Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #63, 13 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Combinatorial aspects of virtually Cohen–Macaulay sheaves

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Abstract. There is an abundance of deep literature on the use of free resolutions to study modules and vector bundle resolutions to study coherent sheaves. When studying a module over the Cox ring of a smooth projective toric variety X, each approach comes with its own challenges. There is geometric information that free resolutions fail to encode, while vector bundle resolutions resist study using algebraic and combinatorial techniques. Recently, Berkesch, Erman, and Smith introduced virtual resolutions, which are amenable to algebraic and combinatorial study and also capture desirable geometric information. In this extended abstract, we continue this program in the combinatorially-rich Stanley–Reisner setting. In particular, when X is a product of projective spaces, we produce a large new class of virtually Cohen–Macaulay Stanley–Reisner rings. After augmenting the simplicial complexes associated to these Stanley–Reisner rings with a coloring that reflects the product structure on X, our primary tool is Reisner's criterion, whose conclusion we interpret in the virtual setting. We also provide two constructions of short virtual resolutions for use beyond the Stanley–Reisner case.

Keywords: Stanley–Reisner Rings, Free Resolutions, Toric Varieties, Homological Algebra

Introduction

Let X be a smooth projective toric variety over an algebraically closed field k with Cox ring S and irrelevant ideal B (see [4, §5.2]). Recall that S is a positively $\mathrm{Pic}(X)$-graded polynomial ring and that there is a correspondence between $\mathrm{Pic}(X)$-graded B-saturated modules M over S and sheaves $\widetilde{M}$ on X [1, 7, 3] (see [8] when X is not smooth). Unfortunately, the numerics of the minimal $\mathrm{Pic}(X)$-graded free resolutions for such S-modules do not obviously provide many geometric insights for M when X is not projective space;

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for example, a minimal $\mathrm{Pic}(X)$-graded free resolution of M may be significantly longer than the dimension of X. However, this failure appears to be a consequence of imposing too much algebraic structure on the resolution. Approaching the problem from the geometric perspective, vector bundle resolutions of $\widetilde{M}$ are bounded in length by the dimension of X and can in principle be used to study the geometry of $\widetilde{M}$, but vector bundles on X quite complicated and notoriously difficult to study. A proposed solution comes from [2], in which the authors introduce a type of resolution of M by free S-modules, which they call a *virtual resolution*. Virtual resolutions capture geometrically meaningful properties of $\mathrm{Pic}(X)$-graded S-modules, such as unmixedness, well-behavedness of deformation theory, and regularity of tensor products, while also being amenable to study by algebraic and combinatorial techniques.

Example 1.1. Consider the variety Y of three points contained in $\mathbb{P}^1 \times \mathbb{P}^1$, where two points have same image under the projection to the first copy of $\mathbb{P}^1$ and all three points have distinct images when projected to the second copy of $\mathbb{P}^1$, as in Figure 1. Letting S be the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ and I_Y denote the B-saturated ideal for Y, then S/I_Y has minimal free resolution

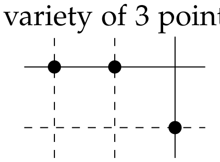
$$S \longleftarrow \begin{array}{c} S(-1,-1) \\ \oplus \\ S(-3,\ \ 0) \\ \oplus \\ S(\ \ 0,-2) \\ \oplus \\ S(-2,-1) \end{array} \longleftarrow \begin{array}{c} S(-2,-2) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-3,-1)^2 \end{array} \longleftarrow S(-3,-2) \longleftarrow 0\,.$$

On the other hand, a virtual resolution of S/I_Y is the shorter chain complex

$$S^1 \longleftarrow \begin{array}{c} S(-1,-1) \\ \oplus \\ S(-2,-1) \end{array} \longleftarrow S(-3,-1) \longleftarrow 0\,.$$

This virtual resolution indicates that Y is the intersection of the variety of a $(1,1)$-form, the solid lines, and the variety of a $(2,1)$-form, the dashed lines.

Figure 1: A variety of 3 points in $\mathbb{P}^1 \times \mathbb{P}^1$



A current goal of this research program is to develop the theory of the virtual Cohen–Macaulay property, in which the minimal length of a virtual resolution of M, denoted $\operatorname{vdim} M$, is equal to $\operatorname{codim} M$, i.e., the codimension of $\mathrm{Spec}(S/\operatorname{Ann}_S(M))$ in $\mathrm{Spec}(S)$. In this extended abstract, we will work in the Stanley–Reisner setting. Our main result provides a large class of virtually Cohen–Macaulay Stanley–Reisner rings.

