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# The saturation problem for refined Littlewood–Richardson coefficients

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**Abstract.** Given partitions $\lambda, \mu, \nu$ with at most $n$ nonzero parts and a permutation $w \in S_n$, the refined Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}(w)$ is the multiplicity of the irreducible $GL_n\mathbb{C}$ module $V(\nu)$ in the so-called Kostant-Kumar submodule $K(\lambda, w, \mu)$ of the tensor product $V(\lambda) \otimes V(\mu)$. We derive a hive model for these coefficients and prove that the saturation property holds if $w$ is 312-avoiding, 231-avoiding or a commuting product of such elements. This generalizes the classical Knutson-Tao saturation theorem.

**Keywords:** hives, saturation, refined Littlewood–Richardson coefficients

## 1 Introduction

The Schur polynomials $s_\lambda(\mathbf{x})$ form a basis of the ring of symmetric polynomials $\mathbb{C}[\mathbf{x}]^{S_n}$ in the $n$ variables $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ as $\lambda$ varies over the set $\mathcal{P}[n]$ of partitions with at most $n$ parts. The Littlewood–Richardson coefficients are the structure constants of this basis:

$$s_\lambda(\mathbf{x}) s_\mu(\mathbf{x}) = \sum_{\nu \in \mathcal{P}[n]} c^{\nu}_{\lambda\mu}\, s_\nu(\mathbf{x})$$

Arguably among the most celebrated numbers in all of algebraic combinatorics, the $c^{\nu}_{\lambda\mu}$ can be explicitly computed by the Littlewood–Richardson rule (and its numerous equivalent formulations). They have been generalized in many directions over the years and in this article, we undertake a closer study of one such generalization.

To define our main objects of study, we recall the Demazure operators $\mathrm{T}_i : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ given by:

$$(\mathrm{T}_i f)(\mathbf{x}) = \frac{x_i f(x_1, x_2, \cdots, x_n) - x_{i+1} f(x_1, \cdots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots, x_n)}{x_i - x_{i+1}}$$

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for $1 \leq i \leq n-1$. For $w \in S_n$, let $\mathrm{T}_w = \mathrm{T}_{i_1}\,\mathrm{T}_{i_2}\cdots\mathrm{T}_{i_k}$ where $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ is any reduced expression for $w$ as a product of simple transpositions $s_i = (i, i+1)$. The $\mathrm{T}_i$ satisfy the braid relations and $\mathrm{T}_w$ is independent of the chosen decomposition. Further if $w_0$ denotes the longest element of $S_n$, then $\mathrm{T}_{w_0}$ is given by

$$\mathrm{T}_{w_0}(f) = \frac{\sum_{w\in S_n}\operatorname{sgn}(w)\, w(\mathbf{x}^\rho f)}{\prod_{1\leq i<j\leq n}(x_i - x_j)} \tag{1.1}$$

where $\rho = (n-1, n-2, \cdots, 1, 0)$ is the staircase partition. The map $\mathrm{T}_{w_0} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]^{S_n}$ is $\mathbb{C}[\mathbf{x}]^{S_n}$-linear and surjective, with $\mathrm{T}_{w_0}(\mathbf{x}^\mu) = s_\mu(\mathbf{x})$ for $\mu \in \mathcal{P}[n]$. Here we use the standard notation $\mathbf{x}^\alpha = \prod_i x_i^{\alpha_i}$ for an $n$-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers.

For $w \in S_n$ and $\mu \in \mathcal{P}[n]$, the Demazure character (or key polynomial) is $\chi_{w,\mu}(\mathbf{x}) := \mathrm{T}_w(\mathbf{x}^\mu)$. The Demazure characters form a basis of $\mathbb{C}[\mathbf{x}]$ as $w, \mu$ vary.

**Definition 1.1.** Given $w \in S_n$ and $\lambda, \mu, \nu \in \mathcal{P}[n]$, the *$w$-refined Littlewood–Richardson coefficient* $c^\nu_{\lambda\mu}(w)$ is the coefficient of $s_\nu(\mathbf{x})$ in the Schur basis expansion

$$\mathrm{T}_{w_0}(\mathbf{x}^\lambda\, \chi_{w,\mu}(\mathbf{x})) = \sum_{\nu\in\mathcal{P}[n]} c^\nu_{\lambda\mu}(w) s_\nu(\mathbf{x}). \tag{1.2}$$

Its key properties are summarized in the following proposition.

**Proposition 1.2.** *(a)* $c^\nu_{\lambda\mu}(w_0) = c^\nu_{\lambda\mu}$ *(b)* $c^\nu_{\lambda\mu}(1) = \delta_{\lambda+\mu,\nu}$ *(c)* $c^\nu_{\lambda\mu}(w) \in \mathbb{Z}_+$ *(d)* $c^\nu_{\lambda\mu}(w) \leq c^\nu_{\lambda\mu}(w')$ *if $w \leq w'$ in the Bruhat order on $S_n$* *(e)* $c^\nu_{\lambda\mu}(w) = c^\nu_{\mu\lambda}(w^{-1})$ *(f)* $c^\nu_{\lambda\mu}(w) = c^\nu_{\lambda\mu}(w')$ *if $W_\lambda w W_\mu = W_\lambda w' W_\mu$ where $W_\lambda, W_\mu$ are the Young subgroups of $S_n$ which stabilize $\lambda, \mu$ respectively.*

Thus, for fixed $\lambda, \mu, \nu$, the map $w \mapsto c^\nu_{\lambda\mu}(w)$ is an increasing function of posets $S_n \to \mathbb{Z}_+$. Figure 1 shows an example for $n = 4$, with the values $c^\nu_{\lambda\mu}(w)$ superimposed on the Bruhat graph of $S_4$.

