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Bi-symmetric multiple equidistributions on ascent sequences

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Abstract. It is well known since the seminal work by Bousquet-Mélou, Claesson, Dukes and Kitaev (2010) that certain refinements of the ascent sequences with respect to several natural statistics are in bijection with corresponding refinements of $(\mathbf{2}+\mathbf{2})$-free posets and permutations that avoid a bivincular pattern. Different multiply-refined enumerations of ascent sequences and other bijectively equivalent structures have subsequently been extensively studied by various authors.

We contribute new bi-symmetric equidistributions to this subject. Our main result is a bijective proof of a bi-symmetric septuple equidistribution of statistics on ascent sequences, involving the number of ascents (asc), the number of repeated entries (rep), the number of zeros (zero), the number of maximal entries (max), the number of right-to-left minima (rmin), and two additional statistics. We further establish a new transformation formula for non-terminating basic hypergeometric ${}_4\phi_3$ series expanded as an analytic function in base q around $q=1$, which is utilized to prove two (bi)-symmetric quadruple equidistributions on ascent sequences.

A by-product of our findings includes the affirmation of a conjecture about the bi-symmetric equidistribution between the quadruples of Euler–Stirling statistics $(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})$ and $(\mathsf{rep}, \mathsf{asc}, \mathsf{max}, \mathsf{zero})$ on ascent sequences, that was motivated by a double Eulerian equidistribution due to Foata (1977) and recently proposed by Fu, Lin, Yan, Zhou and the first author (2018).

Keywords: ascent sequences, equidistributions, Euler–Stirling statistics, Fishburn numbers, basic hypergeometric series

1 Introduction

In the seminal paper [4] by Bousquet-Mélou, Claesson, Dukes and Kitaev, ascent sequences were introduced, as they are in bijection with several different combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets, certain bivincular pattern-avoiding permutations, Stoimenow's involution and regular linearized chord diagrams [25, 26]. Several natural statistics on posets, permutations and sequences are also kept track of by a sequence of bijections established by these authors. Since then, various joint distributions

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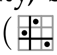
of classical statistics on ascent sequences and many other bijectively equivalent structures including Fishburn matrices [10, 11] and $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences have been intensively explored; see for instance [7, 9, 8, 17, 18, 20, 21, 23].

Recently, Fu, Lin, Yan, Zhou and the first author [13] discovered a new decomposition of ascent sequences which contributes to a systematic study of Eulerian and Stirling statistics on ascent sequences, certain pattern-avoiding permutations and $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences. In particular, their work led them to conjecture the bi-symmetry of a quadruple Euler–Stirling statistics on ascent sequences (see Conjecture 1.4) that is motivated by a double Eulerian equidistribution due to Foata [12]. However, it appears that the use of the new decomposition from [13] is not sufficient to prove this bi-symmetry conjecture.

In our present work, we affirm this conjecture in two different ways: one by developing a second new decomposition of ascent sequences; and the other one by identifying the generating function of the quadruple statistics as a basic hypergeometric series to which a (newly obtained) transformation formula is applied. We start with some necessary definitions and then state the consequences of our results.

An *inversion sequence* $(s_1, s_2, \ldots, s_n)$ is a sequence of non-negative integers such that for all i, $0 \le s_i < i$. We denote by $\mathcal{I}_n$ the set of inversion sequences of length n, which is in one-to-one correspondence with the set $\mathfrak{S}_n$ of permutations of $[n] := \{1, 2, \ldots, n\}$ via the well known Lehmer code σ (see for instance [12, 22]). That is, for $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the map $\sigma : \mathfrak{S}_n \to \mathcal{I}_n$ is defined as

$$\sigma(\pi) = (s_1, s_2, \ldots, s_n), \quad \text{where } s_i := |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

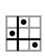
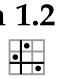
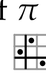
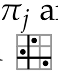
Some restrictions set up on permutations and inversion sequences could produce new sets of equal cardinality, but not necessarily through the Lehmer code. For instance, ascent sequences and ()-avoiding permutations (defined as below) are equinumerous.

