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Finitary affine oriented matroids*

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Abstract. We initiate the study of affine oriented matroids (AOMs) on arbitrary ground sets, extending classical structural features of Oriented Matroids and a natural embedding into the framework of Complexes of Oriented Matroids. The restriction to the finitary case (FAOMs) allows us to study tope graphs and covector posets, as well as to view FAOMs as oriented finitary semimatroids. We show shellability of FAOMs and single out the FAOMs that are affinely homeomorphic to $\mathbb{R}^n$. Finally, we study group actions on AOMs, whose quotients in the case of FAOMs are a stepping stone towards a general theory of affine and toric pseudoarrangements. Our results include applications of the multiplicity Tutte polynomial of group actions of semimatroids, generalizing properties of arithmetic Tutte polynomials of toric arrangements.

Keywords: Oriented matroids, group actions, cell complexes, poset topology, arrangements of submanifolds, arithmetic matroids.

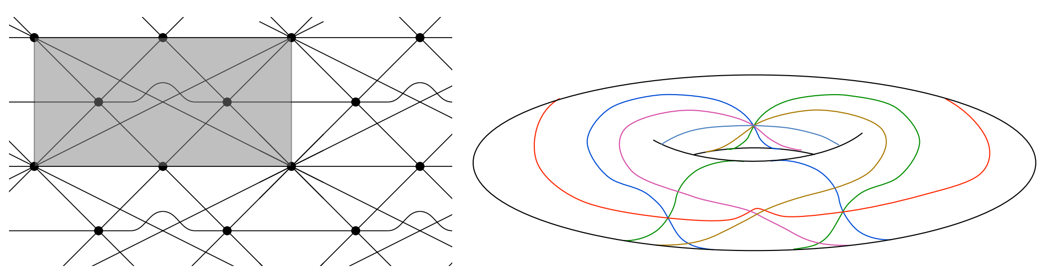

Figure 1: On the left-hand side: a non-stretchable line arrangement with an action of $\mathbb{Z}^2$ defined by letting a lattice basis act as translations by the two sides of the shaded rectangle. Any orientation of it gives rise to a FAOM. The (categorical) quotient of the poset of covectors is the face category of a pseudoarrangement in the 2-dimensional torus (picture on the right-hand side), whose cells are counted by the (arithmetic) Tutte polynomial of the group action on the underlying semimatroid, see Example 4.12.

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1 Introduction

We establish a natural generalization of finite affine oriented matroids to arbitrary ground sets and derive several results about their axiomatics, topology and geometry. Our motivation is twofold: on the one hand we pursue the structural theory of oriented matroids and arithmetic matroids, on the other hand we aim at applications to linear and toric arrangements, as well as to general manifold arrangements (see Remark 1.1).

1.1 Two motivating examples

1.1.1 Arrangements in Euclidean space

Let $\mathscr{A} := \{H_e\}_{e \in E}$ be an *arrangement of hyperplanes*, i.e., a family of codimension 1 affine subspaces of the Euclidean space $\mathbb{R}^d$. We call such an arrangement "oriented" if for every $e \in E$ we are given a labeling by H_e^+ and H_e^- of the two connected components of $\mathbb{R}^d \backslash H_e$. Now, for every $x \in \mathbb{R}^d$ we obtain a *sign vector* $\Sigma_x \in \{+, 0, -\}^E$ as follows.

$$\Sigma_x(e) := \begin{cases} + & \text{if } x \in H_e^+ \\ 0 & \text{if } x \in H_e \\ - & \text{if } x \in H_e^- \end{cases}$$

Let then $\mathscr{L}(\mathscr{A}) := \{\Sigma_x \mid x \in \mathbb{R}^d\}$.