While the first two parts of Proposition 1.2 follow directly from (1.2), the remaining four can be deduced from the underlying representation theory. We now proceed to describe this briefly. Let $V(\lambda)$ denote the finite-dimensional irreducible polynomial representation of $G = GL_n\mathbb{C}$ corresponding to the partition $\lambda \in \mathcal{P}[n]$. Given $\lambda, \mu \in \mathcal{P}[n]$ and $w \in S_n$, let $v_\lambda$ denote the highest weight vector of $V(\lambda)$ and $v'_{w\mu}$ a nonzero vector of weight $w\mu$ in $V(\mu)$. The cyclic $G$-submodule of the tensor product $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v'_{w\mu}$ is called a Kostant-Kumar module [11, 12] and is denoted $K(\lambda, w, \mu)$.

Its character was computed by Kumar [11] (in a more general context):

**Theorem 1.3.** $\operatorname{char} K(\lambda, w, \mu) = \mathrm{T}_{w_0}(\mathbf{x}^\lambda\, \chi_{w,\mu}(\mathbf{x}))$.

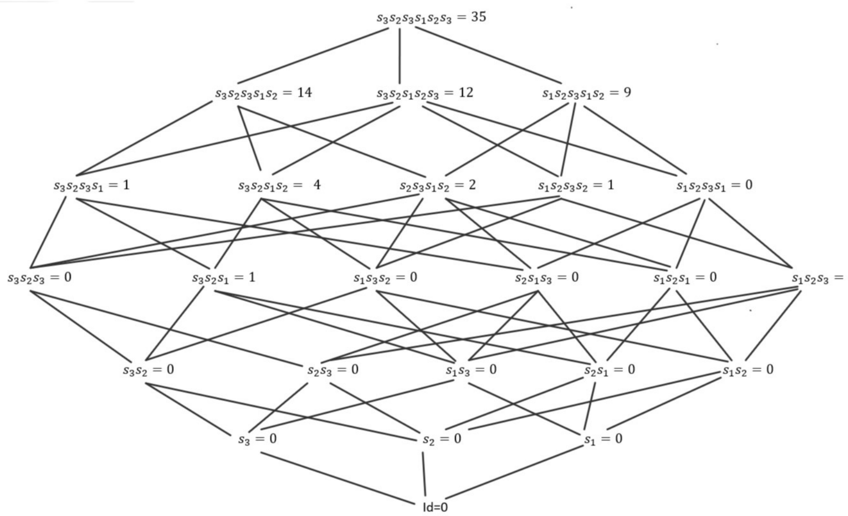


**Figure 1:** Values of $c^{\nu}_{\lambda\mu}(w)$ (superimposed on the Bruhat graph of $S_4$) for $n = 4$, $\lambda = (13, 7, 4)$, $\mu = (13, 7, 2)$, $\nu = (21, 12, 9, 4)$.

Thus $c^{\nu}_{\lambda\mu}(w)$ is the multiplicity of $V(\nu)$ in $K(\lambda, w, \mu)$. This interpretation establishes properties (c)-(f) of Proposition 1.2. A bijective proof of $(e)$ using the hive model will be sketched in §2.3. The $c^{\nu}_{\lambda\mu}(w)$ are also related to the *combinatorial excellent filtrations* of Demazure modules and have descriptions in terms of Lakshmibai-Seshadri paths [14] or crystals [4, 5]. We will return to this point of view in section 2.

In this article, we are interested in the *saturation problem* for the $w$-refined Littlewood–Richardson coefficients. A permutation $w \in S_n$ is said to have the *saturation property* if the following holds for all $\lambda, \mu, \nu \in \mathcal{P}[n]$:

$$c^{k\nu}_{k\lambda, k\mu}(w) > 0 \text{ for some } k \geq 1 \text{ implies } c^{\nu}_{\lambda\mu}(w) > 0 \tag{1.3}$$

Both $w = 1$ and $w = w_0$ have the saturation property. The former is a trivial consequence of Proposition 1.2, while the latter is exactly Klyachko's classical saturation conjecture for the $c^{\nu}_{\lambda\mu}$, established by Knutson-Tao [8] using the honeycomb model. Our main result is the following sufficiency condition for saturation.

**Theorem 1.4.** *(1) Let $w \in S_n$ be either* 312*-avoiding or* 231*-avoiding. Then $w$ has the saturation property.* *(2) More generally, let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p} \subseteq S_n$ be a Young subgroup and $w = w_1 w_2 \cdots w_p \in H$ such that each $w_i \in S_{n_i}$ is either* 312*- or* 231*-avoiding. Then $w$ has the saturation property.*

**Remarks.** (1) Permutations satisfying the conditions of Theorem 1.4 appear in work of Postnikov-Stanley, where explicit formulas for the degree polynomials of the corresponding Schubert varieties were established [18, Theorem 13.4, Corollary 13.5, Remark 15.5]. (2) Since $w_0$ satisfies these conditions, our theorem extends the Knutson-Tao saturation theorem. As in their case, the reverse implication in (1.3) is easy. (3) For $n = 1, 2, 3$, all permutations in $S_n$ are of the form of the theorem. For $n = 4$, our theorem establishes saturation for all *except* the following four permutations of $S_4$ (written in one-line notation): $2413, 3142, 3412, 4231$ (see §4.3).

The rest of the sections are devoted to deriving a hive description of the $c^{\nu}_{\lambda\mu}(w)$ and proving Theorem 1.4. The arguments are sketched to the extent possible subject to the overall space restrictions. The detailed proofs are part of a forthcoming publication [13].

## 2 A hive model for $c^{\nu}_{\lambda\mu}(w)$

Putting together recent results of Fujita [2] and those of [5, 14], one obtains a combinatorial model for $c^{\nu}_{\lambda\mu}(w)$ as a certain subset of integer points in the Gelfand-Tsetlin (GT) polytope. We describe this first, followed by our more succinct reformulation in terms of hives. A word on notation: if $P$ is a (not necessarily bounded) polytope, or a face thereof, then $P_{\mathbb{Z}}$ will denote the set of integer points in $P$.