Theorem 1.2. *Let S be the Cox ring of $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$. If Δ is an r-dimensional simplicial complex and its associated variety $\mathbb{V}(I_\Delta) \subseteq X$ is equidimensional, then S/I_Δ is virtually Cohen–Macaulay.*

Relationships between $\operatorname{vdim}(M)$ and $\dim(X)$ have been of interest since the introduction of virtual resolutions. In [2, Proposition 1.2, Theorem 5.1] a Hilbert Syzygy Theorem-type bound, $\operatorname{vdim}(M) \leq \dim(X)$, was given for an arbitrary $\operatorname{Pic}(X)$-graded S-module M when X is a product of projective spaces and for an arbitrary punctual scheme in any smooth projective toric variety X. Further, [10] shows that $\operatorname{vdim}(S/I) \leq \dim(X)$ when I is a relevant monomial ideal of S and X is a smooth projective toric variety. Our new result most directly compares with a similar theorem in the case of pure and balanced simplicial complexes, which are necessarily dimension of $r-1$ (see [5, Theorem 5.10]). Our proof is constructive, and we illustrate its use in building explicit resolutions in Examples 3.7 and 3.8.

The proof of Theorem 1.2 relies on the addition of faces to Δ that leave the corresponding variety unchanged. The results on this class are hard won through careful application of Reisner's criterion interpreted in a virtual setting together with an analysis of the spectral sequence associated to a certain nerve complex. This combination can be viewed in some respects as a hint towards a virtual Reisner's criterion, especially since the addition of new cells depends surprisingly little on the actual structure of Δ.

We will also discuss methods for building on an understanding of Stanley–Reisner rings to achieve results outside of the squarefree monomial setting. In particular, we provide two methods to construct short virtual resolutions with the goal of establishing virtually Cohen–Macaulay modules outside of the squarefree monomial context. These constructions either improve on a longer resolution of the same module or on a short resolution of a closely-related module.

Acknowledgements

We would like to thank Daniel Erman and Gregory G. Smith for helpful conversations related to this work.

2 Background

2.1 Virtual resolutions and the virtual Cohen–Macaulay property

Throughout this extended abstract, let X be a smooth projective toric variety over the algebraically closed field k, and let $S = \operatorname{Cox}(X)$. All S-modules are assumed to be finitely generated and $\operatorname{Pic}(X)$-graded and all sheaves coherent.

Let M be an S-module. As in [2, Definition 1.1], a free complex $F_\bullet := [F_0 \longleftarrow F_1 \longleftarrow \cdots]$ is a *virtual resolution* of M (or of $\widetilde{M}$) if the corresponding complex $\widetilde{F}_\bullet$ of vector bundles on X is a locally-free resolution of the sheaf $\widetilde{M}$. Next, define the *virtual dimension* of M, denoted $\operatorname{vdim} M$, to be the minimal length of a virtual resolution for M. Recall that $\operatorname{codim} M$ is defined to be the codimension of $\operatorname{Spec}(S/\operatorname{Ann}_S(M))$ in $\operatorname{Spec}(S)$. As noted in [2, Proposition 2.5], working over products of projective space, there is an inequality $\operatorname{vdim} M \geqslant \operatorname{codim} M$; in light of this, we say that M is *virtually Cohen–Macaulay* if $\operatorname{vdim} M = \operatorname{codim} M$, the minimum possible. This definition mirrors the affine case, in which, by the Auslander–Buchsbaum formula, an S-module M is Cohen–Macaulay if and only if its projective dimension is equal to its codimension. Moreover, because every free resolution is a virtual resolution, every Cohen–Macaulay S-module is virtually Cohen–Macaulay. We say that a subscheme $V \subset X$ is *virtually Cohen–Macaulay* if its Cox ring is virtually Cohen–Macaulay as an S-module.

2.2 The Stanley–Reisner correspondence

The now-classical Stanley–Reisner correspondence is a correspondence between quotients of polynomial rings by squarefree monomial ideals and simplicial complexes. It was first developed in the study of the Upper Bound Conjecture and now plays a central role at the intersection of combinatorics, commutative algebra, and algebraic geometry. For a detailed introduction, we refer the reader to [6].