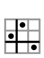
Definition 1.1 (Ascent sequence)**.** For any sequence $s \in \mathcal{I}_n$, let

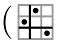
$$\mathsf{asc}(s) := |\{i \in [n-1] : s_i < s_{i+1}\}| \tag{1.1}$$

be the number of **asc**ents of s. An inversion sequence $s \in \mathcal{I}_n$ is an *ascent sequence* if for all $2 \le i \le n$, the s_i satisfy

$$s_i \le \mathsf{asc}(s_1, s_2, \ldots, s_{i-1}) + 1.$$

Definition 1.2 (()-avoiding permutation)**.** We say that a permutation $\pi \in \mathfrak{S}_n$ *avoids the pattern*  if there is no subsequence $\pi_i\pi_{i+1}\pi_j$ of π satisfying both $\pi_i - 1 = \pi_j$ and $\pi_i < \pi_{i+1}$. Otherwise we say π contains the pattern . Sometimes the pattern  is written as $2|3\bar{1}$.

The ()-avoiding permutations, more generally, permutations that avoid a specific bivincular pattern, were introduced and studied by Bousquet-Mélou, Claesson, Dukes and Kitaev [4] as both of them are surprisingly in bijection with other classical combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets [10, 11] and regular linearized chord diagrams [25, 26].

Let $\mathcal{A}_n$ and $\mathfrak{S}_n($$)$ be the sets respectively of ascent sequences and ()-avoiding permutations of length n. Bousquet-Mélou, Claesson, Dukes and Kitaev [4] proved that

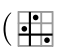
$|\mathcal{A}_n| = |\mathfrak{S}_n($$)| = [t^n]\sum_{k=1}^{\infty}\prod_{i=1}^{k}(1-(1-t)^i),$ (1.2)

and thus, as a consequence of a result by Zagier [26] (who discovered that the series on the right-hand side of (1.2) is the generating functions of the Fishburn numbers), $|\mathcal{A}_n|$ is equal to the n-th Fishburn number (see A022493 of the OEIS [24]). Their explicit values are given as

$$(|\mathcal{A}_n|)_{n\geq 1} = (1, 2, 5, 15, 53, 217, 1014, 5335, 31240, 201608, \ldots),$$

for which no closed form is known. The study of Fishburn numbers and their generalizations has remarkably led to many interesting results, including congruences [2, 14], asymptotic formulas [5, 16, 26], intriguing connections to transformations of hypergeometric series [1], modular forms [5, 26] and a variety of bijections [7, 9, 8, 17, 18, 20, 21, 23]. In particular, various members of the Fishburn family can be viewed as supersets of corresponding members of the Catalan family. Here the Fishburn (resp. Catalan) family refers to classes of combinatorial objects enumerated by the Fishburn (resp. Catalan) numbers.

This extended abstract, which is essentially an abridged version of [19], is devoted to the presentation of new bijective and basic hypergeometric aspects of Fishburn structures.

Let us review some classical statistics on ascent sequences and ()-avoiding permutations. For any sequence $s \in \mathcal{I}_n$, $\mathsf{asc}(s)$ is defined in (1.1). Let furthermore

$$\begin{aligned}
\mathsf{rep}(s) &:= n - |\{s_1, s_2, \ldots, s_n\}|,\\
\mathsf{zero}(s) &:= |\{i \in [n] : s_i = 0\}|,\\
\mathsf{max}(s) &:= |\{i \in [n] : s_i = i-1\}|, \quad \text{and}\\
\mathsf{rmin}(s) &:= |\{s_i : s_i < s_j \text{ for all } j > i\}|,
\end{aligned}$$

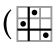
be the respective numbers of **rep**eated entries, **zero**s, **maxi**mal entries (or maximals for short) and **r**ight-to-left **min**ima of s. For instance, when $s = (0,1,2,0,1,3,5) \in \mathcal{I}_7$, then $\mathsf{asc}(s) = 5$, $\mathsf{rep}(s) = 2$, $\mathsf{zero}(s) = 2$, $\mathsf{max}(s) = 3$ and $\mathsf{rmin}(s) = 4$.

