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The polytope algebra of generalized permutahedra

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Abstract. We study the polytope algebra of McMullen relative to a fixed zonotope. We endow the corresponding subalgebra with the structure of a module over the Tits algebra of the corresponding hyperplane arrangement. In the case of Coxeter arrangements of type A and B, we find connections with statistics on (signed) permutations and with the Hopf monoid of generalized permutahedra of Aguiar and Ardila.

Keywords: generalized permutahedra, valuation, hyperplane arrangements, Tits algebra, Eulerian idempotent, zonotope

Introduction

Generalized permutahedra have been a central object of study for many combinatorialists in recent years. They serve as a geometric model for many classical (type A) combinatorial objects. We study generalized permutahedra modulo certain *valuation* and *translation invariance* relations, previously considered in the construction of McMullen's polytope algebra, and give the resulting space the structure of a module over the Tits algebra of the braid arrangement. This structure is compatible with the Hopf monoid structure of Aguiar and Ardila [1].

We review McMullen's construction of the polytope algebra and its main properties in Section 1. In Section 2, we review the Tits algebra $\mathbb{R}\Sigma[\mathcal{A}]$ of a hyperplane arrangement and some generalities about modules over $\mathbb{R}\Sigma[\mathcal{A}]$. In particular, we focus on the action of certain elements of $\mathbb{R}\Sigma[\mathcal{A}]$ that we call *characteristic*. Particular cases of the action of such elements are related to the antipode problem and to the computation of polynomial invariants in Hopf monoids, this is reviewed in Section 2.1. Our main construction, the module structure on *generalized zonotopes* over the Tits algebra of the corresponding arrangement, is given in Section 3. Finally, in Section 4 we specialize the previous construction to the case of generalized permutahedra and type B generalized permutahedra. For full details and proofs see [9].

Preliminaries and notation

Let V be a real vector space of dimension d endowed with an inner product $\langle \cdot , \cdot \rangle$. We let $o \in V$ denote the zero vector of V. Given a polytope $\mathfrak{p} \subseteq V$ and a vector $v \in V$, we

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let $\mathfrak{p}_v$ denote the face of $\mathfrak{p}$ maximized in the direction v. That is, $\mathfrak{p}_v := \{p \in \mathfrak{p} : \langle p, v\rangle \geq \langle q, v\rangle \text{ for all } q \in \mathfrak{p}\}$. The **normal cone** of a face $\mathfrak{f}$ of $\mathfrak{p}$ is the polyhedral cone $N(\mathfrak{f}, \mathfrak{p}) := \{v \in V : \mathfrak{f} \leq \mathfrak{p}_v\}$. The **normal fan** of $\mathfrak{p}$ is the collection $\Sigma_{\mathfrak{p}} := \{N(\mathfrak{f}, \mathfrak{p}) : \mathfrak{f} \leq \mathfrak{p}\}$ of all normal cones of $\mathfrak{p}$.

Recall that a fan Σ **refines** Σ' if every cone in Σ' is a union of cones in Σ. We say that a polytope $\mathfrak{q}$ is a **deformation** of $\mathfrak{p}$ if $\Sigma_{\mathfrak{p}}$ refines $\Sigma_{\mathfrak{q}}$. The **Minkowski sum** of two polytopes $\mathfrak{p}, \mathfrak{q} \subseteq V$ is the polytope $\mathfrak{p} + \mathfrak{q} := \{p + q : p \in \mathfrak{p}, q \in \mathfrak{q}\}$. We say that $\mathfrak{q}$ is a **Minkowski summand** of $\mathfrak{p}$ if $\mathfrak{p} = \mathfrak{q} + \mathfrak{q}'$ for some polytope $\mathfrak{q}'$. The normal fan of $\mathfrak{p} + \mathfrak{q}$ is the common refinement of $\Sigma_{\mathfrak{p}}$ and $\Sigma_{\mathfrak{q}}$. Hence, $\Sigma_{\mathfrak{p}}$ refines the normal fan of any of its Minkowski summands.

1 McMullen's polytope algebra

The **polytope algebra** $\Pi(V)$ is generated as a group by elements $[\mathfrak{p}]$, one for each polytope $\mathfrak{p} \subseteq V$. These generators satisfy the following *valuation* and *translation invariance* relations:

$$[\mathfrak{p} \cup \mathfrak{q}] + [\mathfrak{p} \cap \mathfrak{q}] = [\mathfrak{p}] + [\mathfrak{q}] \qquad \text{and} \qquad [\mathfrak{p} + \{t\}] = [\mathfrak{p}] \tag{1.1}$$

whenever $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{p} \cup \mathfrak{q}$ are polytopes; and for any translation vector $t \in V$.

The product of $\Pi(V)$ is defined on generators by means of the Minkowski sum

$$[\mathfrak{p}] \cdot [\mathfrak{q}] := [\mathfrak{p} + \mathfrak{q}]. \tag{1.2}$$

It readily follows from (1.1) that the class of a point $1 := [\{o\}]$ is the unit of $\Pi(V)$.

Lemma 1.1 ([14, Lemma 13]). *Let $\mathfrak{p}$ be a k-dimensional polytope. Then,*

$$([\mathfrak{p}] - 1)^k \neq 0 \qquad \textit{and} \qquad ([\mathfrak{p}] - 1)^r = 0 \quad \textit{for } r > k.$$

Thus, we can define the **log-class** of a polytope $\mathfrak{p}$ by means of the usual power series of $\log(x)$ centered at $x = 1$. If $\mathfrak{p}$ has dimension k, then

$$\log[\mathfrak{p}] := \sum_{r=1}^{k} \tfrac{(-1)^{r-1}}{r}([\mathfrak{p}] - 1)^r.$$

For $r \geq 0$ let $\Xi_r(V)$ be the subgroup of $\Pi(V)$ generated by elements of the form $(\log[\mathfrak{p}])^r$.