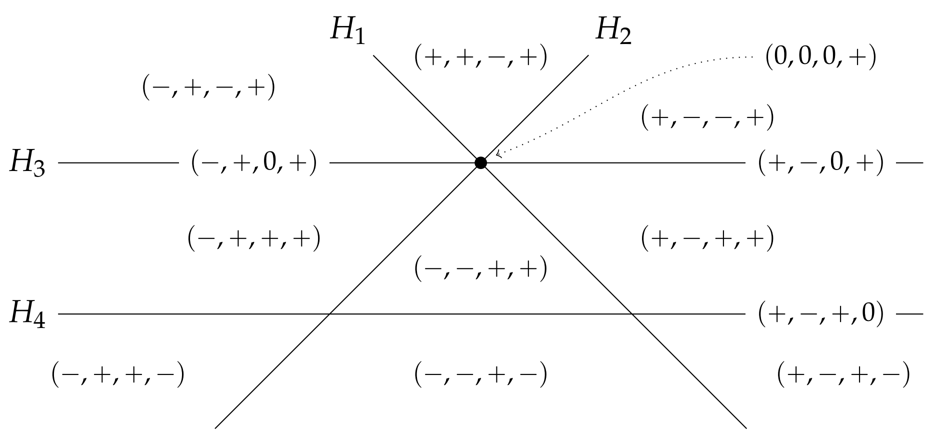

Figure 2: An affine arrangement of hyperplanes in $\mathbb{R}^2$ with some cells labeled by the respective sign vector.

The covector axioms of *oriented matroids* (OMs) abstract some properties of $\mathscr{L}(\mathscr{A})$ in the case where $\mathscr{A}$ is finite and $\cap\mathscr{A} \neq \emptyset$. While not all oriented matroids arise from an arrangement of hyperplanes, the "Topological representation theorem" of Folkman and

Lawrence asserts that the set of covectors of any oriented matroid is the set of sign vectors determined by an arrangement of oriented pseudospheres in the sphere (obtained as the boundary of the order complex of the covector poset, see [5, Chapter 5]).

If $\mathscr{A}$ is finite, but $\cap\mathscr{A}$ is not necessarily non-empty, then $\mathscr{L}(\mathscr{A})$ is the set of covectors of a *finite affine oriented matroid*. Finite affine oriented matroids can be defined either intrinsically or as subsets of covector sets of oriented matroids, see [4, 13]. The latter point of view allows us to interpret every finite affine oriented matroid as an arrangement of pseudoplanes in Euclidean space, via the order complex of its covector poset. It is open which arrangements arise from finite affine oriented matroids, see [12] and §5.1 .

More generally, if $\mathscr{A}$ is only assumed to be *finitary*, meaning that every $x \in \mathbb{R}^d$ has a neighborhood meeting finitely many H_i, then every element of $\mathscr{L}(\mathscr{A})$ indexes an open cell in $\mathbb{R}^d$. These open cells are the relative interiors of the *faces* of the polyhedral subdivision of $\mathbb{R}^d$ induced by $\mathscr{A}$. The faces are naturally ordered by inclusion, and this partial order corresponds to the (abstract) natural order among sign vectors.

- Our "Finitary Affine Oriented Matroids" axiomatize properties of the polyhedral stratification of Euclidean space induced by finitary hyperplane arrangements. Not every FAOM is realizable as $\mathscr{L}(\mathscr{A})$ for a finitary arrangement. Still, some familiar geometric and topological features generalize nicely to the non-realizable case.
- Our topological representation of FAOMs is a step towards the currently open problem of a topological characterization of affine pseudoarrangements (see §5.1).

1.1.2 Toric arrangements

Let now $\mathscr{A}$ be a finite family of level sets of characters of the compact torus $T = (S^1)^d$. Such *toric arrangements* have been in the focus of recent research motivated by work of De Concini, Procesi and Vergne on partition functions and splines. For further background see, e.g., [6, 7, Introduction]. A toric arrangement defines a polyhedral CW-structure $K(\mathscr{A})$ on the torus. The face category of this cell complex is central in the study of the topology of the associated arrangement in the complex torus [7, §2] and of arrangements in products of elliptic curves [9]. It can be regarded as the "toric" counterpart of the poset of faces of a linear arrangement[1].

Notice that, by passing to the universal cover of the torus, a toric arrangement can be seen as a quotient of an infinite, periodic arrangement of hyperplanes by the action of the deck transformation group.

The current impulse towards the combinatorial study of toric arrangements already led to substantial algebraic-combinatorial developments such as arithmetic Tutte polynomials and arithmetic matroids [6]. However, the only available results about the structure

[1]The broadening from face posets to face categories is necessary since the CW-complex $K(\mathscr{A})$ is not necessarily regular, see [8, Appendix].

of face categories to date are an explicit description in the case of toric Weyl arrangements by means of “labeled necklaces" [1].