Given a partition $\mu \in \mathcal{P}[n]$, let $\mathrm{Tab}(\mu)$ denote the set of semistandard Young tableaux of shape $\mu$ with entries in $1, 2, \cdots, n$. To each $T \in \mathrm{Tab}(\mu)$ we associate its reverse row word $b_T$ obtained by reading the entries of $T$ from right to left and top to bottom (in English notation), for example, $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$ and $b_T = 31132$. The crystal raising and lowering operators $e_i, f_i$ $(1 \leq i < n)$ act on the set $\mathrm{Tab}(\mu)$, and more generally on words in the alphabet $\{1, 2, \cdots, n\}$ (we refer to [16, Chapter 5] for all undefined terms). Let $T^{\circ}_{\mu}$ denote the highest weight element of $\mathrm{Tab}(\mu)$, satisfying $e_i T^{\circ}_{\mu} = 0$ for all $i$, and let $b^{\circ}_{\mu}$ denote its reverse row word. The weight of a word $u$ in the alphabet $\{1, 2, \cdots, n\}$ is the tuple $(a_1, a_2, \cdots, a_n)$ where $a_i$ is the number of occurrences of $i$ in $u$.

Given $w \in S_n$, fix a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_k}$. The set $\mathrm{Dem}(\mu, w) := \{f^{m_1}_{i_1} f^{m_2}_{i_2} \cdots f^{m_k}_{i_k} T^{\circ}_{\mu} : m_j \geq 0\}$ is called a *Demazure crystal*. We now have:

**Theorem 2.1** ([4, 5, 14])**.** *$c^{\nu}_{\lambda\mu}(w)$ is the cardinality of the set*

$$\mathrm{Dem}^{\nu}_{\lambda}(\mu, w) := \{T \in \mathrm{Dem}(\mu, w) : b^{\circ}_{\lambda} * b_T \text{ is a dominant word of weight } \nu\}.$$

Here $*$ denotes concatenation, and a word $u$ is said to be *dominant* (or a ballot sequence) if every left subword of $u$ contains more occurrences of $i$ than $i+1$ for all $1 \leq i < n$. We note that we could replace $b^{\circ}_{\mu}$ with any other dominant word $b^{+}$ of weight $\mu$ (these are Knuth equivalent), define $\mathrm{Dem}(\mu, w) := \{f^{m_1}_{i_1} f^{m_2}_{i_2} \cdots f^{m_k}_{i_k} b^{+} : m_j \geq 0\}$, and the theorem still holds, appropriately modified. We will use this in §4.2.

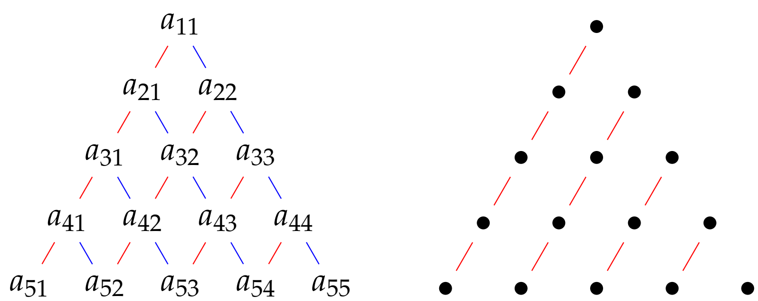


**Figure 2:** Gelfand-Tsetlin array for $n = 5$. The red edges $a_{ij} \longrightarrow a_{i-1,j}$ are labelled by $s_{i-j}$

## 2.1 Kogan faces of GT polytopes

A GT pattern of size $n$ is a triangular array $A = (a_{ij})_{n \geq i \geq j \geq 1}$ of real numbers (figure 2) satisfying the following ("North-East" and "South-East") inequalities for all $i > j$: $NE_{ij} = a_{ij} - a_{i-1,j} \geq 0$ and $SE_{ij} = a_{i-1,j} - a_{i,j+1} \geq 0$. For $\mu \in \mathcal{P}[n]$, the GT polytope $\mathrm{GT}(\mu)$ is the set of all GT patterns with $a_{ni} = \mu_i$ for $1 \leq i \leq n$. We have the standard bijection $A \mapsto \mathbf{\Gamma}(A)$ from $\mathrm{GT}_{\mathbb{Z}}(\mu)$ to $\mathrm{Tab}(\mu)$, with the tableau $\mathbf{\Gamma}(A)$ uniquely specified by the condition that for all $i \geq j$, the number of $\boxed{i}$ in row $j$ equals $NE_{ij}$ (with $a_{i-1,i} := 0$).

Fix a subset $F \subset \{(i,j) : n \geq i > j \geq 1\}$. Consider the face of $GT(\mu)$ obtained by setting $NE_{ij} = 0$ for $(i,j) \in F$ and leaving all other inequalities untouched. We call this the *Kogan face*[1] $\mathrm{K}(\mu, F)$. To each pair $i > j$, associate the simple transposition $s_{i-j} \in S_n$. We list the elements of $F$ in lexicographically increasing order: $(i,j)$ precedes $(i',j') \Leftrightarrow$ either $i < i'$, or $i = i'$ and $j < j'$. Denote the product of the corresponding $s_{i-j}$ in this order by $\sigma(F)$. If $\operatorname{len} \sigma(F) = |F|$, i.e., this word is reduced, we say that $F$ is *reduced* and set [2, Definition 5.1]:

$$\varpi(F) = w_0 \, \sigma(F) \, w_0$$

For $w \in S_n$, let $\mathrm{K}(\mu, w) := \cup \mathrm{K}(\mu, F)$, the union over reduced $F$ for which $\varpi(F) = w$. We can now state [2, Corollary 5.19]:

**Proposition 2.2.** *The bijection* $\mathbf{\Gamma} : \mathrm{GT}_{\mathbb{Z}}(\mu) \to \mathrm{Tab}(\mu)$ *restricts to a bijection* $\mathrm{K}_{\mathbb{Z}}(\mu, w_0 w) \xrightarrow{\sim} \mathrm{Dem}(\mu, w)$.