Theorem 1.2 ([14, Theorem 1]). *$\Pi(V)$ is almost a graded $\mathbb{R}$-algebra, in the following sense:*

i. as an abelian group, $\Pi(V)$ admits a direct sum decomposition $\Pi(V) = \bigoplus_{r=0}^{d} \Xi_r(V)$;

ii. under multiplication, $\Xi_r(V) \cdot \Xi_s(V) = \Xi_{r+s}(V)$, with $\Xi_r(V) = 0$ for $r > d$;

iii. $\Xi_0(V) \cong \mathbb{Z}$, and for $r = 1, \ldots, d$, $\Xi_r(V)$ is a real vector space;

iv. the product of elements in $\bigoplus_{r\geq 1}\Xi_r(V)$ is bilinear.

Convention 1.3. As in later work of McMullen [15], we replace $\Xi_0(V)\cong\mathbb{Z}$ with the tensor product $\Xi_0(V)_{\mathbb{R}} := \Xi_0(V)\otimes_{\mathbb{Z}}\mathbb{R}$ to get a genuine graded $\mathbb{R}$-algebra $\Pi(V)_{\mathbb{R}}$. To simplify notation, we drop the subscript $\mathbb{R}$.

For each scalar $\lambda\in\mathbb{R}$, define the **dilation** $\delta_\lambda : \Pi(V)\to\Pi(V)$ on generators by $\delta_\lambda[\mathfrak{p}] := [\lambda\mathfrak{p}]$, where $\lambda\mathfrak{p} := \{\lambda p : p\in\mathfrak{p}\}$. One can easily verify that δ_λ preserves relations (1.1) and defines an algebra morphism. The following result characterizes the graded components of $\Pi(V)$ as the eigenspaces of any positive nontrivial dilation.

Lemma 1.4 ([14, Lemma 20])**.** *Let $x\in\Pi(V)$ and $\lambda>0$, with $\lambda\neq 1$. Then,*

$$x\in\Xi_r(V) \qquad \text{if and only if} \qquad \delta_\lambda x=\lambda^r x. \tag{1.3}$$

Definition 1.5. Fix a polytope $\mathfrak{p}\subseteq V$. The **subalgebra relative to** $\mathfrak{p}$, denoted $\Pi(\mathfrak{p})$, is the subalgebra of $\Pi(V)$ generated by classes $[\mathfrak{q}]$ of deformations $\mathfrak{q}$ of $\mathfrak{p}$.

Remark 1.6. McMullen [15] defines $\Pi(\mathfrak{p})$ in terms of Minkowski summands of $\mathfrak{p}$. A result of Shephard [13, Section 15.2.7] implies that both definitions are equivalent.

The grading of $\Pi(V)$ induces a grading of $\Pi(\mathfrak{p})$. We let $\Xi_r(\mathfrak{p})=\Pi(\mathfrak{p})\cap\Xi_r(V)$. The dimension of these spaces was described by McMullen in the case of simple polytopes.

Theorem 1.7 ([15, Theorem 6.1])**.** *If $\mathfrak{p}$ is a simple polytope, then $\dim_{\mathbb{R}}(\Xi_r(\mathfrak{p}))=h_r(\mathfrak{p})$ for all r.*

Fix a face $\mathfrak{f}$ of $\mathfrak{p}$ and a direction $v\in V$ such that $\mathfrak{p}_v=\mathfrak{f}$. If $\mathfrak{q}$ is a Minkowski summand of $\mathfrak{p}$, say $\mathfrak{p}=\mathfrak{q}+\mathfrak{q}'$, then $\mathfrak{f}=\mathfrak{q}_v+\mathfrak{q}'_v$. That is, $\mathfrak{q}_v$ is a Minkowski summand of $\mathfrak{f}$. It then follows by [14, Theorem 7] that there is a well-defined algebra morphism

$$\psi_{\mathfrak{f}}:\Pi(\mathfrak{p})\to\Pi(\mathfrak{f}) \tag{1.4}$$

sending a generator $[\mathfrak{q}]$ of $\Pi(\mathfrak{p})$ to $[\mathfrak{q}_v]\in\Pi(\mathfrak{f})$. One easily checks that $\psi_{\mathfrak{f}}$ is independent on the particular choice of $v\in\operatorname{relint}(N(\mathfrak{f},\mathfrak{p}))$.

Theorem 1.8 ([15, Theorem 2.4])**.** *Let $\mathfrak{p}$ be a simple polytope and $\mathfrak{f}$ a face of $\mathfrak{p}$. Then, the morphism $\psi_{\mathfrak{f}}:\Pi(\mathfrak{p})\to\Pi(\mathfrak{f})$ is surjective.*

2 The Tits algebra of a linear hyperplane arrangement

Let $\mathcal{A}$ be a linear hyperplane arrangement in V. The hyperplanes in $\mathcal{A}$ split V into a collection $\Sigma[\mathcal{A}]$ of convex cones called **faces** of $\mathcal{A}$. The set $\Sigma[\mathcal{A}]$ has the structure of a monoid under the *Tits product* illustrated in Figure 1. The product of two faces F and G, denoted FG, is the first face you encounter after moving a small positive distance from an interior point of F to an interior point of G. The unit of this product is the **central face** $O\in\Sigma[\mathcal{A}]$: the intersection of all the hyperplanes in $\mathcal{A}$.