Definition 2.1. Let Δ be a simplicial complex on $\{1, 2, \ldots, n\}$ and $R = k[x_1, \ldots, x_n]$. Define the *Stanley–Reisner ideal* of Δ to be

$$I_\Delta = \langle x_{i_i} \cdots x_{i_k} \mid \{i_i, \ldots, i_k\} \notin \Delta \rangle$$

and the *Stanley–Reisner ring* of Δ to be R/I_Δ.

Reisner showed in his thesis that R/I_Δ is Cohen–Macaulay if and only if Δ is Cohen–Macaulay as a simplicial complex. He used what is now known as Reisner's criterion, which detects if R/I_Δ Cohen–Macaulay by the vanishing of certain reduced simplicial homology modules of links of the faces of Δ. Careful application of this result in the virtual setting is key to our proof of Theorem 1.2.

3 Virtually Cohen–Macaulay Stanley–Reisner rings

In this section, we will highlight the main steps in the proof of Theorem 1.2 and give examples illustrating those steps.

For a vector $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ with $|\vec{n}| = \sum_{i=1}^r (n_i + 1)$, let $X = \mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. As a product of projective spaces, X is a smooth projective toric variety. As usual throughout this extended abstract, we will take $S = \mathrm{Cox}(X)$ and B to be the irrelevant ideal of S. In this section, we consider simplicial complexes with the vertex set $\mathcal{X}$ corresponding to the $|\vec{\boldsymbol{n}}|$-variables $(x_{i,j})_{1\le i\le r, 0\le j\le n_i}$ in S. Endow the vertices in $\mathcal{X}$ corresponding to $x_{i,\bullet}$ with *color i*.

Let Δ be a simplicial complex with vertices in $\mathcal{X}$. Define the *color set of a simplex* $\sigma \in \Delta$ to be the set of the colors of the vertices of σ, denoted by $\mathrm{color}(\sigma)$. We say a simplex $\sigma \in \Delta$ is *relevant* if $\mathrm{color}(\sigma) = \{1, 2, \ldots, r\}$ and *irrelevant* otherwise. A simplicial complex Δ is *relevant* if it contains at least one relevant face, and it is *irrelevant* otherwise. Note that if Δ is an irrelevant simplicial complex on $\mathcal{X}$, then S/I_Δ is irrelevant, meaning that the support of S/I_Δ is contained in $\mathbb{V}(B) = \{P \in \mathrm{Spec}(S) \mid B \subseteq P\}$. A relevant simplicial complex Δ is *relevant-connected* if its geometric realization is (topologically) connected after removing the realization of $\mathcal{B} := \{\sigma \subseteq \mathcal{X} \mid \sigma \text{ is irrelevant}\}$. Further, a subcomplex of Δ is called a *relevant-connected component* if it is maximal among relevant-connected subcomplexes of Δ.

The first step in the proof of Theorem 1.2 is to prove the result in the relevant-connected case, a case we will then reduce to.

Theorem 3.1. *If Δ is an r-dimensional relevant-connected simplicial complex on $\mathcal{X}$, then S/I_Δ is virtually Cohen–Macaulay.*

The core principle behind the proof of this theorem is that if Δ and Δ' differ by only irrelevant simplicies and $S/I_{\Delta'}$ is Cohen–Macaulay, then S/I_Δ is virtually Cohen–Macaulay. Conveniently, the only modification we will need to make to our simplicial complex is to take the union with another simplicial complex, dependent only on the ambiant toric variety, namely $\mathcal{B}_r := \{\sigma \subseteq \mathcal{X} \mid \dim \sigma \le r, \sigma \text{ is irrelevant}\}$. One might additionally notice that this is precisely the r-skeleton of the simplicial complex $\mathcal{B}$ corresponding to the irrelevant ideal.

In particular, the simplicial complex $\mathcal{B}_r$ is itself close to being Cohen–Macualay, and the precise way in which it fails to be Cohen–Macualay is captured in Lemma 3.2, which in the proof of Theorem 1.2 is used along with the long exact sequence of a pair to compute the reduced homology modules necessary to apply Reisner's Criterion.