- We obtain an abstract characterization of the face category of toric arrangements as the quotient of the poset of covectors of an infinite affine oriented matroid by a suitable class of group actions. This can be seen as an “oriented" version of the theory of group actions on semimatroids [10].
- We obtain a notion of pseudoarrangements in the torus whose topology and geometry is amenable to treatment via the existing combinatorial toolkit. We leave the relation to Pagaria’s orientable arithmetic matroids to future research, see §5.2.

1.2 Results and structure

- We present axiom systems for covectors of **Affine Oriented Matroids** (AOMs) over arbitrary ground sets (Section 2). These support canonical operations such as reorientation and taking minors. In particular our axiomatization, derived from results of Baum and Zhu [4], allows us to see AOMs as part of the theory of Complexes of Oriented Matroids (COMs) – a recent generalization of oriented matroids [3].
- To obtain a theory that more closely encapsulates geometric features of finitary affine hyperplane arrangements, in Section 3 we state axioms for **Finitary Affine Oriented Matroids** (FAOMs), i.e., AOMs with local cardinality restrictions. A main theoretical feature of this setting is that FAOMs are “orientations of finitary semimatroids”, i.e.: the zero sets of covectors of an FAOM constitute the geometric semilattice of flats of a finitary semimatroid (see Wachs and Walker [16], Ardila [2] and Kawahara [14], and [10] for the infinite setting). After a basic study of **tope graphs and covector posets** of FAOMs we focus on topological properties. We prove that order complexes of covector posets of FAOMs are shellable, describe their homeomorphism type, and introduce a geometric parallelism relation. This allows us to single out a class of FAOMs whose covector poset is affinely homeomorphic to Euclidean space.
- In Section 4 we take FAOMs as a stepping stone in order to extend the theory of arrangements of pseudospheres (and -planes) beyond the Euclidean setting, towards **pseudoarrangements in the torus**. See §1.1 below for some motivating context. A main ingredient are **group actions** on AOMs and, in particular, a class of group actions for which the quotient of the covector poset is homeomorphic to a torus. In this torus, the one-element contractions of the FAOM determine an arrangement of tamely embedded tori. Such “toric pseudoarrangements” are strictly more general than toric arrangements defined by level sets of characters, (which we call “stretchable" extrapolating the Euclidean terminology), see Figure 1. The faces of

the corresponding dissection of the torus are enumerated by the Tutte polynomial associated to the induced group action on the underlying semimatroid [10], generalizing enumerative results by Moci on arithmetic Tutte polynomials associated to toric arrangements [6]. Pagaria [15] proposes a notion of orientable arithmetic matroid, asking for an interpretation in terms of pseudoarrangements on the torus.

Remark 1.1. Ehrenborg and Readdy ask in [11] for a natural class of submanifold arrangements where an "arithmetic" Tutte polynomial can be meaningfully defined. Our first answer to this question is the class of arrangements in Euclidean space or in tori obtained from (possibly trivial) "sliding" group actions on FAOMs (Definition 4.2). Theorem 4.11 shows that the Tutte polynomial of the group action induced on the underlying semimatroid provides the desired topological enumeration (see Example 4.12), together with the structural properties studied in [10]. For "standard" toric arrangements we recover the arithmetic Tutte polynomial, see [6].

Infinite affine oriented matroids are at the crossroads of several topics in structural, algebraic and topological combinatorics. Thus AOMs offer new tools for existing **open problems**, and create some new ones in their own right, which we outline in Section 5.

2 Affine oriented matroids (AOM)

In the non-finite context it is essential to view AOMs with an intrinsic axiomatization instead of as halfspaces of oriented matroids as described in §1.1.1. The covector axiomatization of finite AOMs is due to Karlander [13], whose proof was corrected recently by Baum and Zhu [4]. We present a simplified equivalent form, which puts AOMs into the context of (complexes) of oriented matroids, (C)OMs [3]. Notions of minors and parallelism generalize to the infinite setting. For the purpose of the present section no assumption on the ground set E is made.