It was previously shown in [6] (for regular $\mu$) and [18] (for $w$ 312-avoiding) that $\mathrm{K}_{\mathbb{Z}}(\mu, w_0 w)$ and $\mathrm{Dem}(\mu, w)$ have the same character. This weaker statement is however inadequate for our present purposes.

[1] These are often called dual Kogan faces in the literature, with *Kogan faces* reserved for ones defined by South-East equalities. We will not be needing this other kind here.

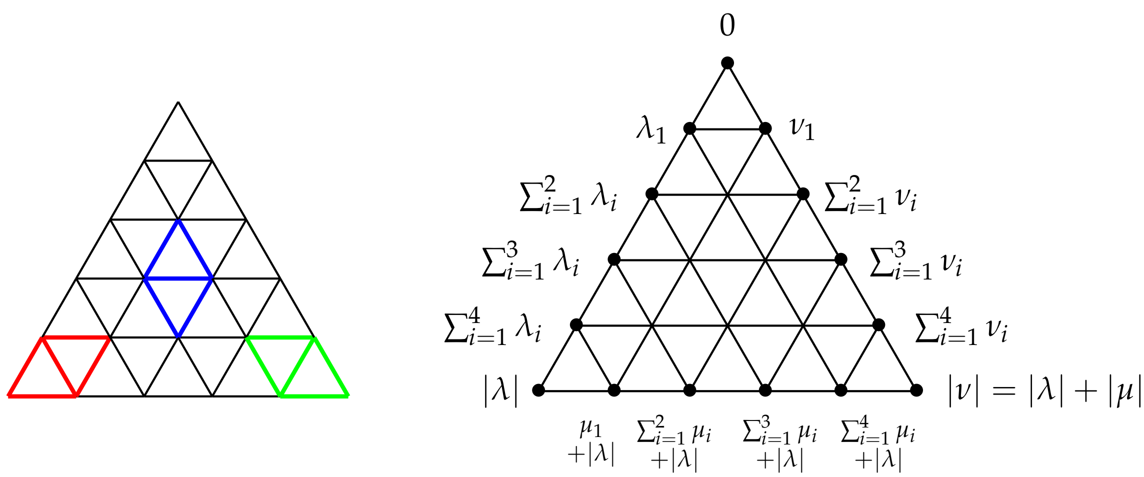


**Figure 3:** (a) The big hive triangle $\mathbf{\Delta}$ for $n = 5$, with the three kinds of rhombi marked. (b) The border labels of hives in $\mathrm{Hive}(\lambda, \mu, \nu)$ as functions of $\lambda, \mu, \nu$.

## 2.2 Hives

We begin with a quick overview. The *big hive triangle* $\mathbf{\Delta}$ is the array of Figure 3, with $(n+1)$ vertices on each boundary edge, and $(n-2)(n-1)/2$ interior vertices. Given $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$, let $|\lambda| = \sum_i \lambda_i$. Define the $(n+1)$-tuple of partial sums $\overline{\lambda} = (0, \lambda_1, \sum_{i=1}^{2} \lambda_i, \cdots, \sum_{i=1}^{n} \lambda_i)$ and the $(n-1)$-tuple of successive differences $\partial\lambda = (\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \cdots, \lambda_n - \lambda_{n-1})$, so that $\partial\overline{\lambda} = \lambda$.

For $(\lambda, \mu, \nu) \in (\mathbb{R}^n)^3$ with $|\lambda| + |\mu| = |\nu|$, the hive polytope $\mathrm{Hive}(\lambda, \mu, \nu)$ is the set of all labellings of the vertices of $\mathbf{\Delta}$ with real numbers such that: (i) the boundary labels are $\overline{\lambda}$ (left edge, top to bottom), $|\lambda| + \overline{\mu}$ (bottom edge, left to right) and $\overline{\nu}$ (right edge, top to bottom) (figure 3) (ii) content$(R) \geq 0$ for each rhombus $R$ in $\mathbf{\Delta}$, where content$(R)$ is the sum of the labels on the obtuse angled vertices of $R$ minus the sum of labels on its acute angled vertices. A *hive* is an element of $\mathrm{Hive}(\lambda, \mu, \nu)$ for some $\lambda, \mu, \nu$. We note that there are 3 types of rhombi in $\mathbf{\Delta}$ (figure 3), the NE slanted (in red), the SE slanted (in green) and the vertical diamonds (in blue).

We now fix $\lambda, \mu, \nu \in \mathcal{P}[n]$ with $|\lambda| + |\mu| = |\nu|$. Since each $h \in \mathrm{Hive}(\lambda, \mu, \nu)$ is an $\mathbb{R}$-labelled triangular array (of size $n+1$), its horizontal sections (marked in blue in figure 4) form a sequence of vectors $h_0, h_1, \cdots, h_n$ (listed from top to bottom), with $h_i \in \mathbb{R}^{i+1}$. Consider the ("row-wise successive differences") map $h \mapsto \boldsymbol{\partial} h$ where $\boldsymbol{\partial} h$ is the sequence of vectors $\partial h_1, \partial h_2, \cdots, \partial h_n$, thought of again as a triangular array (of size $n$ this time) (figure 4). One sees immediately that $\boldsymbol{\partial} h \in \mathrm{GT}(\mu)$ [1, Appendix], [19].