For any permutation $\pi \in \mathfrak{S}_n$, let

$$\begin{aligned}
\mathsf{des}(\pi) &:= |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|,\\
\mathsf{iasc}(\pi) &:= \mathsf{asc}(\pi^{-1}) = |\{i \in [n-1] : \pi_i + 1 \text{ appears to the right of } \pi_i\}|,
\end{aligned}$$

be the number of **des**ents and **i**nverse **asc**ents of π, respectively. Similar to rmin, the statistics lmin, lmax and rmax represent the numbers of **l**eft-to-right **min**ima, **l**eft-to-right **max**ima and **r**ight-to-left **max**ima, respectively.

Previous bijections developed in [4, 9, 13] preserve natural statistics on posets, permutations, sequences and matrices. As examples, we list below five pairs of equidistributed statistics that were established in those papers.

(asc, zero) on ascent sequences $\overset{1-1}{\longleftrightarrow}$ (des, lmax) on ()-avoiding permutations,
$\overset{1-1}{\longleftrightarrow}$ (mag, min) on $(\mathbf{2}+\mathbf{2})$-free posets,
$\overset{1-1}{\longleftrightarrow}$ $(\mathsf{dim}, \mathsf{rowsum}_1)$ on Fishburn matrices,
$\overset{1-1}{\longleftrightarrow}$ (rep, max) on $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences.

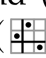
Remark 1.3. The statistics mag, min are abbreviations for **mag**nitude and the number of **min**imal elements of a poset; the statistics dim and rowsum_1 refer to **dim**ension and the **sum** of entries in the **first row** of a matrix.

In a recent paper [13] by Fu, Lin, Yan, Zhou and the first author, a joint *symmetric* distribution of statistics asc and rep over ascent sequences was discovered. The motivation came from a symmetric distribution of (asc, rep) on inversion sequences

$$\sum_{s\in\mathcal{I}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)} = \sum_{s\in\mathcal{I}_n} u^{\mathsf{rep}(s)}x^{\mathsf{asc}(s)}. \tag{1.3}$$

This is a direct consequence of a double Eulerian equidistribution due to Foata [12]:

$$\sum_{s\in\mathcal{I}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)} = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}(\pi)}x^{\mathsf{iasc}(\pi)}. \tag{1.4}$$

It turns out that not only (1.3) and (1.4) are true if $\mathcal{I}_n$ and $\mathfrak{S}_n$ are replaced by the corresponding subsets $\mathcal{A}_n$ and $\mathfrak{S}_n$(), but an even stronger result on a bi-symmetric equidistribution of Euler–Stirling statistics[1] over ascent sequences was conjectured.

Conjecture 1.4 ([13])**.** *For each $n\geq 1$, the following bi-symmetric quadruple equidistribution holds:*

$$\sum_{s\in\mathcal{A}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)}z^{\mathsf{zero}(s)}y^{\mathsf{max}(s)} = \sum_{s\in\mathcal{A}_n} u^{\mathsf{rep}(s)}x^{\mathsf{asc}(s)}z^{\mathsf{max}(s)}y^{\mathsf{zero}(s)}.$$

Remark 1.5. Conjecture 1.4 is equivalent to a bi-symmetric equidistribution between the quadruples (des, iasc, lmax, lmin) and (iasc, des, lmin, lmax) on ()-avoiding permutations, according to [13, Theorem 12].