We introduce some notations, see e.g. [3, 4, 5]. A *system of sign vectors* is a subset $\mathscr{L} \subseteq \{+,-,0\}^E$. Every system of sign vectors carries a natural partial order, defined by $X \leqslant Y$ if and only if $X(e) \leqslant Y(e)$ for all $e \in E$ where $0 < +$, $0 < -$, $+$ and $-$ incomparable. The poset $(\mathscr{L}, \leqslant)$ will be denoted by $\mathscr{F}(\mathscr{L})$. The *support* of a sign vector X is $\underline{X} := \{e \in E \mid X(e) \neq 0\}$ and its *zero set* is $\mathrm{ze}(X) := \{e \in E \mid X(e) = 0\}$. The *separator* of two sign vectors X, Y is $S(X,Y) := \{e \in \underline{X} \cap \underline{Y} \mid X(e) \neq Y(e)\}$, and their *composition* is

$$X \circ Y(e) := \begin{cases} X(e) & \text{if } e \in \underline{X} \\ Y(e) & \text{otherwise.} \end{cases} \quad \text{for all } e \in E.$$

Let X, Y be sign vectors on E, let $e \in E$, and let $\mathscr{L}$ be a given system of sign vectors on E. Define $I_e(X,Y;\mathscr{L}) := \{Z \in \mathscr{L} \mid Z(e) = 0, \forall f \notin S(X,Y) : Z(f) = X(f) \circ Y(f)\}$ and set $I(X,Y;\mathscr{L}) := \bigcup_{e \in S(X,Y)} I_e(X,Y;\mathscr{L})$.

Moreover, let $X \oplus Y$ be the sign vector:

$$X \oplus Y(e) := \begin{cases} 0 & \text{if } e \in S(X,Y) \\ X \circ Y(e) & \text{otherwise.} \end{cases} \quad \text{for all } e \in E.$$

and set $\mathcal{P}(\mathscr{L}) := \{X \oplus (-Y) \mid X, Y \in \mathscr{L},\ I(X,-Y;\mathscr{L}) = I(-X,Y;\mathscr{L}) = \varnothing\}$.

Definition 2.1 (AOM)**.** A system of sign vectors $\mathscr{L}$ is an affine oriented matroid if and only if

(FS) $\mathscr{L} \circ (-\mathscr{L}) \subseteq \mathscr{L}$,

(SE) $X, Y \in \mathscr{L} \implies \forall e \in S(X,Y) : I_e(X,Y;\mathscr{L}) \neq \varnothing$,

(P) $\mathcal{P}(\mathscr{L}) \circ \mathscr{L} \subseteq \mathscr{L}$.

Remark 2.2. This axiomatization is stated in [8, Proposition 2.4], where it is proved that for finite E it is equivalent to Karlander's axiomatization as reviewed by Baum and Zhu, see [8, Definition 2.2]. In particular, by [4, Theorem 1.2], finite AOMs are exactly affine oriented matroids in the sense, e.g., of [5].

Following [3] a COM is a system of sign vectors on finite E satisfying (FS) and (SE) and an OM is a COM that has the all-zeroes vector $\mathbf{0} \in \mathscr{L}$. This yields:

Corollary 2.3 ([8, Corollary 2.5 and 2.6])**.** *Every OM is an AOM and every AOM is a COM.*

2.1 Reorientations and minors

The notions of reorientations, and minors are crucial in the study of OMs, finite AOMs, and COMs. We establish these operations for (possibly infinite) AOMs.

Let $(E, \mathscr{L})$ be any system of sign vectors. A *reorientation* of $\mathscr{L}$ is a set $\mathscr{L}^{(h)} := \{h \cdot X \mid X \in \mathscr{L}\}$ for a given $\tau \in \{+1,-1\}^E$, where $(\tau \cdot X)(e) := \tau(e)X(e)$. Moreover, for any $A \subseteq E$ define the *contraction* of A in $\mathscr{L}$ as $\mathscr{L}/A := \{X_{|E\setminus A} \mid X \in \mathscr{L},\ X(A) = \{0\}\}$, and the *deletion* of A from $\mathscr{L}$ as $\mathscr{L}\setminus A := \{X_{|E\setminus A} \mid X \in \mathscr{L}\}$. Moreover, we call *restriction to* A the set $\mathscr{L}[A] := \mathscr{L}\setminus(E\setminus A)$. A system of sign vectors $(E', \mathscr{L}')$ is a *minor* of another system of sign vectors $(E, \mathscr{L})$ if there are disjoint sets $A, B \subseteq E$ such that $(E', \mathscr{L}') = (E\setminus A\setminus B, \mathscr{L}\setminus A/B)$.