Note that each NE edge difference $NE_{ij}$ of $\boldsymbol{\partial} h$ (§2.1) equals the content of a corresponding NE-slanted rhombus $R_{ij}$ of $h$ (figure 5). Thus $NE_{ij} = 0$ (in $\boldsymbol{\partial} h$) if and only if content$(R_{ij}) = 0$ (in $h$). A rhombus with zero content is said to be *flat*.

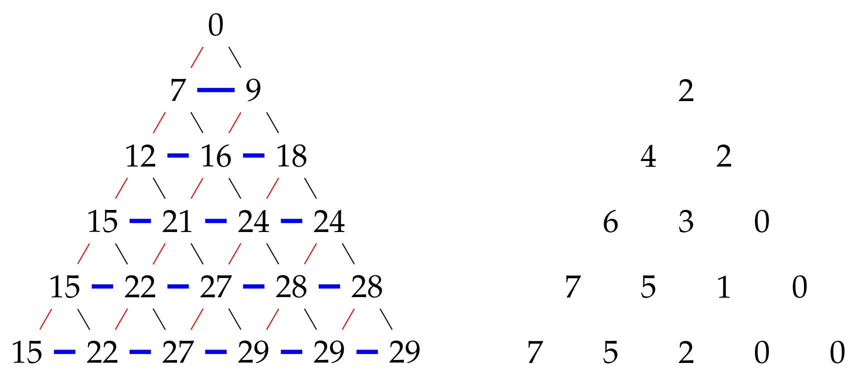


**Figure 4:** The hive on the left maps under $\boldsymbol{\partial}$ to the GT pattern on the right (example borrowed from [19])

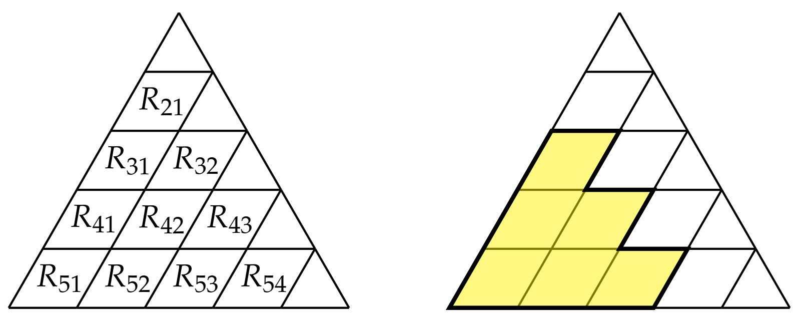


**Figure 5:** (a) Labelling of North-East slanted rhombi (shown for $n = 5$). (b) A typical configuration of rhombi in $F_w$.

**Proposition 2.3.** *(1)* $\boldsymbol{\partial} : \mathrm{Hive}(\lambda, \mu, \nu) \to \mathrm{GT}(\mu)$ *is an injective, linear map.*
*(2)* $\boldsymbol{\partial} h \in \mathrm{GT}_{\mathbb{Z}}(\mu) \Leftrightarrow h \in \mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$. *(3)* $\boldsymbol{\Gamma} \circ \boldsymbol{\partial}$ *is a bijection between* $\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$ *and* $\{T \in \mathrm{Tab}(\mu) : b^{\circ}_{\lambda} * b_T$ *is a dominant word of weight* $\nu\}$.

This proposition is easily verified [13]. Note that the last assertion implies that $|\,\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)| = c^{\nu}_{\lambda\mu}$ (and is a variation of proofs in [1], [17]).

Given $F \subset \{(i, j) : n \geq i > j \geq 1\}$, recall that $\mathrm{K}(\mu, F)$ is the face of $\mathrm{GT}(\mu)$ on which $NE_{ij}$ vanishes for all $(i, j) \in F$. The inverse image $\boldsymbol{\partial}^{-1}\mathrm{K}(\mu, F)$ is thus the face $\{h \in \mathrm{Hive}(\lambda, \mu, \nu) : R_{ij}$ is flat in $h$ for all $(i, j) \in F\}$. We denote this ("hive Kogan") face by $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F)$, and term it *reduced* if $F$ is. For $w \in S_n$, define $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, w) := \boldsymbol{\partial}^{-1}(\mathrm{K}(\mu, w))$. Putting together Theorem 2.1 and Propositions 2.2, 2.3, we obtain our hive description of the $c^{\nu}_{\lambda\mu}(w)$:

**Theorem 2.4.** $c^{\nu}_{\lambda\mu}(w) = |(\boldsymbol{\Gamma} \circ \boldsymbol{\partial})^{-1}(\mathrm{Dem}^{\nu}_{\lambda}(\mu, w))| = \#\,\mathrm{K}^{\mathrm{Hive}}_{\mathbb{Z}}(\lambda, \mu, \nu, w_0 w)$.

### 2.3 Right keys

The symmetry $c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda}$ was first studied via hives in [3]. We briefly touch upon another point-of-view stemming from Proposition 2.3, which leads to a bijective proof of the general symmetry property $c^{\nu}_{\lambda\mu}(w) = c^{\nu}_{\mu\lambda}(w^{-1})$. This uses Lascoux-Schutzenberger's notion of the *right key* of a tableau (or more generally of any word in the alphabet $\{1, 2, \cdots, n\}$ - the right key is invariant under Knuth moves) [15]. Right keys of tableaux correspond to certain permutations in $S_n$ [15, Definition 2.12, Proposition 5.2] (actually to minimal coset representatives of the stabilizer of the shape of the tableau).