Two results in approaching this conjecture were presented in [13]: one is an explicit formula for the generating function of ascent sequences with respect to the four

[1]We adopt the classification of statistics from [13]: any statistic whose distribution over a member of the Fishburn family equals the distribution of asc (resp. zero) on ascent sequences is called *an Eulerian* (resp. a Stirling) statistic. So according to Theorem 1.7, asc, rep are Eulerian statistics and zero, max, rmin are Stirling statistics.

statistics asc, rep, zero, max (see Theorem 1.6); and the other one is a quadruple equidistribution between (asc, rep, zero, max) and (rep, asc, rmin, zero) on ascent sequences (see Theorem 1.7).

Let $\mathcal{G}(t;x,y,u,z)$ denote the generating function of ascent sequences counted by the length (variable t), asc (variable u), rep (variable x), max (variable y) and zero (variable z). That is,

$$\mathcal{G}(t;x,y,u,z) := \sum_{n=1}^{\infty} t^n \sum_{s\in\mathcal{A}_n} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} u^{\mathsf{asc}(s)} z^{\mathsf{zero}(s)}. \tag{1.5}$$

Theorem 1.6 ([13])**.** *The generating function $\mathcal{G}(t;x,y,u,z)$ of ascent sequences is*

$$\mathcal{G}(t;x,y,u,z) = \sum_{m=0}^{\infty} \frac{zyrx^m(1-yr)(1-r)^m(x+u-xu)}{[x(1-u)+u(1-yr)(1-r)^m][x+u(1-x)(1-yr)(1-r)^m]} \times \prod_{i=0}^{m-1} \frac{1+(zr-1)(1-yr)(1-r)^i}{x+u(1-x)(1-yr)(1-r)^i}, \tag{1.6}$$

where $r = t\,(x+u-xu)$.

Theorem 1.7 ([13])**.** *There is a bijection $\Upsilon : \mathcal{A}_n \to \mathcal{A}_n$ which transforms the quadruple*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}) \ \textit{to}\ \ (\mathsf{rep}, \mathsf{asc}, \mathsf{rmin}, \mathsf{zero}).$$

Conjecture 1.4 can be settled, with the the help of Theorems 1.6 and 1.7, by showing either (I) or (II), described as follows.

(I) $\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y)$;

(II) the quadruple (asc, rep, zero, max) has the same distribution as (asc, rep, zero, rmin) over ascent sequences.

We are able to settle Conjecture 1.4 independently in both ways, (I) and (II).

2 Main results

2.1 A bi-symmetric septuple equidistribution

Our first main result (Theorem 2.1) is a bijective proof of a bi-symmetric *septuple* equidistribution on ascent sequences, which significantly generalizes (II) and consequently affirms Conjecture 1.4.

The five statistics asc, rep, zero, max, and rmin on inversion sequences were defined in the previous section. We now define two additional statistics on ascent sequences.

First we recall the definition of the statistic ealm on ascent sequences which was introduced in [13]. Let s be an ascent sequence with $\mathsf{max}(s) \neq |s|$, then $\mathsf{ealm}(s) =$

$s_{\mathsf{max}(s)+1}$, i.e., the entry right after the last maximal entry. For the ascent sequence $s = (0, 1, \ldots, |s| - 1)$ that has $\mathsf{max}(s) = |s|$, we set $\mathsf{ealm}(s) = 0$.

For example, $\mathsf{ealm}(0, 1, \mathbf{0}, 1, 3, 0, 2) = 0$.

Next, we introduce the new statistic rpos on ascent sequences. For the sake of convenience, we index all right-to-left minima from left to right starting from 0 (rather than from 1). For any ascent sequence s with $\mathsf{rmin}(s) \neq |s|$, define $\mathsf{rpos}(s) = m$ if m is the maximal index such that the m-th right-to-left minimum appears at least twice after the $(m-1)$-th right-to-left minimum. If no such m exists or $\mathsf{rmin}(s) = |s|$, then set $\mathsf{rpos}(s) = 0$.

For example, $\mathsf{rpos}(\mathbf{0}, \mathbf{0}, 1, 2, 3, 4) = 0$ and $\mathsf{rpos}(0, 0, 1, 2, 0, 1, 2, 1, \mathbf{3}, \mathbf{3}, 4) = 2$.