Figure 1: Product of faces in arrangements in dimension 2 and 3. The second arrangement has been intersected with a sphere around the origin.

The **Tits algebra** of $\mathcal{A}$ is the monoid algebra $\mathbb{R}\Sigma[\mathcal{A}]$. See [5, Chapters 1 and 9] for more details. We let $\mathtt{H}_F$ denote the basis element of $\mathbb{R}\Sigma[\mathcal{A}]$ associated to the face F of $\mathcal{A}$.

An arbitrary intersection of hyperplanes in $\mathcal{A}$ is a **flat** of the arrangement. The set $\mathcal{L}[\mathcal{A}]$ of flats is a lattice with maximum $\top := V$ and minimum $\bot$, the intersection of all hyperplanes in $\mathcal{A}$. The **support** of a face F is the smallest flat $\mathrm{s}(F)$ containing it.

We view $\mathcal{L}[\mathcal{A}]$ as a commutative monoid with the join operation for the product. Then, the support map $\mathrm{s} : \Sigma[\mathcal{A}] \to \mathcal{L}[\mathcal{A}]$ is a morphism of monoids.

The monoid algebra $\mathbb{R}\mathcal{L}[\mathcal{A}]$ is the maximal (split-)semisimple quotient of $\mathbb{R}\Sigma[\mathcal{A}]$ via the support map. We let $\mathtt{H}_{\mathrm{X}}$ denote the basis element of $\mathbb{R}\mathcal{L}[\mathcal{A}]$ associated to the flat X of $\mathcal{A}$, so that $\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} = \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}}$. It follows that the simple modules over the Tits algebra are one-dimensional and are indexed by the flats of $\mathcal{A}$. This rests on the fact that $\mathbb{R}\mathcal{L}[\mathcal{A}]$ has a unique complete system of orthogonal idempotents $\{\mathtt{Q}_{\mathrm{X}}\}_{\mathrm{X}\in\mathcal{L}[\mathcal{A}]}$ determined by

$$\mathtt{H}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y}\geq\mathrm{X}} \mathtt{Q}_{\mathrm{Y}} \quad \text{or equivalently} \quad \mathtt{Q}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\mathtt{H}_{\mathrm{Y}}, \tag{2.1}$$

where μ denotes the MÃűbius function of the lattice $\mathcal{L}[\mathcal{A}]$. This is a result of Solomon.

For every flat X, let χ_{X} denote the character of the simple module of $\mathbb{R}\Sigma[\mathcal{A}]$ indexed by X. An element $w \in \mathbb{R}\Sigma[\mathcal{A}]$ is **characteristic of parameter** $t \in \mathbb{R}$ if $\chi_{\mathrm{X}}(w) = t^{\dim(\mathrm{X})}$ for every flat $\mathrm{X} \in \mathcal{L}[\mathcal{A}]$. Characteristic elements determine the characteristic polynomial of $\mathcal{A}$ and of its contractions. See [2] and [5, Section 12.4] for more information.

A main motivation to study characteristic elements and their action on modules over the Tits algebra comes from the theory of Hopf monoids. We expand on this next.

2.1 Hopf monoids and the braid arrangement

The theory of Hopf monoids in the category of species was developed by Aguiar and Mahajan [3, 4], and has received significant attention in recent years. It provides a unified framework to study families of combinatorial objects that have a natural way to *merge* and *break* structures.

A **species H** consists of a vector space $\mathbf{H}[I]$ for each finite set I and *relabeling maps* $\mathbf{H}[I] \to \mathbf{H}[J]$ for each bijection $I \to J$. A **Hopf monoid** is a species **H** with *compatible* product and coproduct maps

$$\mu_{S_1,\dots,S_k} : \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \to \mathbf{H}[I] \qquad \text{and} \qquad \Delta_{S_1,\dots,S_k} : \mathbf{H}[I] \to \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k]$$

indexed by set compositions $(S_1, \dots, S_k)$ of I, and an **antipode** map $\mathbf{s}_I : \mathbf{H}[I] \to \mathbf{H}[I]$, which generalizes the notion of inversion in a group. A fundamental problem in Hopf monoid theory is to find a reduced formula for the antipode map of a given Hopf monoid **H**.

Let $\mathcal{A}_d$ denote the **braid arrangement** in $\mathbb{R}^d$. It consists of the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq d$. We identify faces and flats of $\mathcal{A}_d$ with set compositions $F \vDash [d]$ and set partitions $\mathrm{X} \vdash [d]$ of $[d]$ in the usual way. For instance, the set composition $F = (\{1,3\}, \{2\}, \{4,5\})$ represents the face with equations $x_1 = x_3 > x_2 > x_4 = x_5$ of $\mathcal{A}_5$. Similarly, the set partition $\mathrm{X} = \big\{\{1,3\}, \{2\}, \{4,5\}\big\}$ represents the flat satisfying $x_1 = x_3$ and $x_4 = x_5$. Note that $\mathrm{s}(F) = \mathrm{X}$.

The **Takeuchi element** $\tau \in \mathbb{R}\Sigma[\mathcal{A}_d]$ and the **Adams element** $\alpha_t \in \mathbb{R}\Sigma[\mathcal{A}_d]$ are defined by

$$\tau = \sum_F (-1)^{\dim(F)} \mathtt{H}_F \qquad \text{and} \qquad \alpha_t = \sum_F \binom{t}{\dim(F)} \mathtt{H}_F.$$

They are characteristic elements of parameter -1 and t, respectively.