By Proposition 2.3, an integral hive $h \in \mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$ determines a tableau $T = \boldsymbol{\Gamma} \circ \boldsymbol{\partial}(h)$ of shape $\mu$. Consider now the "North-Easterly" version $\boldsymbol{\partial}^{NE}$ of $\boldsymbol{\partial}$, which takes successive differences of labels along the $NE - SW$ direction (red edges of Figure 4) (see [19, Example 2.8] and [1, Appendix], whose hive-drawing conventions differ from ours and from each other!). This produces a GT pattern $\boldsymbol{\partial}^{NE}(h)$ of shape $\lambda$, which can be interpreted as a *contretableau* $T^{\dagger}$ of shape $\lambda$ [1]. It can be shown that if $w \in S_n$ corresponds to the right key of $T$ and $w^{\dagger}$ to the right key of $T^{\dagger}$ (or the unique element of $\mathrm{Tab}(\lambda)$ in the Knuth equivalence class of $T^{\dagger}$), then $w^{\dagger} = w^{-1}$ [13].

### 2.4 Hive Kogan faces for 312-avoiding permutations

Let $w \in S_n$ be 312-avoiding. Then, $w_0 w$ is 132-avoiding and there exists a unique reduced $F_w \subset \{(i, j) : n \geq i > j \geq 1\}$ such that $\varpi(F_w) = w_0 w$ (§2.1). Further, it has the following form $F_w = \{(i, j) : p \leq i \leq n, 1 \leq j \leq m_i\}$ for some $1 \leq p \leq n$, $1 \leq m_p \leq m_{p+1} \leq \cdots \leq m_n$ with $m_i < i$ for all $i$ [10, 18]. Pictorially, the union of the rhombi $R_{ij}$, $(i, j) \in F_w$ forms a left-and-bottom justified region in the big hive triangle $\boldsymbol{\Delta}$ (figure 5). Thus, for such $w$, $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, w_0 w)$ is just the single Kogan face $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$ on which the $R_{ij}$ are flat for all $(i, j) \in F_w$.

## 3 Increasable subsets for hives

Let $\lambda, \mu, \nu \in \mathbb{R}^n$ with $|\lambda| + |\mu| = |\nu|$ and let $h \in \mathrm{Hive}(\lambda, \mu, \nu)$. A subset $S$ of the interior vertices of $\boldsymbol{\Delta}$ is said to be *increasable* for $h$ if those vertex labels of $h$ can be simultaneously increased by some $\epsilon > 0$ to obtain another element of $\mathrm{Hive}(\lambda, \mu, \nu)$. Formally, let $I_S$ denote the indicator function of $S$ (1 on $S$ and 0 elsewhere); then $S$ is increasable if there is an $\epsilon > 0$ such that $h' = h + \epsilon I_S \in \mathrm{Hive}(\lambda, \mu, \nu)$. This notion is one of the central ideas of Knutson-Tao's proof of the saturation conjecture via hives [7, 1]:

**Proposition 3.1.** *(Knutson-Tao) Let $\lambda, \mu, \nu \in \mathbb{R}^n$ be regular (i.e., $\lambda_i \neq \lambda_j$ if $i \neq j$, and likewise for $\mu, \nu$) with $|\lambda| + |\mu| = |\nu|$. Let $h$ satisfy the following properties: (i) $h$ is a vertex of the hive polytope $\mathrm{Hive}(\lambda, \mu, \nu)$, (ii) $h$ has no increasable subsets. Then each interior label of $h$ is* *an integral linear combination of its boundary labels. In particular, if $\lambda, \mu, \nu \in \mathcal{P}[n]$, then $h \in \mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$.*

### 3.1 Increasable subsets for hives in 312-avoiding Kogan faces

Let $w$ be 312-avoiding and let $F_w$ be as in §2.4. Let $\lambda, \mu, \nu \in \mathbb{R}^n$ with $|\lambda| + |\mu| = |\nu|$. The following simple observation is a crucial step in extending the Knutson-Tao method to our problem.

**Lemma 3.2.** *Let $h \in \mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$ and let $S$ be an increasable subset for $h$, say $h' = h + \epsilon I_S \in \mathrm{Hive}(\lambda, \mu, \nu)$ for some $\epsilon > 0$. Then $h' \in \mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$.*

*Proof.* We will show that $S$ is disjoint from the set of vertices of the rhombi $R_{ij}$ for $(i, j) \in F_w$. This would imply that the $R_{ij}$ remain flat in $h'$, which is the desired conclusion. This is trivial if $F_w$ is empty. If $F_w$ is non-empty, then $(n, 1) \in F_w$. The rhombus $R_{n1}$ has three vertices on the boundary, and these cannot be in $S$. The fourth vertex is acute-angled, and if it belongs to $S$, then $\mathrm{content}(R_{n1}) < 0$ in $h'$, a contradiction. Moving on to the next rhombus $R_{n2}$ (if $(n, 2) \in F_w$), again three of its vertices cannot be in $S$ since they are either on the boundary or shared with $R_{n1}$. Neither can its fourth vertex, since it is acute-angled as before. Proceed in this fashion, left-to-right along the rows, from the bottom row to the top. □

**Remark 3.3.** The following converse holds too. If $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F)$ is a hive Kogan face for which the conclusion of Lemma 3.2 holds for all $\lambda, \mu, \nu \in \mathbb{R}^n$, then $F = F_w$ for some 312-avoiding permutation $w$.

## 4 Proof of the main theorem

With Lemma 3.2 in place, we can use Knutson-Tao's arguments to complete the proof of Theorem 1.4 for $w$ 312-avoiding.