Lemma 3.2. *The ring $S/I_{\mathcal{B}_r}$ is Cohen–Macaulay on the punctured spectrum. Also, $\widetilde{H}_{r-2}(\mathcal{B}_r; k) = k$ and $\widetilde{H}_i(\mathcal{B}_r; k) = 0$ for all $i < r$ with $i \neq r-2$.*

Now with this simplicial complex, we can introduce the notion of interior and exterior faces, which are not only essential ingredients in the proof but also give rise to pictures that build our intuition for which aspects of a simplicial complex may or may not contribute enough simplicial homology to preclude its being virtually Cohen–Macaulay.

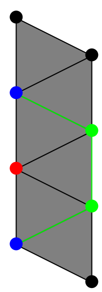
If σ is a face of the simplicial complex Δ, define the *link of σ in* Δ to be

$$\mathrm{lk}_\sigma(\Delta) = \{\sigma' \in \Delta \mid \sigma \cup \sigma' \in \Delta, \sigma \cap \sigma' = \varnothing\}.$$

For a relevant simplicial complex Δ and $\sigma \neq \emptyset$ a face of Δ, let $\mathrm{Ex}(\sigma, \Delta) = \mathrm{lk}_\sigma(\Delta) \cap \mathrm{lk}_\sigma(\mathcal{B}_r)$, and call these simplices the *exterior* faces of $\mathrm{lk}_\sigma(\Delta)$. Call the remaining faces of $\mathrm{lk}_\sigma(\Delta)$ the *interior* faces of $\mathrm{lk}_\sigma(\Delta)$. Then via some homological computations, questions about the homologies of links in $\Delta \cup \mathcal{B}_r$ can be reduced to careful analysis of the interior and exterior faces.

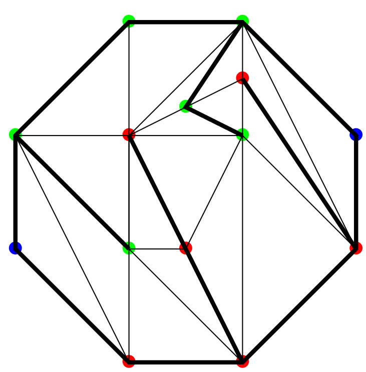
Example 3.3. Consider the following example in $\mathbb{P}^3 \times \mathbb{P}^3$, where the left vertical line in Figure 2 is colored by the first copy of $\mathbb{P}^3$ and the right vertical line is colored by the second copy of $\mathbb{P}^3$. Consider the link of the red vertex, which is the union of the green and blue faces, where the interior faces are in green and the exterior faces are in blue. Notice that the green faces on the right hand edge of the diagram are irrelevant, but are still interior faces.

Figure 2: Each column of vertices corresponds to a copy of $\mathbb{P}^3$.



The idea of interior and exterior faces can become considerably more complex. Consider the following illustration of the link of a cell in an example Δ on $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell$. In Figure 3, the vertices corresponding to each of the parts of the product are colored red, blue, and green. Only the link is illustrated, and it is the link of a vertex that would be colored blue. Then the bold faces are the exterior faces, and the remainder are interior faces.

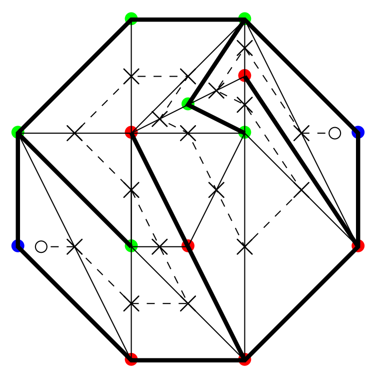
Figure 3: The link in some Δ of a certain blue vertex on $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell$.



Lemma 3.4. *Let Δ be a pure, relevant simplicial complex of dimension r and $\sigma \neq \emptyset$ be a face of Δ. Then every facet of the link $\mathrm{lk}_\sigma(\Delta)$ has at most two facets of its own that are interior faces of $\mathrm{lk}_\sigma(\Delta)$. Moreover, a face of $\mathrm{lk}_\sigma(\Delta)$ has exactly one interior face if it shares a color with σ.*

Example 3.5. Continuing with Example 3.3, one of the critical steps in the proof of Theorem 3.1 is the reduction of the some of the more troublesome homology groups to the homology of a graph by the construction of the graph given by the interior faces of the link. It is Lemma 3.4 that allows such a graph to be constructed. In Figure 4 that graph is shown with the vertices given by $\times$ symbols, the edges given by dashed lines, and the half edges illustrated with a edge terminated by a $\circ$ symbol.