Example/Theorem 2.1 ([4, Section 13.1])**.** Let **H** be a commutative Hopf monoid. Then, the space $\mathbf{H}[[d]]$ is a right $\mathbb{R}\Sigma[\mathcal{A}_d]$-module. The action of the basis element $\mathtt{H}_F$ corresponding to a face $F = (S_1, \dots, S_k)$ is given by the composition $\mu_{S_1,\dots,S_k} \circ \Delta_{S_1,\dots,S_k}$.

Then, the antipode map **s** is determined by the action of the Takeuchi element. Explicitly, $\mathbf{s}_{[d]}(x) = x \cdot \tau$. On the other hand, when $t = n$ is an integer, the action of the Adams element α_n agrees with the n-th *convolution power of the identity*. If ζ is a *character* of **H** and $P(x,t)$ denotes the corresponding polynomial invariant on an element $x \in \mathbf{H}[[d]]$ (see [1, Section 16]), then $P(x,t) = \zeta(x \cdot \alpha_t)$.

2.2 Eulerian idempotents and diagonalization

An **Eulerian family** of $\mathcal{A}$ is a collection $\{\mathtt{E}_\mathrm{X}\}_{\mathrm{X} \in \mathcal{L}[\mathcal{A}]}$ of idempotent and mutually orthogonal elements of $\mathbb{R}\Sigma[\mathcal{A}]$ of the form

$$\mathtt{E}_\mathrm{X} = \sum_{F : \mathrm{s}(F) \geq \mathrm{X}} a^F \mathtt{H}_F,$$

with $a^F \neq 0$ for at least one F with $\mathrm{s}(F) = \mathrm{X}$. It follows [5, Theorem 11.20] that $\{\mathtt{E}_\mathrm{X}\}_\mathrm{X}$ is a complete system of orthogonal idempotents and that

$$\mathrm{s}(\mathtt{E}_\mathrm{X}) = \mathtt{Q}_\mathrm{X}.$$

A characteristic element w of *non-critical*[1] parameter t uniquely determines an Eulerian family $\mathtt{E} = \{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$, which satisfies

$$w = \sum_{\mathrm{X}} t^{\dim(\mathrm{X})} \mathtt{E}_{\mathrm{X}}. \tag{2.2}$$

It follows that the action of w on any $\mathbb{R}\Sigma[\mathcal{A}]$-module M is diagonalizable.

Let M be a (right) $\mathbb{R}\Sigma[\mathcal{A}]$-module, $w \in \mathbb{R}\Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t and $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be the corresponding Eulerian Family. Then, we have a decomposition

$$M = \bigoplus_{\mathrm{X}} M \cdot \mathtt{E}_{\mathrm{X}}$$

of vector spaces. Expression (2.2) shows that w acts on $M \cdot \mathtt{E}_{\mathrm{X}}$ by multiplication by $t^{\dim(\mathrm{X})}$. We define

$$\eta_{\mathrm{X}}(M) := \dim_{\mathbb{R}}(M \cdot \mathtt{E}_{\mathrm{X}}). \tag{2.3}$$

The character $\chi_M : \mathbb{R}\Sigma[\mathcal{A}] \to \mathbb{R}$ of M factors through $\mathbb{R}\mathcal{L}[\mathcal{A}]$:

$$\begin{tikzcd}
\mathbb{R}\Sigma[\mathcal{A}] \arrow[d, "\mathrm{s}"'] \arrow[dr, "\chi_M"] & \\
\mathbb{R}\mathcal{L}[\mathcal{A}] \arrow[r, dashed, "\overline{\chi}_M"'] & \mathbb{R}
\end{tikzcd}$$

Thus, $\eta_{\mathrm{X}}(M) = \dim_{\mathbb{R}}(M \cdot \mathtt{E}_{\mathrm{X}}) = \chi_M(\mathtt{E}_{\mathrm{X}}) = \overline{\chi}_M(\mathtt{Q}_{\mathrm{X}})$ is independent of the characteristic element w. Furthermore, using relations (2.1) and the linearity of $\overline{\chi}_M$ we deduce

$$\begin{aligned}
\eta_{\mathrm{X}}(M) = \overline{\chi}_M(\mathtt{Q}_{\mathrm{X}}) &= \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \overline{\chi}_M(\mathtt{H}_{\mathrm{Y}}) \\
&= \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \chi_M(\mathtt{H}_{F_{\mathrm{Y}}}) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}}(M \cdot \mathtt{H}_{F_{\mathrm{Y}}})
\end{aligned} \tag{2.4}$$

where $F_{\mathrm{Y}} \in \Sigma[\mathcal{A}]$ is such that $\mathrm{s}(F_{\mathrm{Y}}) = \mathrm{Y}$. The last equality follows since $\mathtt{H}_{F_{\mathrm{Y}}}$ is an idempotent element, and thus $\chi_M(\mathtt{H}_{F_{\mathrm{Y}}}) = \dim_{\mathbb{R}}(M \cdot \mathtt{H}_{F_{\mathrm{Y}}})$.

Moreover, the number of composition factors M_{i+1}/M_i isomorphic to the simple module indexed by X in a composition series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$ of M is precisely $\eta_{\mathrm{X}}(M)$.

3 The McMullen module of generalized zonotopes

Fix a hyperplane arrangement $\mathcal{A}$ in V. Take a normal vector v_{H} for each hyperplane $\mathrm{H} \in \mathcal{A}$, and consider the zonotope $\mathfrak{z} := \sum_{\mathrm{H}} \mathrm{Conv}\{o, v_{\mathrm{H}}\}$. Its normal fan $\Sigma_{\mathfrak{z}}$ coincides with the collection of faces $\Sigma[\mathcal{A}]$ of the arrangement. We say that a polytope $\mathfrak{p}$ is a **generalized zonotope** of $\mathcal{A}$ if it is a deformation of $\mathfrak{z}$.

