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