Figure 4: The graph associated to the complex of interior faces of the link from Figure 3.



In light of Theorem 3.1, we see that Theorem 1.2 will follow if we can show that that S/I_Δ is virtually Cohen–Macaulay on each of the components of its support. The required result follows.

Proposition 3.6. *Suppose that M is module with equidimensional support $X = \bigsqcup X_i$ with disjoint components X_i, then M is virtually Cohen–Macaulay if each $M|_{X_i}$ is virtually Cohen–Macaulay.*

The behavior in Proposition 3.6 can be illustrated in the following example, a union of two disjoint lines in a 3-dimensional projective space, which is virtually Cohen–Macaulay but not arithmetically Cohen–Macaulay.

Example 3.7. Let $S = k[x_0, \dots, x_3]$ be the Cox ring of $X = \mathbb{P}^3$, and consider the ideal $J = \langle x_0, x_1 \rangle \cap \langle x_2, x_3 \rangle$. Notice that $S/\langle x_0, x_1 \rangle$ and $S/\langle x_2, x_3 \rangle$ are Cohen–Macaulay but S/J is not, as can be seen from its minimal free resolution, whose length is 3, exceeding

$2 = \operatorname{codim} S/J$:

$$S^1 \xleftarrow{\begin{bmatrix} x_0x_2 & x_1x_2 & x_0x_3 & x_1x_3 \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} -x_1 & 0 & -x_3 & 0 \\ x_0 & 0 & 0 & -x_3 \\ 0 & -x_1 & x_2 & 0 \\ 0 & x_0 & 0 & x_2 \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} x_3 \\ -x_2 \\ -x_1 \\ x_0 \end{bmatrix}} S^1 \leftarrow 0.$$

Since J corresponds to a 1-dimensional simplicial complex with a single color, Theorem 1.2 implies that S/J is virtually Cohen–Macaulay, with a short virtual resolution of the form

$$S^2 \xleftarrow{\begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_3 \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} -x_1 & 0 \\ x_0 & 0 \\ 0 & -x_3 \\ 0 & x_2 \end{bmatrix}} S^2 \leftarrow 0.$$

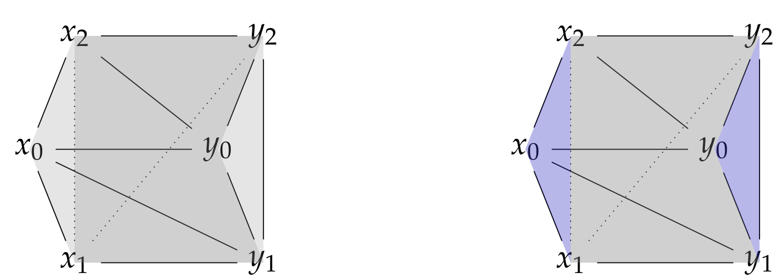
We now give an example illustrating the difference between the virtually Cohen–Macaulay and arithmetically Cohen–Macaulay properties when working over the Cox ring of a product of projective spaces.

Example 3.8. Let $X = \mathbb{P}^2 \times \mathbb{P}^2$, and consider the simplicial complex Δ that is homeomorphic to a a cylinder, as shown in Figure 5.

The Stanley–Reisner ideal corresponding to Δ is $I_\Delta = \langle x_0y_2, x_1y_0, x_2y_1, x_0x_1x_2, y_0y_1y_2 \rangle$. Since $\widetilde{H}_1(\Delta; k) \neq 0$ and $\dim \Delta = 2$, Reisner's criterion implies that S/I_Δ is not Cohen–Macaulay.

Recall that the essential ingredient of our technique is to take the simplicial complex given by the union of Δ and an $\mathcal{B}_r$, so here we can consider $\mathcal{B}_2 \cup \Delta$. This is illustrated in Figure 5 and corresponds to the ideal $J = \langle x_0y_2, x_1y_0, x_2y_1 \rangle$, then one can check that Reisner's criterion is satisfied in this case. Since $\widetilde{I_\Delta} = \widetilde{J}$, we conclude that S/I_Δ is virtually Cohen–Macaulay.

Figure 5: A cylindrical Δ on $\mathbb{P}^2 \times \mathbb{P}^2$, to which adding irrelevant faces produces a Cohen–Macaulay complex



We conclude this section with an example showing that, even when J is a monomial ideal, S/J being virtual Cohen–Macaulay is not determined by the property holding for $S/\sqrt{J}$. As such, being virtual Cohen–Macaulay is a scheme-theoretic property rather than a set-theoretic one.