[1] $t \in \mathbb{R}$ is non-critical if it is not a root of the characteristic polynomial of $\mathcal{A}$ nor any of its contractions.

We now consider the algebra $\Pi(\mathfrak{z})$ introduced in Definition 1.5. It is generated by the classes of generalized zonotopes of $\mathcal{A}$. It only depends on the arrangement $\mathcal{A}$ and not on the particular choice normal vectors v_{H}.

Faces of a zonotope $\mathfrak{z}$ are Minkowski summands of $\mathfrak{z}$. Therefore, for each face $\mathfrak{f}$ of $\mathfrak{z}$, the algebra $\Pi(\mathfrak{f})$ is a subalgebra of $\Pi(\mathfrak{z})$. Moreover, if $\mathfrak{f} = \mathfrak{z}_v$ for some $v \in V$ and $\mathfrak{q}$ is a Minkowski summand of $\mathfrak{f}$, then $\mathfrak{q}_v = \mathfrak{q}$. Hence, the composition $\Pi(\mathfrak{f}) \hookrightarrow \Pi(\mathfrak{z}) \xrightarrow{\psi_{\mathfrak{f}}} \Pi(\mathfrak{f})$, where $\psi_{\mathfrak{f}}$ is the morphism (1.4), is the identity map. We have proved the following.

Proposition 3.1. *Let $\mathfrak{z}$ be a zonotope and $\mathfrak{f}$ a face of $\mathfrak{z}$. Then, the morphism $\psi_{\mathfrak{f}}$ is surjective.*

Remark 3.2. Compare this to Theorem 1.8 and note that we do not assume that $\mathfrak{z}$ is simple. For an arbitrary polytope $\mathfrak{p}$ and a face $\mathfrak{f}$ of $\mathfrak{p}$, there is no natural morphism $\Pi(\mathfrak{f}) \to \Pi(\mathfrak{p})$, unlike in the case of zonotopes.

Theorem 3.3. *The algebra $\Pi(\mathfrak{z})$ is a right $\mathbb{R}\Sigma[\mathcal{A}]$-module under the following action. For a generator $[\mathfrak{q}]$ of $\Pi(\mathfrak{z})$ and a basis element $\mathtt{H}_F$ of $\mathbb{R}\Sigma[\mathcal{A}]$,*

$$[\mathfrak{q}] \cdot \mathtt{H}_F := [\mathfrak{q}_v], \tag{3.1}$$

where $v \in \operatorname{relint}(F)$. Moreover, each graded component $\Xi_r(\mathfrak{z})$ is a $\mathbb{R}\Sigma[\mathcal{A}]$-submodule and the action of basis elements $\mathtt{H}_F$ on $\Pi(\mathfrak{z})$ is by (graded) algebra endomorphisms.

Sketch of proof. It follows from [13, Section 3.1.5] that for a polytope $\mathfrak{q} \subseteq V$ and vectors $v, w \in V$, $(\mathfrak{q}_v)_w = \mathfrak{q}_{v+\lambda w}$ for any small enough $\lambda > 0$. Similarly, if $v \in \operatorname{relint}(F)$ and $w \in \operatorname{relint}(G)$ for $F, G \in \Sigma[\mathcal{A}]$, then $v + \lambda w \in \operatorname{relint}(FG)$ for any small enough $\lambda > 0$.

The last statement follows form [14, Theorem 7] and the characterization of the graded components Ξ_r in (1.3). □

Let $w \in \mathbb{R}\Sigma[\mathcal{A}]$ be a characteristic element of parameter t and $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ the corresponding Eulerian family. It follows from (1.3) and (2.2) that for any $\lambda > 0$,

$$\text{If } x \in \Xi_r(\mathfrak{z}) \cdot \mathtt{E}_{\mathrm{X}}, \qquad \text{then} \qquad \delta_\lambda x = \lambda^r x \qquad \text{and} \qquad x \cdot w = t^{\dim(\mathrm{X})} x.$$

We say that elements in $\Xi_r(\mathfrak{z}) \cdot \mathtt{E}_{\mathrm{X}}$ are **double-eigenvectors**. We proceed to compute the dimension of these *double-eigenspaces* $\eta_{\mathrm{X}}(\Xi_r(\mathfrak{z})) = \dim_{\mathbb{R}}(\Xi_r(\mathfrak{z}) \cdot \mathtt{E}_{\mathrm{X}})$ defined in (2.3).

Let $F \in \Sigma[\mathcal{A}]$ and $\mathfrak{f}$ be the corresponding face of $\mathfrak{z}$. Observe that for any $x \in \Pi(\mathfrak{z})$, $x \cdot \mathtt{H}_F = \psi_{\mathfrak{f}}(x)$. Applying Proposition 3.1 to (2.4), we have

$$\eta_{\mathrm{X}}(\Xi_r(\mathfrak{z})) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}}(\Xi_r(\mathfrak{z}) \cdot \mathtt{H}_{F_{\mathrm{Y}}}) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}}(\Xi_r(\mathfrak{z}_{\mathrm{Y}})),$$

where $\mathfrak{z}_{\mathrm{Y}}$ is any face of $\mathfrak{z}$ perpendicular to Y. That is, such that $N(\mathfrak{z}_{\mathrm{Y}}, \mathfrak{z}_{\mathrm{Y}}) = \mathrm{Y}$. If in addition $\mathcal{A}$ is a simplicial arrangement (i.e. $\mathfrak{z}$ is simple), Theorem 1.7 implies that

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\mathfrak{z})) z^r = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) h(\mathfrak{z}_{\mathrm{Y}}, z), \tag{3.2}$$

where $h(\mathfrak{z}_{\mathrm{Y}}, z)$ denotes the h-polynomial of $\mathfrak{z}_{\mathrm{Y}}$.