Consider the set of all $\mathbb{R}$-labellings of vertices of the big hive triangle $\boldsymbol{\Delta}$ (with the boundary labels also allowed to vary) subject to (i) the inequalities: $\mathrm{content}(R) \geq 0$ for all rhombi $R$ in $\boldsymbol{\Delta}$, and (ii) the equalities: $\mathrm{content}(R_{ij}) = 0$ for all $(i, j) \in F_w$. This set forms a polyhedral cone, denoted $\mathrm{K}^{\mathrm{Hive}}(-, w_0 w)$. Given $h \in \mathrm{K}^{\mathrm{Hive}}(-, w_0 w)$, consider the projection $\pi : \mathrm{K}^{\mathrm{Hive}}(-, w_0 w) \to (\mathbb{R}^n)^3$ defined by $\pi(h) = (\lambda, \mu, \nu)$, where the boundary labels of $h$ are $\overline{\lambda}, |\lambda| + \overline{\mu}, \overline{\nu}$ as in Figure 3. The image of $\pi$ is a polyhedral cone in $\mathbb{R}^{3n}$ [20, lecture 1], which we denote by $\mathrm{Horn}(w_0 w)$ (adapting the notation of [7]). For $w = w_0$, $\mathrm{Horn}(1)$ is the cone of spectra of triples $(A, B, C)$ of $n \times n$ Hermitian matrices with $C = A + B$ [9].

We note that the saturation property (1.3) is equivalent to the statement that:

$$\mathrm{K}_{\mathbb{Z}}^{\mathrm{Hive}}(-, w_0 w) \cap \pi^{-1}(\lambda, \mu, \nu) \text{ is non-empty for all } \lambda, \mu, \nu \in \mathrm{Horn}_{\mathbb{Z}}(w_0 w). \tag{4.1}$$

This follows from Theorem 2.4 and the scaling property:
$\mathrm{K}^{\mathrm{Hive}}(p\lambda, p\mu, p\nu, w_0 w) = p\ \mathrm{K}^{\mathrm{Hive}}(\lambda,\mu,\nu,w_0 w)$ for all positive real numbers $p$.

## 4.1 The largest lift map

Following [7, 1], choose a functional $\zeta$ on the cone $\mathrm{K}^{\mathrm{Hive}}(\,-, w_0 w)$ which maps each hive $h$ to a generic positive linear combination of its vertex labels. Then, for each $\lambda,\mu,\nu \in \mathrm{Horn}(w_0 w)$, the maximum value of $\zeta$ on $\pi^{-1}(\lambda,\mu,\nu)$ is attained at a unique point; this point will be called its *largest lift*. The map $\ell : \mathrm{Horn}(w_0 w) \to \mathrm{K}^{\mathrm{Hive}}(\,-, w_0 w)$, $(\lambda,\mu,\nu) \mapsto$ largest lift of $(\lambda,\mu,\nu)$, is continuous and piecewise-linear.

It is also clear that $\ell(\lambda,\mu,\nu)$ is a vertex of $\mathrm{K}^{\mathrm{Hive}}(\lambda,\mu,\nu,w_0 w)$, thereby satisfying the first condition of Proposition 3.1. We claim that it also satisfies the second condition there, i.e., that $h = \ell(\lambda,\mu,\nu)$ has no increasable subsets. For if $S$ is an increasable subset, let $h' = h + \epsilon I_S \in \mathrm{Hive}(\lambda,\mu,\nu)$ for some $\epsilon > 0$. By Lemma 3.2, $h' \in \mathrm{K}^{\mathrm{Hive}}(\lambda,\mu,\nu,w_0 w)$. But $\zeta(h') > \zeta(h)$, violating maximality of $\zeta(h)$.

So Proposition 3.1 implies that for $\lambda,\mu,\nu$ regular, each label of $\ell(\lambda,\mu,\nu)$ is an integer linear combination of the $\lambda_i, \mu_i, \nu_i$, $1 \le i \le n$. As in [1, §4] and [7], by the continuity of $\ell$, it follows that each piece of $\ell$ is a linear function of $(\lambda,\mu,\nu) \in \mathbb{R}^{3n}$ with $\mathbb{Z}$-coefficients. As a corollary:

$$\ell(\mathrm{Horn}_{\mathbb{Z}}(w_0 w)) \subseteq \mathrm{K}^{\mathrm{Hive}}_{\mathbb{Z}}(\lambda,\mu,\nu,w_0 w)$$

This proves Theorem 1.4 for $w$ 312-avoiding.

## 4.2 Completing the proof

We now complete the proof of Theorem 1.4. If $w$ is 231-avoiding, then $w^{-1}$ is 312-avoiding. Proposition 1.2(e) finishes the argument in this case.

To handle the remaining $w$, note that only the $p = 2$ case needs to be established (in the notation of Theorem 1.4), with induction doing the rest. We sketch the contours of the argument. It is more natural to work with the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ here. Suppose $I^1, I^2 \subset \{1, 2, \cdots, n-1\}$ are such that $s_i$ and $s_j$ commute for all $i \in I^1, j \in I^2$. Let $I^0 = \{1, 2, \cdots, n-1\} - \bigcup_{r=1}^{2} I^r$. Let $W^r$ be the subgroup of $S_n$ generated by the $\{s_i : i \in I^r\}$ for $r = 1, 2$. Let $\mu$ be a dominant integral weight $\mu = \sum_{i=1}^{n-1} c_i \Lambda_i$ of $\mathfrak{sl}_n\mathbb{C}$, where $\Lambda_i$ are its fundamental weights. Define $\mu = \mu^0 + \mu^1 + \mu^2$ where $\mu^r = \sum_{i \in I^r} c_i \Lambda_i$ for $r = 0, 1, 2$. Let $b(\mu^r)$ denote the reverse row word of the highest weight tableau $T^{\circ}_{\mu^r}$ of shape $\mu^r$ for $r = 0, 1, 2$. The concatenation $\eta = b(\mu^0) * b(\mu^1) * b(\mu^2)$ is a dominant word of weight $\mu$ (§2).