Example 3.9. If $S = k[x_0, x_1, x_2]$ is the Cox ring of $\mathbb{P}^2$ and $M = S/\langle x_0^2, x_0x_1\rangle$, then there is no $f \in S$ so that $(S/\langle f\rangle)^\sim \cong (S/I)^\sim$. Therefore, the virtual dimension of M is at least 2 while $\operatorname{codim} M = 1$, and so M is not virtually Cohen–Macaulay. However, $\sqrt{\langle x_0^2, x_0x_1\rangle} = \langle x_0\rangle$ and $S/\langle x_0\rangle$ is clearly arithmetically Cohen–Macaulay and so also virtually Cohen–Macaulay. We can view this example as suggesting that the Stanley–Reisner theory in this section is capturing some well-behavedness of virtual resolutions in the combinatorial setting of simplicial complexes.

4 New virtual resolutions from old

In this section, we give a construction for shortening a known virtual resolution in Theorem 4.1. We then propose a notion of a *virtually regular element* f, which allows us to describe cases when we can construct a virtual resolution for $S/(J + \langle f\rangle)$ from one for S/J in Proposition 4.5. These tools allow us to expand on the foundation we built in the previous section to move beyond the squarefree monomial setting in our consideration of the virtual Cohen–Macaulay property.

4.1 The mapping cone construction

We will begin by stating our main result to construct a shorter virtual resolution from a longer one. In our full paper, the construction, which makes use of a mapping cone, is described in detail. We will then return to Example 3.7 to compare and contrast the virtual resolution construction using Stanley–Reisner theory in Section 3 and the one presented in this section using mapping cones. We will then give an example of the mapping cone construction applied to an example that does not come from a squarefree monomial ideal.

Let X be a smooth projective toric variety with $\operatorname{Pic}(X)$-graded Cox ring S, and let M be a finitely generate $\operatorname{Pic}(X)$-graded S-module. For instance, X can be $\mathbb{P}^{\vec{n}}$, as in Section 3.

Theorem 4.1. *Let $F_\bullet$ be a virtual resolution of M of length t such that $\operatorname{Ext}^t_S(M, S)^\sim = 0$. If $\operatorname{Ext}^t_S(M, S)$ admits a free resolution of length at most $t + 2$, then we can construct a virtual resolution of M of length $t - 1$.*

Example 4.2. Referring again to Example 3.7 of the ideal $J = \langle x_0x_2, x_0x_3, x_1x_2, x_1x_3\rangle$, the mapping cone construction of Theorem 4.1 also yields a short virtual resolution of

S/J. To give the reader a picture of how the mapping cone construction merges the resolutions of M and $\operatorname{Ext}^3_S(S/J, S)$ without stating the full details, we will record the degree shifts of the free S-modules involved in these resolutions. In that format, the minimal free resolution of S/J is

$$S \leftarrow S(-2)^4 \leftarrow S(-3)^4 \leftarrow S(-4) \leftarrow 0.$$

Take the mapping cone of the following map of chain complexes, where the bottom chain complex is the dual of the free resolution of $\operatorname{Ext}^3(S/J, S) \cong k$.

$$\begin{array}{ccccccccccccc}
\cdots \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & S & \longleftarrow & S(-2)^4 & \longleftarrow & S(-3)^4 & \longleftarrow & S(-4) & \longleftarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longleftarrow & 0 & \longleftarrow & S & \longleftarrow & S(-1)^4 & \longleftarrow & S(-2)^6 & \longleftarrow & S(-3)^4 & \longleftarrow & S(-4) & \longleftarrow 0.
\end{array}$$

The mapping cone yields

$$S^2 \leftarrow \begin{array}{c} S(-1)^4 \\ \oplus \\ S(-2)^4 \end{array} \leftarrow \begin{array}{c} S(-2)^6 \\ \oplus \\ S(-3)^4 \end{array} \leftarrow \begin{array}{c} S(-3)^4 \\ \oplus \\ S(-4) \end{array} \leftarrow S(-4) \leftarrow 0,$$

which after minimizing provides the following virtual resolution of S/J of length $\operatorname{codim} S/J = 2$:

$$S^2 \leftarrow S(-1)^4 \leftarrow S(-2)^2 \leftarrow 0. \tag{4.1}$$

Note that this resolution can also be constructed using the techniques of sheaves over simplicial complexes of [9].