4 Generalized permutahedra and Eulerian polynomials

4.1 Type A

Let $\mathcal{A}_d$ denote the braid arrangement in $\mathbb{R}^d$, see Section 2.1. It is the Coxeter arrangement of type A, corresponding to the symmetric group $\mathfrak{S}_d$. This group acts on $\mathbb{R}^d$ by permuting coordinates $\sigma(x_1, x_2, \ldots, x_d) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(d)})$. The **permutahedron** $\pi_d \subseteq \mathbb{R}^d$ is the convex hull of the points obtained by permuting the coordinates of $(1, 2, \ldots, d)$. It is a zonotope of $\mathcal{A}_d$ and has dimension $d-1$. Deformations of π_d are called **generalized permutahedra**. In this section we study the algebra/$\mathbb{R}\Sigma[\mathcal{A}_d]$-module $\Pi(\pi_d)$.

4.1.1 Hopf monoid structure

Aguiar and Ardila introduced in [1] the Hopf monoid of generalized permutahedra **GP**. For any finite set I, $\mathbf{GP}[I] = \mathbb{R}\{\mathfrak{p} \subseteq \mathbb{R}^I \,:\, \mathfrak{p} \text{ is a generalized permutahedron}\}$. The product and coproduct maps of **GP** are completely determined by

$$\mu_{S,T}(\mathfrak{p} \otimes \mathfrak{q}) = \mathfrak{p} \times \mathfrak{q} \qquad \text{and} \qquad (\mu_F \circ \Delta_F)(\mathfrak{p}) = \mathfrak{p}_v$$

where $v \in \operatorname{relint}(F)$. Compare this to (3.1). Let $\mathbf{\Pi}$ be the species defined by setting $\mathbf{\Pi}[I] := \Pi(\pi_I)$, where π_I denotes the permutahedron in $\mathbb{R}^I$.

Theorem 4.1. *The species* $\mathbf{\Pi}$ *is the Hopf monoid quotient of* **GP** *induced by the map* $\mathfrak{p} \mapsto [\mathfrak{p}]$. *Moreover,* $\mathbf{\Pi}$ *has the structure of a (2,1)-monoid in the category of species with the Cauchy and Hadamard product.*

A result similar to the first statement was recently obtained by Ardila and Sanchez in [8]. The *higher monoidal structure* is a consequence of the last statement in Theorem 3.3. For more information on higher monoidal categories see [3, Chapter 7].

4.1.2 Module structure

We let $\mathrm{s}(\sigma)$ denote the subspace of fixed points by the action of σ on $\mathbb{R}^d$; it is a flat of $\mathcal{A}_d$. Recall that $i \in [d]$ is an **excedance** of a permutation $\sigma \in \mathfrak{S}_d$ if $\sigma(i) > i$. Let $\operatorname{exc}(\sigma)$ denote the number of excedances of σ. We completely determine the numbers $\eta_{\mathrm{X}}(\Xi_r(\pi_d))$.

Theorem 4.2. *For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d]$ *and* $r = 0, 1, \ldots, d-1$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d)) = \left|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \operatorname{exc}(\sigma) = r\}\right|.$$

Building on top of work by BjÃűrner, Brenti showed that for any Coxeter group W, the h-polynomial of the W-permutahedron is the corresponding W-Eulerian polynomial [10, Theorem 2.3]. The *classical* **Eulerian polynomial** $A_d(z)$ is

$$A_d(z) = \sum_{k=0}^{d-1} A_{d,k} z^k = \sum_{\sigma \in \mathfrak{S}_d} z^{\operatorname{exc}(\sigma)}.$$

That is, the coefficient $A_{d,k}$ counts the number of permutations of $[d]$ with exactly k excedances. These coefficients are called *Eulerian numbers* (OEIS: A008292). It is customary to define $A_d(z)$ in terms of the *descents* of permutations in $\mathfrak{S}_d$. However, Foata's *fundamental transformation* shows that the distribution of these two statistics are equal.

The exponential generating function for the classical Eulerian polynomials was first described by Euler:

$$A(z,x) = \sum_{d\geq 0} A_d(z)\frac{x^d}{d!} = \frac{z-1}{z-e^{x(z-1)}}. \tag{4.1}$$

Under the identification between flats of $\mathcal{A}_d$ and set partitions of $[d]$, $\mathrm{s}(\sigma)$ is the partition of $[d]$ underlying the cycle decomposition of σ. Given a finite set S, we let $\mathfrak{C}(S)$ denote the collection of cyclic permutations on S. For a block $S \in \mathrm{s}(\sigma)$, we let $\sigma|_S \in \mathfrak{C}(S)$ be the restriction of σ to S.

Lemma 4.3. $\displaystyle\sum_{\mathrm{X}=\{S_1,\ldots,S_k\}\vdash[d]} \mu(\perp,\mathrm{X})A_{|S_1|}(z)\cdot\ldots\cdot A_{|S_k|}(z) = \sum_{\sigma\in\mathfrak{C}(d)} z^{\mathrm{exc}(\sigma)}.$

Sketch of proof. We verify that the exponential generating functions of both sides of the equation above is $\log(A(z,x))$. Recall that $\mu(\perp,\mathrm{X}) = (-1)^{k-1}(k-1)!$, where $k = |\mathrm{X}|$, and that $\log(1+x) = \sum_{d\geq 1}(-1)^{d-1}(d-1)!\frac{x^d}{d!}$. An application of the Compositional Formula [16, Theorem 5.1.4] yields the result for the terms on the left.