Theorem 2.4 ([8, §2.1])**.** *AOMs are closed under reorientation and under taking minors. In particular finite restrictions of AOMs are finite AOMs.*

2.2 Parallelism

We introduce a "geometric" parallelism relation that is inspired by the intuitive notion in the case of pseudoarrangements (e.g., the pseudoline arrangement on the l.-h. s. of Figure 1 has 5 parallelism classes).

Definition 2.5. Given two elements $e, f \in E$, we say that e and f are parallel, written $e \parallel f$, if there is no $X \in \mathscr{L}$ with $e, f \in \mathrm{ze}(X)$.

This should not be confused with the standard matroid-theoretical definition leading to simplicity, see [8, §2.2]. An AOM is *simple* if $\{X(e) \mid X \in \mathscr{L}\} = \{+, 0, -\}$ and $\{X(e)X(f) \mid X \in \mathscr{L}\} = \{+, 0, -\}$ for all $e \neq f \in E$.

Proposition 2.6 (See [8, Corollary 2.27])**.** *In every simple AOM with ground set E, the reflexive closure of parallelism is an equivalence relation on E. We call $\pi(e)$ the parallelism class of $e \in E$.*

On every parallelism class $\pi \subseteq E$ there is a total order $<_\pi$ and a reorientation of π, such that

$$\text{for all } x, y \in \pi,\ x <_\pi y \text{ if and only if } X(f) = Y(f) = 0 \Rightarrow X(e) = Y(e) = + \text{ for all } X, Y \in \mathscr{L}.$$

The (order-)isomorphism type of $<_\pi$ does not depend on the reorientation.

Write 1_π (0_π) for the unique maximal (minimal) element of $<_\pi$ if they exist. Assume after possibly reversing $<_\pi$, that if an 1_π exists, then 0_π exists. Proposition 2.6 allows us to define a partition of the ground set of an AOM, invariant under reorientation.

Definition 2.7. Let $(E, \mathscr{L})$ be an AOM and define the following partition of E.

$$\begin{aligned} E^{01} &:= \{e \in E \mid \text{both } 1_{\pi(e)} \text{ and } 0_{\pi(e)} \text{ exist}\}, \\ E^{0*} &:= \{e \in E \mid 0_{\pi(e)} \text{ exists}\} \backslash E^{01}, \\ E^{**} &:= E \backslash (E^{01} \cup E^{0*}). \end{aligned}$$

3 Finitary affine oriented matroids (FAOM)

We move to a more restrictive definition, towards a topological study of covector posets and a closer connection to semimatroid theory.

Definition 3.1 (FAOM)**.** A pair $(E, \mathscr{L})$ is a *Finitary* Affine Oriented Matroid if ((FS), (SE) and (P) hold and if, moreover,

(S) $X, Y \in \mathscr{L} \implies |S(X, Y)| < \infty$ (finite separators),

(Z) $X \in \mathscr{L} \implies |\mathrm{ze}(X)| < \infty$ (finite zero sets),

(I) $|\mathscr{F}(\mathscr{L})_{\leqslant X}| < \infty$ (finite intervals).

An analysis of topes and tope graphs of FAOMs can be carried out – see [8, §3.1], where in particular the following corollary is proved.

Corollary 3.2 ([8, Corollary 3.12 and 3.14])**.** *If $\mathscr{L}$ is the set of covectors of a FAOM, then it has countable cardinality, and the poset $\mathscr{F}(\mathscr{L})$ augmented by a minimum and maximum is graded of finite length.*

3.1 Topology of covector posets

We study the topology of the order complexes of posets of covectors of FAOMs. This is an extension of the known results about OMs and finite AOMs.

Lemma 3.3 ([8, Lemma 3.18])**.** *Let $\mathscr{L}$ be the set of covectors of an FAOM. Let ω be a maximal chain in $\mathscr{F}(\mathscr{L})$, let $X \in \omega$ and write $\omega' := \omega\setminus\{X\}$. Let $\mathscr{Y}$ be the set of all $Y \in \mathscr{L}$ such that $\omega' \cup \{Y\}$ is a chain in $\mathscr{F}(\mathscr{L})$. Then*

(1) $|\mathscr{Y}| \leqslant 2$, and

(2) the boundary of $\|\mathscr{F}(\mathscr{L})\|$ is generated by all chains of the form ω' with $|\mathscr{Y}| = 1$.