Given $w^r \in W^r$ for $r = 1, 2$ and $w = w^1 w^2$, consider the Demazure crystal (cf. remarks following Theorem 2.1) $\mathrm{Dem}(\mu, w) := \{f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} \eta : m_j \ge 0\}$. where $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced word of $w$ obtained by concatenating reduced words of $w^1$ and $w^2$. It follows from the hypotheses and the properties of the crystal operators [16] that:

$$\mathrm{Dem}(\mu, w) = b(\mu^0) * \mathrm{Dem}(\mu^1, w^1) * \mathrm{Dem}(\mu^2, w^2)$$

Given dominant weights $\lambda, \nu$ of $\mathfrak{sl}_n\mathbb{C}$, we decompose them likewise into $\lambda^r, \nu^r$ for $r = 0, 1, 2$. Let $\pi \in \mathrm{Dem}(\mu, w)$, say $\pi = b(\mu^0) * \pi^1 * \pi^2$ with $\pi^r \in \mathrm{Dem}(\mu^r, w^r)$ for $r = 1, 2$. Then, $\pi \in \mathrm{Dem}^{\nu}_{\lambda}(\mu, w) \Leftrightarrow \lambda^0 + \mu^0 = \nu^0$ and $\pi^r \in \mathrm{Dem}^{\nu^r}_{\lambda^r}(\mu^r, w^r)$ for $r = 1, 2$. We have thus proved that $c^{\nu}_{\lambda\mu}(w) = \delta_{\lambda^0+\mu^0,\nu^0}\, c^{\nu^1}_{\lambda^1\,\mu^1}(w^1)\, c^{\nu^2}_{\lambda^2\,\mu^2}(w^2)$. It is easy to see that this equation establishes that if $w^1$ and $w^2$ have the saturation property, then so does $w$. This concludes the proof of Theorem 1.4. □

## 4.3 Concluding remarks

For $n = 4$, the only permutations in $S_4$ which are not of the form of Theorem 1.4 are $3412, 3142, 2413, 4231$ (in one-line notation). For $w = 3142$, we have $w_0 w = 2413$ with reduced decompositions $s_3 s_1 s_2 = s_1 s_3 s_2$. There is a unique reduced $F$ such that $\varpi(F) = 2413$, but the rhombi $R_{ij}$, $(i, j) \in F$ are not left-and-bottom justified, and Lemma 3.2 fails (Remark 3.3). For the other three $w$, there exist two reduced faces each. In these cases, $\mathrm{K}^{\mathrm{Hive}}(\,-, w_0 w)$ is a union of two polyhedral cones.

While our methods do not apply to a general $w \in S_n$ (beyond those covered by Theorem 1.4), we do not know if the saturation property fails there. In particular, a preliminary search using *Sage* for $n = 4, 5$ and small $\lambda, \mu, \nu, k$ did not turn up any counterexamples.

# References


[1] A. S. Buch. "The saturation conjecture (after A. Knutson and T. Tao)". *Enseign. Math. (2)* **46**.1-2 (2000). With an appendix by William Fulton, pp. 43–60.

[2] N. Fujita. "Schubert calculus from polyhedral parametrizations of Demazure crystals" (2020). arXiv:2008.04599.

[3] A. Henriques and J. Kamnitzer. "The octahedron recurrence and $\mathfrak{gl}_n$ crystals". *Advances in Mathematics* **206**.1 (2006), pp. 211 –249.

[4] A. Joseph. "A decomposition theorem for Demazure crystals". *J. Algebra* **265**.2 (2003), pp. 562–578.

[5] A. Joseph. "Modules with a Demazure flag". *Studies in Lie theory*. Vol. 243. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 131–169.

[6] V. A. Kirichenko, E. Y. Smirnov, and V. A. Timorin. "Schubert calculus and Gelfand-Tsetlin polytopes". *Uspekhi Mat. Nauk* **67**.4(406) (2012), pp. 89–128.

[7] A. Knutson and T. Tao. "Apiary views of the Berenstein-Zelevinsky polytope, and Klyachko's saturation conjecture" (1998). arXiv:math/9807160v1.

[8] A. Knutson and T. Tao. "The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture". *J. Amer. Math. Soc.* **12**.4 (1999), pp. 1055–1090.

[9] A. Knutson, T. Tao, and C. Woodward. "The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone". *J. Amer. Math. Soc.* **17**.1 (2004), pp. 19–48.

[10] M. Kogan. "Schubert Geometry of Flag Varieties and Gelfand-Cetlin Theory". PhD thesis. MIT, 2000.

[11] S. Kumar. "Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture". *Invent. Math.* **93**.1 (1988), pp. 117–130.

[12] M. S. Kushwaha, K. N. Raghavan, and S Viswanath. "A study of Kostant-Kumar modules via Littelmann paths". *Advances in Mathematics* **381** (2021), p. 107614.

[13] M. S. Kushwaha, K. N. Raghavan, and S Viswanath. "Refined Littlewood-Richardson coefficients, hives and the saturation problem". In preparation. 2021.

[14] V Lakshmibai, P Littelmann, and P Magyar. "Standard monomial theory for Bott-Samelson varieties". *Compositio Math.* **130**.3 (2002), pp. 293–318.

[15] A. Lascoux and M.-P. Schützenberger. "Keys & standard bases". *Invariant theory and tableaux (Minneapolis, MN, 1988)*. Vol. 19. IMA Vol. Math. Appl. Springer, New York, 1990, pp. 125–144.

[16] M. Lothaire. *Algebraic combinatorics on words*. Vol. 90. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2002.

[17] I. Pak and E. Vallejo. "Combinatorics and geometry of Littlewood-Richardson cones". *European J. Combin.* **26**.6 (2005), pp. 995–1008.

[18] A. Postnikov and R. P. Stanley. "Chains in the Bruhat order". *J. Algebraic Combin.* **29**.2 (2009), pp. 133–174.

[19] I. Terada, R. C. King, and O. Azenhas. "The symmetry of Littlewood-Richardson coefficients: a new hive model involutory bijection". *SIAM J. Discrete Math.* **32**.4 (2018), pp. 2850–2899.

[20] G. M. Ziegler. *Lectures on polytopes*. Vol. 152. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.