Having defined the two additional statistics ealm and rpos, we are now ready to state our main result.

Theorem 2.1. *There is a bijection $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ such that for all $s \in \mathcal{A}_n$,*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos})s = (\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{rmin}, \mathsf{rpos}, \mathsf{max}, \mathsf{ealm})\Phi(s). \tag{2.1}$$

The main idea to prove Theorem 2.1 relies on two *parallel* decompositions of ascent sequences that are in close relation to the two respective statistics ealm and rpos. The first decomposition was discovered in [13], while the second decomposition is new and plays an essential role in the proof of Theorem 2.1. The proof itself consists of a sequence of bijections with delicate subdivisions into cases. See [19] for the details.

2.2 A new basic hypergeometric transformation with applications

Our second main result (Theorem 2.2) is a new transformation formula of non-terminating basic hypergeometric ${}_4\phi_3$ series, valid as an identity expanded in base $q = 1 - r$ around $q = 1$, or, equivalently, $r = 0$. Basic hypergeometric series in base q expanded around $q = 1$ typically appear as (multiply-refined) generating functions of objects of the Fishburn family (compare with (1.2)) which is the reason of our interest in identities for such series, and, indeed, we successfully apply special cases of the new ${}_4\phi_3$ transformation to prove equidistribution results.

For convenience, we recall some standard notions from the theory of basic hypergeometric series, cf. [15].

For indeterminates a and q (the latter is referred to as the base), and non-negative integer k, the basic shifted factorial (or q-shifted factorial) is defined as

$$(a; q)_k := \prod_{j=1}^{k} (1 - aq^{j-1}).$$

This also makes sense for $k = \infty$, where the infinite product is viewed as a formal power series in q (whereas, viewed as an analytic expression in q, we would need to insist on $|q| < 1$, for convergence). When dealing with products of q-shifted factorials, it is convenient to use the following short notation,

$$(a_1, \ldots, a_m; q)_k := (a_1; q)_k \cdots (a_m; q)_k,$$

where again k is a non-negative integer or ∞.

An ${}_\alpha\phi_\beta$ basic hypergeometric series with α upper parameters $a_1, \ldots, a_\alpha$, and β lower parameters $b_1, \ldots, b_\beta$, base q and argument z is defined as

$$ {}_\alpha\phi_\beta\left[\begin{matrix} a_1, \ldots, a_\alpha \\ b_1, \ldots, b_\beta \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_\alpha; q)_k}{(q, b_1, \ldots, b_\beta; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+\beta-\alpha} z^k. \tag{2.2} $$

The series in (2.2) (where the lower parameters are assumed to be chosen such that no poles occur in the summands of the series) terminates if one of the upper parameters, say a_1, is of the form q^{-n}. Since $(q^{-n}; q)_k = 0$ for $k > n$, the series in that case contains only finitely many non-vanishing terms. If the series does not terminate, one usually imposes $|q| < 1$. See [15, Sec. 1.2] for conditions under which the series converges.

One of the most important identities in the theory of basic hypergeometric series is the Sears transformation [15, (III.15)],

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right] = \frac{(e/a, de/bc; q)_n}{(e, de/abc; q)_n} \, {}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, de/bc \end{matrix}; q, q\right]. \tag{2.3} $$

In (2.3), a, b, c, d, e and q are indeterminates and n is a non-negative integer (which is responsible that both ${}_4\phi_3$ series are actually finite sums and each contains only $n+1$ non-vanishing terms).

With the relevant definitions and ingredients of proof given, we are ready to state the main result of this subsection.