For the terms on the right, observe that the number of excedances of σ is the sum of the excedances on each of its cycles. That is,

$$\mathrm{exc}(\sigma) = \sum_{S\in\mathrm{s}(\sigma)} \mathrm{exc}(\sigma|_S) \tag{4.2}$$

An application of the Exponential formula then shows that $A(z,x) = \exp(\widetilde{A}(z,x))$ where $\widetilde{A}(z,x)$ is the exponential generating function of the terms on the right. □

Proof of Theorem 4.2. Sketch. Faces of π_d are products of permutahedra of lower dimension. Explicitly, if $\mathrm{X} = \{S_1,\ldots,S_k\}$, then $(\pi_d)_{\mathrm{X}} \cong \pi_{|S_1|}\times\cdots\times\pi_{|S_k|}$. On the other hand, flats Y containing $\mathrm{X} = \{S_1,\ldots,S_k\}$ correspond to a choice of partitions $\mathrm{Y}|_{S_i}$ for each S_i, and in this case $\mu(\mathrm{X},\mathrm{Y}) = \mu(\{S_1\},\mathrm{Y}|_{S_1})\cdots\mu(\{S_k\},\mathrm{Y}|_{S_k})$.

Using that $h(\mathfrak{p}\times\mathfrak{q},z) = h(\mathfrak{p},z)h(\mathfrak{q},z)$, formula (3.2) then yields

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi))z^r = \prod_{S\in\mathrm{X}}\Big(\sum_{\mathrm{Y}=\{T_1,\ldots,T_\ell\}\vdash S}\mu(\perp,\mathrm{Y})A_{|T_1|}(z)\cdot\ldots\cdot A_{|T_\ell|}(z)\Big).$$

Applying Lemma 4.3, we get

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi))z^r = \prod_{S\in\mathrm{X}}\Big(\sum_{\sigma\in\mathfrak{C}(S)} z^{\mathrm{exc}(\sigma)}\Big) = \sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)}.$$

Finally, taking the coefficient of z^r on both sides yields the result. □

Let $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be the Eulerian family corresponding to the Adams element α_t from Section 2.1. The particular case $r = 1$ of Theorem 4.2 reads

$$\eta_{\mathrm{X}}(\Xi_1(\pi_d)) = \begin{cases} 1 & \text{if X has exactly 1 non-singleton singleton block,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Let X be such a flat and let $J \subseteq [d]$ be the corresponding non-singleton block.

Proposition 4.4. *The space $\Xi_1(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$ is spanned by the element $\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}}$, where Δ_J denotes the simplex with vertex set $\{e_j : j \in J\}$.*

In view of (4.3), the proof of this boils down to showing that $\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}} \neq 0$. This rests on a result of Ardila, Benedetti and Doker [6, Proposition 2.4] that implies that $\{\log[\Delta_J] : |J| \geq 2\}$ is a linear basis of $\Xi_1(\pi_d)$.

Their result shows that every generalized permutahedron $\mathfrak{p}$ can be written uniquely as a **signed Minkowski sum** of the simplices $\{\Delta_J, \emptyset \neq J \subseteq [d]\}$. That is, there are unique coefficients $\{y_J\}_J \subseteq \mathbb{R}$ such that $\mathfrak{p} + \sum_{y_J<0} |y_J|\Delta_J = \sum_{y_J>0} y_J \Delta_J$. Taking log-classes, we get that

$$\log[\mathfrak{p}] = \sum_J y_J \log[\Delta_J].$$

Since $\log[\Delta_J] = \log(1) = 0$ for $|J| = 1$, we deduce that every log-class $\log[\mathfrak{p}]$ can be written **uniquely** as a linear combination of $\{\log[\Delta_J], |J| \geq 2\}$.

4.2 Type B

Let $\mathcal{B}_d$ denote the Coxeter arrangement of type B in $\mathbb{R}^d$. It consists of hyperplanes $x_i = x_j$, $x_i = -x_j$ and $x_i = 0$. Flats of $\mathcal{B}_d$ are in correspondence with signed set partitions $\mathrm{X} \vdash^B [d]$ of $[d]$. A **signed set partitions** of $[d]$ is a weak partition $\{S_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\}$ of $\pm[d] := \{i, \bar{i} : i \in [d]\}$ such that $S_0 = \overline{S_0}$, $S_i \neq \emptyset$ and $S_i \cap \overline{S_i} = \emptyset$ for $i \neq 0$. For instance, the type B partition $\big\{\{1,\bar{1},4,\bar{4},5,\bar{5}\},\{2,\bar{3}\},\{\bar{2},3\}\big\}$ represents the flat of $\mathcal{B}_5$ with equations $x_1 = x_4 = x_5 = 0$, $x_2 = -x_3$. For $i \in [d]$, write $i = |\bar{i}| = |i|$.

The **hyperoctahedral group** $\mathfrak{B}_d$ consists of **signed permutations** $\tau : \pm[d] \to \pm[d]$ satisfying $\tau(i) = j$ if and only if $\tau(\bar{i}) = \bar{j}$. Signed permutations act on $\mathbb{R}^d$ by permuting coordinates and changing signs. For example, let $\tau = (1\overline{41}4)(5\bar{5})(2\bar{3})(\bar{2}3)$. Then, $\tau(x_1, x_2, x_3, x_4, x_5) = (-x_4, -x_3, -x_2, x_1, -x_5)$. Note that $\mathrm{s}(\tau)$ is the flat in the previous paragraph.