The resolution in (4.1) is different from that obtained using Stanley–Reisner theory as Example 3.7. This shows us that these two constructions, even applied to the same relatively small example, are substantively different. It also highlights the fact, known well since [2], that virtual resolutions of minimal length are typically not unique.

The next example shows the strength of the mapping cone construction outside of the squarefree monomial setting.

Example 4.3. Consider the hyperelliptic curve C of genus 4 that can be embedded as a curve of bidegree $(2, 8)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ found in [2, Example 1.4]. If I_C denotes the B-saturated ideal for C, then

$$I_C = \left\langle \begin{array}{l} x_{1,1}^3x_{2,0} - x_{1,1}^3x_{2,1} + x_{1,0}^3x_{2,2},\ x_{1,0}^2x_{2,0}^2 + x_{1,1}^2x_{2,1}^2 + x_{1,0}x_{1,1}x_{2,2}^2,\ x_{1,1}^2x_{2,0}^3 - x_{1,1}^2x_{2,0}^2x_{2,1} - x_{1,0}x_{1,1}x_{2,1}^2x_{2,2} - x_{1,0}^2x_{2,2}^3,\ x_{1,0}x_{1,1}x_{2,0}^3 + \\ x_{1,0}x_{1,1}x_{2,0}^2x_{2,1} - x_{1,0}^2x_{2,1}^2x_{2,2} + x_{1,1}^2x_{2,0}x_{2,2}^2 + x_{1,1}^2x_{2,1}x_{2,2}^2,\ x_{1,1}x_{2,0}^3x_{2,1}^2 + x_{1,1}x_{2,0}^2x_{2,1}^3 - x_{1,0}x_{2,1}^4x_{2,2} - x_{1,0}x_{2,0}^3x_{2,2}^2 + x_{1,0}x_{2,0}^2x_{2,1}x_{2,2}^2 - \\ x_{1,1}x_{2,0}x_{2,2}^4 - x_{1,1}x_{2,1}x_{2,2}^4,\ x_{1,1}x_{2,0}^5 + x_{1,1}x_{2,0}^4x_{2,1} - x_{1,0}x_{2,0}^2x_{2,1}^2x_{2,2} + x_{1,1}x_{2,1}^2x_{2,2}^3 + x_{1,0}x_{2,2}^5,\ x_{1,0}x_{2,0}^5 + x_{1,0}x_{2,0}^4x_{2,1} + x_{1,1}x_{2,1}^4x_{2,2} + \\ x_{1,1}x_{2,0}^3x_{2,2}^2 + x_{1,1}x_{2,0}^2x_{2,1}x_{2,2}^2 + x_{1,0}x_{2,1}^2x_{2,2}^3,\ x_{2,0}^8 + 2x_{2,0}^7x_{2,1} + x_{2,0}^6x_{2,1}^2 + x_{2,1}^6x_{2,2}^2 + 3x_{2,0}^3x_{2,1}^2x_{2,2}^3 + 3x_{2,0}^2x_{2,1}^3x_{2,2}^3 - x_{2,0}x_{2,2}^7 - x_{2,1}x_{2,2}^7 \end{array} \right\rangle.$$

Now S/I_C has minimal free resolution

$$S^1 \longleftarrow \begin{array}{c} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{array} \longleftarrow \begin{array}{c} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{array} \longleftarrow \begin{array}{c} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{array} \longleftarrow S(-3,-7)^1 \longleftarrow 0\,.$$

Applying Theorem 4.1 to this free resolution and then again on the virtual resolution constructed by that process produces a virtual resolution for S/I_C of the form

$$\begin{array}{c} S^1 \\ \oplus \\ S(0,-1)^2 \\ \oplus \\ S(0,-2)^1 \\ \oplus \\ S(-0,-2)^1 \end{array} \leftarrow \begin{array}{c} S(-1,-1)^2 \\ \oplus \\ S(-1,-2)^1 \\ \oplus \\ S(0,-3)^1 \\ \oplus \\ S(-1,-2)^1 \\ \oplus \\ S(0,-3)^1 \\ \oplus \\ S(-1,-1)^1 \\ \oplus \\ S(0,-3)^2 \end{array} \leftarrow S(-1,-3)^5 \longleftarrow 0.$$

Because $\operatorname{codim} S/I_C = 2$, this construction shows that S/I_C is virtually Cohen–Macaulay.