Theorem 2.2. *Let a, b, c, d, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominator factors in (2.4) have vanishing constant term in r, we have the following transformation of convergent power series in a and r:*

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} (1-r)^j, 1-a, b, c \\ d, e, (1-r)^{j+1}(1-a)bc/de \end{matrix}; 1-r, 1-r\right] \\ &= \frac{((1-r)/e, (1-r)(1-a)bc/de; 1-r)_j}{((1-r)(1-a)/e, (1-r)bc/de; 1-r)_j} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} (1-r)^j, 1-a, d/b, d/c \\ d, de/bc, (1-r)^{j+1}(1-a)/e \end{matrix}; 1-r, 1-r\right]. \end{aligned} \tag{2.4} $$

While for non-terminating basic hypergeometric series in base q we usually consider expansions around $q = 0$, here (and more generally, when dealing with generating functions of members of the Fishburn family) we are dealing with power series in r, which can be written as basic hypergeometric series in base $q = 1 - r$, thus can be viewed as functions analytic around $q = 1$. We need to be cautious when we resort to non-terminating identities for basic hypergeometric series.

The first part of the argument in the proof of Theorem 2.2, as our main result in this section, is similar to that used by Andrews and Jelínek in [1] for establishing q-series identities around $q = 1$.

Proof of Theorem 2.2. Observe that both sides of the identity converge as power series in a, thus are analytic functions in a. Indeed, for each $m \geq 0$ the expansion of $(1-a;1-r)_m$ in monomials $a^i r^l$ only involves terms with $i+l \geq m$ and each factor in the denominator of the series has a non-vanishing constant term. Thus, in the expansion of the series in the variables a and r the contribution of coefficients for each monomial $a^i r^l$ is finite.

Now both sides of (2.4) agree for $a = 1-(1-r)^{-n}$ where $n = 0,1,2,\ldots$ by the $(q,a,b,c,d,e) \mapsto (1-r,(1-r)^j,b,c,d,e)$ special case of the transformation in (2.3). Since we have shown (2.4) for infinitely many values of a accumulating at $a=-\infty$, i.e. $1-a=\infty$, by the identity theorem in complex analysis the transformation (2.4) is true for all a in its domain of analyticity. □

We utilize special cases of Theorem 2.2 to give analytic proofs of two different quadruple (bi)-symmetric equidistributions of Euler–Stirling statistics on ascent sequences, collected in Theorem 2.3. The first application of Theorem 2.2 is a proof of (I) by making use of the explicit form of the generating function in Theorem 1.6, and thus constitutes a non-combinatorial proof of the bi-symmetric equidistribution in Conjecture 1.4, while the second application establishes a symmetric equidistribution by employing a new explicit generating function obtained by a refined recursive construction of ascent sequences from [13].

Theorem 2.3. *For the generating function defined in (1.5), we have the bi-symmetry*

$$\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y). \tag{2.5}$$

Furthermore, define

$$\mathfrak{G}(t;x,y,u,v) := \sum_{n=1}^{\infty} t^n \sum_{s\in\mathcal{A}_n} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} u^{\mathsf{asc}(s)} v^{\mathsf{rmin}(s)}, \tag{2.6}$$

then we have, with $r = t(x+u-xu)$,

$$\begin{aligned}\mathfrak{G}(t;x,y,u,v) &= \frac{vyt}{1-vytu} + \sum_{m=0}^{\infty} \frac{rv(1-yr)(1-r)^m}{(x-xu+u(1-yr)(1-r)^m)(1-tuvy)} \\ &\times \prod_{i=0}^{m} \frac{x(1-(1-yr)(1-r)^i)(x-xu+u(1-yr)(1-r)^i)}{(x-u(x-1)(1-yr)(1-r)^i)(x-xu+u(1-rv)(1-yr)(1-r)^i)},\end{aligned} \tag{2.7}$$

and the symmetry

$$\mathfrak{G}(t;x,y,u,v) = \mathfrak{G}(t;x,v,u,y), \tag{2.8}$$

Remark 2.4. In the language of bijections, the (bi)-symmetric equidistributions in Theorem 2.3 mean that for any ascent sequence $s \in \mathcal{A}_n$,

$$\begin{aligned}(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max})s &= (\mathsf{rep},\mathsf{asc},\mathsf{max},\mathsf{zero})\Upsilon^{-1}(\Phi(s)),\\ (\mathsf{asc},\mathsf{rep},\mathsf{max},\mathsf{rmin})s &= (\mathsf{asc},\mathsf{rep},\mathsf{rmin},\mathsf{max})\Phi(s),\\ (\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{rmin})s &= (\mathsf{rep},\mathsf{asc},\mathsf{rmin},\mathsf{zero})\Upsilon(\Phi(s)),\end{aligned}$$

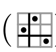
where Υ and Φ are the bijections respectively in Theorems 1.7 and 2.1.