Theorem 3.4 ([8, Theorem 3.19])**.** *Let $\mathscr{L}$ be the set of covectors of an FAOM. Then $\|\mathscr{F}(\mathscr{L})\|$ is a shellable, contractible PL d-manifold whose boundary is described in Lemma 3.3.(2). Moreover,*

(1) If $\mathscr{L}$ is finite, then $\|\mathscr{F}(\mathscr{L})\|$ is a PL-ball.

(2) If $\|\mathscr{F}(\mathscr{L})\|$ has no boundary, then it is PL-homeomorphic to $\mathbb{R}^{\operatorname{rk}\mathscr{L}}$.

If $\mathscr{L}$ is the set of covectors of an oriented matroid (cf. Corollary 2.3), the classical associated PL-sphere is the boundary of the PL ball from part (1) of the theorem.

3.2 The underlying semimatroid

We show that zero sets of covectors of an FAOM define a semimatroid on the same ground set. Given a system of sign vectors $(E, \mathscr{L})$ define

$$L(\mathscr{L}) := \{\mathrm{ze}(X) \mid X \in \mathscr{L}\} \qquad \mathcal{K}(\mathscr{L}) := \bigcup_{A\in L(\mathscr{L})} 2^A.$$

Elements of $L(\mathscr{L})$ are called *flats*, elements of $\mathcal{K}(\mathscr{L})$ "central sets" of $\mathscr{L}$. For $A, B \in L(\mathscr{L})$, let $A \leqslant B :\Leftrightarrow A \subseteq B$ and define $F(\mathscr{L}) := (L(\mathscr{L}), \leqslant)$.

Theorem 3.5 ([8, Theorem 3.22])**.** *Let $(E, \mathscr{L})$ be an AOM satisfying (Z). Then $F(\mathscr{L}) \subseteq 2^E$ is the geometric semilattice of flats of a finitary semimatroid $\mathscr{S}(\mathscr{L})$, unique up to isomorphism.*

Note that $X \leqslant Y$ implies $\mathrm{ze}(X) \supseteq \mathrm{ze}(Y)$, where $\leqslant$ is the partial order of $\mathscr{F}$. Hence, taking zero sets induces an order reversing poset map $\mathrm{ze}(\cdot) : \mathscr{F}(\mathscr{L}) \to F(\mathscr{L})$ that can be shown to be rank-preserving. Thus, for any FAOM $\mathscr{L}$ we write rk for both the rank function of $\mathscr{F}(\mathscr{L})$ and for the rank function of its underlying semimatroid. We write $\mathrm{rk}(\mathscr{L}) = \mathrm{rk}(\mathscr{S}(\mathscr{L}))$ for the rank of either (i.e., the length of $\mathscr{F}(\mathscr{L})$ and $F(\mathscr{L})$).

4 Group actions

If a group G acts on a set E by permutations, for every $g \in G$ and $e \in E$ we write $g(e)$ for the image of e under the action of g. Moreover, for every $X \in \{+,-,0\}^E$ we define a sign vector $g.X$ by setting $g.X(e) := X(g^{-1}(e))$ for all $e \in E$. This extends the action of G on E to an action on the set of sign vectors. For $\mathscr{X} \subseteq \{+,-,0\}^E$ write $g.\mathscr{X} := \{g.X \mid X \in \mathscr{X}\}$.

Remark 4.1. If $\mathscr{L}$ is an oriented matroid, then requiring $g.\mathscr{L} = \mathscr{L}$ for all $g \in G$ amounts to saying that G acts on $\mathscr{L}$ by strong maps, see [5, Proposition 7.7.1]

Definition 4.2. A group G acts on an AOM $(E, \mathscr{L})$ if G acts by permutations on E and $g.\mathscr{L} = \mathscr{L}$ for all $g \in G$. An action of G on $\mathscr{L}$ will be denoted by $\alpha : G \circlearrowright \mathscr{L}$. The action of G is called *sliding* if $g(e) \in \pi(e)$ for all $e \in E$.

A group action on an AOM $\mathscr{L}$ induces a natural group action on reorientations and minors of $\mathscr{L}$. If the given action is sliding, then so are the induced ones.