The **type B permutahedron** $\pi_d^B \subseteq \mathbb{R}^d$ is the convex hull of all the signed permutations of the point $(1, 2, \dots, d)$. It is full-dimensional and a zonotope of $\mathcal{B}_d$, its h-polynomial is the **type B Eulerian polynomial** $B_d(z)$, it keeps track of the statistic $\mathrm{exc}_B(\tau)$ on $\mathfrak{B}_d$. Let $B(z, x)$ denote the type B generating function of the polynomials $B_d(z)$. Deformations of π_d^B are called **type B generalized permutahedra**. We now consider the module $\Pi(\pi_d^B)$.

Theorem 4.5. *For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{B}_d]$ *and* $r = 0, 1, \ldots, d$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d^B)) = \big|\{\tau \in \mathfrak{B}_d \,:\, \mathrm{s}(\tau) = \mathrm{X},\ \mathrm{exc}_B(\tau) = r\}\big|.$$

The statistic exc_B is closely related to the *flag-excedance* of a signed permutation, see [12]. We do not go into the details of its definition, but we highlight a key property of exc_B analogous to (4.2). If $\mathrm{s}(\tau) = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$, then $\tau|_{S_0}$ is a signed permutation with $\mathrm{s}(\tau|_{S_0}) = \perp$ and each $\tau|_{S_i} \in \mathfrak{C}(S_i)$ for $i \geq 1$ is a cyclic permutation. Moreover, $\mathrm{exc}_B(\tau) = \mathrm{exc}_B(\tau|_{S_0}) + \sum_{i\geq 1} \mathrm{exc}_{\prec}(\tau|_{S_i})$, where $\mathrm{exc}_{\prec}$ denotes the usual excedance but computed with respect to a particular order $\prec$ on S_i.

Sketch of proof. The analogous result to Lemma 4.3 is

$$\sum_{\{S_0,\ldots,S_k,\overline{S_k}\} \vdash^B [d]} \mu(\perp, \mathrm{X}) B_{|S_0|/2} A_{|S_1|} \ldots A_{|S_k|} = \sum_{\substack{\tau \in \mathfrak{B}_d \\ \mathrm{s}(\tau) = \perp}} z^{\mathrm{exc}_B(\tau)}. \tag{4.4}$$

We again proceed by comparing the *type B* exponential generating function of both sides and verify it equals $\frac{B(z,x)}{\sqrt{A(z,x)}}$. The proof involves a type B analogous of the Compositional and Exponential formula. With X as above, $\mu(\perp, \mathrm{X}) = (-1)^k(2k-1)!!$ (compare with the generating function of $(1+x)^{-1/2}$). On the other hand, the type B exponential formula together with the observations following the statement of Theorem 4.5 yield $B(z,x) = \widetilde{B}(z,x)\exp(\frac{\log(A(z,x))}{2})$, where $\widetilde{B}(z,x)$ is the type B exponential generating function of the right side of (4.4). See [9, Section 6.2] for all the details. □

As a byproduct of the proof, we obtain the following formula. To the best of our knowledge, this is a new formula. The analogous result for the symmetric group was described by Brenti in [11, Proposition 7.3].

Corollary 4.6. $\displaystyle\sum_{d\geq 0} \Big(\sum_{\tau \in \mathfrak{B}_d} t^{\dim(\mathrm{s}(\tau))} z^{\mathrm{exc}_B(\tau)} \Big) \frac{x^d}{2^d\, d!} = B(z,x)\, A(z,x)^{\frac{t-1}{2}}.$

In [7], the authors ask for a nice set of generators for the family of type B generalized permutahedra, in the sense of Ardila, Benedetti and Doker in the previous section. The special case $r = 1$ of Theorem 4.5 implies the following.

Corollary 4.7. *Any set of generators for the family of type B generalized permutahedra must contain at least* 2^{d-1} *full-dimensional polytopes.*

Proof. As discussed after Proposition 4.4, the log-classes of any such set would linearly generate $\Xi_1(\pi_d^B)$. Let $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be an Eulerian family of $\mathcal{B}_d$.

Let $\mathfrak{p}$ be a type B generalized permutahedron that is not full-dimensional. Let F be a maximal face in the flat $N(\mathfrak{p}, \mathfrak{p})$. Then, $\mathrm{s}(F) \neq \perp$. It follows from [5, Lemma 11.12] that $\mathtt{H}_F \cdot \mathtt{E}_{\perp} = 0$. Since the action of basis elements $\mathtt{H}_F$ on $\Pi(\pi_d^B)$ is by endomorphisms, we get $\log[\mathfrak{p}] \cdot \mathtt{H}_F = \log([\mathfrak{p}] \cdot \mathtt{H}_F) = \log[\mathfrak{p}]$ and consequently $\log[\mathfrak{p}] \cdot \mathtt{E}_{\perp} = (\log[\mathfrak{p}] \cdot \mathtt{H}_F) \cdot \mathtt{E}_{\perp} = 0$. The result follows since, by Theorem 4.5, $\dim_{\mathbb{R}}(\Xi_1(\pi_d^B) \cdot \mathtt{E}_{\perp}) = \eta_{\perp}(\Xi_1(\pi_d^B)) = 2^{d-1}$. □

This is in sharp contrast with the type A case, where the set of generators is formed by the collection of faces of the standard simplex.

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