4.2 The quotient by a virtually regular element

We conclude by giving an example of how one can take virtual resolution of some quotient ring S/J and a virtually regular element f and construct a virtual resolution of $S/(J + \langle f \rangle)$ of length exactly one greater than the original resolution. This process gives us the capacity to build from a squarefree monomial example that we understand to the setting of more general quotients of polynomial rings.

Definition 4.4. Let f be a $\operatorname{Pic}(X)$-graded element of S and M an S-module. If $\operatorname{Ann}_M f$ is irrelevant and $\dim M/fM = \dim M - 1$, then we say that f is *virtually regular on* M or that f is a *virtually regular element on* M.

It is immediate that any regular element on M is virtually regular and that no element of a minimal prime of M can be virtually regular. The additional flexibility gained in considering virtually regular elements over regular elements alone is that an element of an embedded associated prime of M can be virtually regular if its annihilator is sufficiently well controlled. Notice also that if M' is an S-module satisfying $\widetilde{M'} = \widetilde{M}$, then f is virtually regular on M if and only if f is virtually regular on M'.

Proposition 4.5. *If M has a virtual resolution of length ℓ and f is a virtually regular element on M, then M/fM has a virtual resolution of length $\ell + 1$. In particular, if M is virtually Cohen–Macaulay, then M/fM is virtually Cohen–Macaulay.*

Example 4.6. Let $S = k[x_0, \ldots, x_5]$ be the Cox ring of $\mathbb{P}^5$, $\mathfrak{m} = \langle x_0, \ldots, x_5 \rangle$, and

$$J = \langle x_0, x_1, x_2 \rangle \cap \langle x_3, x_4, x_5 \rangle.$$

With $M = S/J$ and $F_\bullet$ equal to the minimal free resolution of M, the construction in Theorem 4.1 yields a virtual resolution of M of length $\operatorname{codim} M = 3$, which shows that M is virtually Cohen–Macaulay. We claim that $x_2 - x_5$ is a virtually regular element on M, that $x_1 - x_4$ is a virtually regular element on $M/\langle x_2 - x_5 \rangle M$, and that $x_0 - x_3$ is a virtually regular element on $M/\langle x_2 - x_5, x_1 - x_4 \rangle M$. Because $x_2 - x_5$ is a regular element, it is automatically a virtually regular element. Observe that

$$\begin{aligned} \overline{M} := \frac{M}{\langle x_2 - x_5 \rangle M} &\cong \frac{S}{\langle x_0x_3, x_0x_4, x_0x_2, x_1x_3, x_1x_4, x_1x_2, x_2x_3, x_2x_4, x_2^2, x_5 \rangle} \\ &\cong \frac{S}{\langle x_0, x_1, x_2, x_5 \rangle \cap \langle x_2, x_3, x_4, x_5 \rangle \cap \langle x_0, x_1, x_2^2, x_3, x_4, x_5 \rangle}. \end{aligned}$$

Now $x_1 - x_4$ is not a regular element on $\overline{M}$, but it is not in either minimal prime of $\overline{M}$, and so $\dim \overline{M} = 1 + \dim \overline{M}/(x_1 - x_4)\overline{M}$. The isomorphism presented above is given by $x_i \mapsto x_i$ for $i \neq 5$ and $x_5 \mapsto x_2 - x_5$. After application of this isomorphism, it is easy to see that $\operatorname{Ann}_{\overline{M}}(x_1 - x_4) = \langle x_2 \rangle \overline{M}$, which is irrelevant. Hence, $x_1 - x_4$ is a virtually regular element that is not a regular element on $\overline{M}$. A similar computation shows that $x_0 - x_3$ is in an embedded prime of $M/\langle x_2 - x_5, x_1 - x_4 \rangle M$ and has, after applying an analogous isomorphism to the one described above, an irrelevant annihilator generated by x_1 and x_2.

Therefore, since M is virtual Cohen–Macaulay, so are each of the modules $M/\langle x_2 - x_5 \rangle M$, $M/\langle x_2 - x_5, x_1 - x_4 \rangle M$, and $M/\langle x_2 - x_5, x_1 - x_4, x_0 - x_3 \rangle M$ by Proposition 4.5.

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