Lemma 4.3 ([8, Lemma 5.5])**.** *Any group action on an FAOM $(E, \mathscr{L})$ induces an action on the underlying semimatroid $\mathscr{S}(\mathscr{L})$. If the action is sliding, the action on $\mathscr{S}(\mathscr{L})$ is translative.*

Every action $\alpha : G \circlearrowright \mathscr{L}$ induces an action $\alpha : G \circlearrowright \mathscr{F}(\mathscr{L})$ by poset automorphisms. We view posets as acyclic categories to define the following:

Definition 4.4. Let $\alpha : G \circlearrowright \mathscr{L}$ be a group action on an AOM. Let

$$q_\alpha : \mathscr{F}(\mathscr{L}) \to \mathscr{F}(\mathscr{L}) /\!\!/ G$$

be the quotient functor in the category of acyclic categories (see [8, §A.1.4]). [2]

[2] Since, by Corollary 3.2, $\mathscr{F}(\mathscr{L})$ is ranked, the category $\mathscr{F}(\mathscr{L}) /\!\!/ G$ can be described explicitly via [9, Lemma A.18]. It has object set $\mathrm{Ob}(\mathscr{F}(\mathscr{L}) /\!\!/ G) = \mathrm{Ob}(\mathscr{F}(\mathscr{L}))/G$, the set of orbits of objects. The morphisms of $\mathscr{F}(\mathscr{L}) /\!\!/ G$ are orbits of morphisms of $\mathscr{F}(\mathscr{L})$: the orbit of $\phi : X \to Y$ is a morphism $G\phi : GX \to GY$ and composition between orbits $G\phi$ and $G\psi$ is defined as the orbit of the composition of ϕ and ψ, if it exists.

4.1 Topological aspects

We consider the topology of quotients of covector posets of AOMs under a group action.

Definition 4.5. If a group G acts on an FAOM $(E, \mathscr{L})$, let

$$Q_\alpha : \|\mathscr{F}(\mathscr{L})\| \to \|\mathscr{F}(\mathscr{L})\|/G$$

denote the topological quotient map.

Definition 4.6. Call an action $\alpha : G \circlearrowright \mathscr{L}$ *free* if the induced action on $\mathscr{L}$ is free.

If α is free, then Q_α is a (topological) covering map. Moreover, the following holds.

Theorem 4.7 ([8, Theorem 5.16]). *Let $(E, \mathscr{L})$ be a nonempty FAOM with a distinguished basis $B \subseteq E^{*,*}$, and let a free abelian group $G \cong \mathbb{Z}^{|B|}$ act freely on $(E, \mathscr{L})$. If the action is sliding, then $\|\mathscr{F}(\mathscr{L}) /\!\!/ G\|$ is homeomorphic to the $|B|$-torus $(S^1)^{|B|}$ and, for every $A \in \mathcal{K}(\mathscr{L})$, $Q_\alpha(\|\mathscr{F}(\mathscr{L}/A)\|)$ is homeomorphic to the $(|B| - \mathrm{rk}(A))$-torus.*

4.2 Toric pseudoarrangements

Throughout this section let $\alpha : G \circlearrowright \mathscr{L}$ be a free and sliding action of a finitely generated free abelian group G on an FAOM $\mathscr{L}$, and suppose that $\mathscr{S}(\mathscr{L})$ has a basis $B \in E^{*,*}$.

Definition 4.8. Let $\mathscr{H}_e$ be the subcomplex of of $\|\mathscr{F}(\mathscr{L})\|$ given by all covectors whose zero set contains e (notice that $\mathscr{H}_e$ is in fact a copy of $\|\mathscr{F}(\mathscr{L}/e)\|$).

Moreover, let $T := \|\mathscr{F}(\mathscr{L})\|/G$ and, for every $a \in E/G$, define

$$\mathscr{A}_\alpha := \{\mathscr{T}_a \mid a \in E/G\} \quad \text{with} \quad \mathscr{T}_a := Q_\alpha(\mathscr{H}_e).$$

where e is a representative of a (this is well-defined by [8, Lemma 5.8]). We call the arrangement $\mathscr{A}_\alpha$ a *toric pseudoarrangement*, and *proper* if $\mathscr{S}(\mathscr{L})$ has no loops.

Lemma 4.9 ([8, Lemma 5.19]). *Every toric pseudoarrangement $\mathscr{A}_\alpha$ as in Definition 4.8 defines a CW-complex structure $K(\mathscr{A}_\alpha)$ on T, with one cell for every object in $q_\alpha(\mathscr{F}(\mathscr{L}))$.*

If $\mathscr{A}_\alpha$ is proper, the union $\cup\mathscr{A}_\alpha$ is the $(d-1)$-skeleton of $K(\mathscr{A})$, here $d = \dim T = \mathrm{rk}(\mathscr{L})$. In particular, the complement of $\cup\mathscr{A}_\alpha$ in T is a union of open d-cells.

Lemma 4.9 allows us to apply Zaslavsky's theory of topological dissections together with the theory of group actions on semimatroids, in order to enumerate the open cells constituting the complement of a toric pseudoarrangement $\mathscr{A}_\alpha$. The stepping stone is determining the poset of connected components of intersections of $\mathscr{A}_\alpha$.

Proposition 4.10 ([8, Proposition 5.20]). *The poset of connected components of intersections of $\mathscr{A}_\alpha$ is isomorphic to the quotient poset $F(\mathscr{L})/G$. Moreover, every intersection is topologically a torus $(S^1)^{d-r}$ where r is the rank of the corresponding element in $F(\mathscr{L})/G$.*

Theorem 4.11 ([8, Theorem 5.21]). *Let $\underline{\alpha}$ denote the induced G-semimatroid $\underline{\alpha} : G \circlearrowright \mathscr{S}(\mathscr{L})$, and let $T_{\underline{\alpha}}(x,y)$ be the associated Tutte polynomial (see [8, §A.2.3] or [10]). Then $T_{\underline{\alpha}}(x,y)$ computes the number of connected components of the arrangement's complement:*

$$|\pi_0\left(T \setminus \cup \mathscr{A}_\alpha\right)| = T_{\underline{\alpha}}(1,0).$$

Example 4.12. The multiplicity Tutte polynomial of the arrangement of Figure 1 is

$$T_{\underline{\alpha}}(x,y) = 7x^2 - 14x + 24 + 5y + 2y^2 + y^3$$

and in fact the associated toric pseudoarrangement has $T_\alpha(1,0) = 17$ regions. Note that in this case the multiplicity function is arithmetic, but this is not true in general – not even when restricting to (periodic) pseudoline arrangements (see e.g., [10, Figure 11]).

5 Open questions

5.1 Pseudoarrangements in Euclidean space

No topological characterization of arrangements of pseudohyperplanes that appear as realizations of a (finite) AOM is known [12]. Our work suggests to define *pseudoarrangement* as any collection $\mathscr{A} = \{H_e\}_{e \in E}$ of pseudohyperplanes of $\mathbb{R}^n$ that is an affine topoplane arrangement (in the sense of [12]) such that every $p \in \mathbb{R}^d$ has a neighborhood that intersects only finitely many H_i, chambers have finitely many walls, and parallelism ($H_i \| H_j$ iff $H_i \cap H_j = \emptyset$) is an equivalence relation; see [8, §6.1]). An "oriented" pseudoarrangement $\mathscr{A}$ gives rise to a set $\mathscr{L}(\mathscr{A})$ of sign vectors on E as in §1.1.1.

(Q1) **Conjecture:** For every oriented pseudoarrangement $\mathscr{A}$, $\mathscr{L}(\mathscr{A})$ is the set of covectors of a unique simple FAOM. Conversely, every simple FAOM arises this way.

5.2 Toric oriented matroids and pseudoarrangements

Section 4 suggests the categories $q_\alpha(\mathscr{F}(\mathscr{L}))$ associated to a free and sliding action on an FAOM $\mathscr{L}$ as the counterpart of covector posets for toric arrangements, see Section 1.1.2.

(Q2) Find an intrinsic axiomatic description of the class of acyclic categories that can be obtained as $q_\alpha(\mathscr{F}(\mathscr{L}))$ for a free and sliding action on an FAOM $\mathscr{L}$. This framework should include Aguiar and Petersen's posets of labeled necklaces [1].

Pagaria [15] proposed an algebraic notion of *orientable arithmetic matroid* and asked whether it can be interpreted in terms of pseudoarrangements in the torus. Every toric pseudoarrangement in the sense of Definition 4.2 has an associated matroid with multiplicity.

(Q3) Does every orientable arithmetic matroid as defined by Pagaria [15] arise from a toric pseudoarrangement in the sense of §4.2?

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