Remark 2.5. We are not the first ones to study equivalent forms for generating functions of objects of the Fishburn family using tools from basic hypergeometric series. Initiating with work of Zagier [26] who established the basic hypergeometric series in (1.2) as a concrete form of the generating function $\mathcal{G}(t;1,1,1,1)$ for the Fishburn numbers, Andrews and Jelínek [1] subsequently proved three equivalent forms of $\mathcal{G}(t;1,1,1,z)$ by applying the Rogers–Fine identity. However, to the best of our knowledge, no algebraic or analytic arguments to determine equivalent forms of the generating functions $\mathcal{G}(t;x,y,u,z)$ or $\mathfrak{G}(t;x,y,u,v)$ were known, not even, say, for the special case $\mathcal{G}(t;1,1,u,z)$. Our analytic proofs of $\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y)$ and $\mathfrak{G}(t;x,y,u,v) = \mathfrak{G}(t;x,v,u,y)$ strengthen the already known existing ties between (refined) generating functions of objects of the Fishburn family with basic hypergeometric series that are expanded in base $q = 1 - r$ around $r = 0$. At the same time it demonstrates the benefit of having equivalent forms of generating functions, and the power of basic hypergeometric machinery.

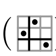
3 Discussion

3.1 Reformulations

All aforementioned (bi)-symmetric distributions on ascent sequences have counterparts over other members of the Fishburn family.

Corollary 3.1. *There are three bijections between* $\mathfrak{S}_n($$)$ *and itself such that the following three (bi)-symmetric equidistributions hold, respectively:*

$$(\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin}, \mathsf{rmax})\pi = (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{rmax}, \mathsf{lmin})(\Psi^{-1} \circ \Phi \circ \Psi)(\pi),$$
$$(\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin})\pi = (\mathsf{iasc}, \mathsf{des}, \mathsf{lmin}, \mathsf{lmax})(\Psi^{-1} \circ \Upsilon^{-1} \circ \Phi \circ \Psi)(\pi),$$
$$(\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{rmax})\pi = (\mathsf{iasc}, \mathsf{des}, \mathsf{rmax}, \mathsf{lmax})(\Psi^{-1} \circ \Upsilon \circ \Phi \circ \Psi)(\pi),$$

where Υ, Φ *are the bijections respectively in Theorems 1.7 and 2.1, and* $\Psi : \mathfrak{S}_n($$) \to \mathcal{A}_n$ *is the bijection from [13, Theorem 12].*

Let us recall the definition of Fishburn matrices and associated three Stirling statistics.

Any cell (i,j) of a matrix M is called a *weakly north-east cell* if $M_{i,j} \neq 0$ and $M_{s,t} = 0$ for all $s \leq i$ and $t \geq j$. A matrix is a *Fishburn* matrix if all of its entries are non-negative integers such that neither row nor column contains only zero entries. Let $\mathcal{F}_n$ be the set of Fishburn matrices whose sum of entries equals n, then for any $M \in \mathcal{F}_n$, let

$$\begin{aligned}
\mathsf{rowsum}_1(M) &:= \text{the sum of entries in the first row of } M,\\
\mathsf{ne}(M) &:= \text{the number of weakly north-east cells of } M,\\
\mathsf{mtr}(M) &:= \text{the smallest index } i \text{ such that } (M_{1,i}, \dots, M_{i-1,i}, M_{i,i}) \neq (0, \dots, 0, 1).\\
&\quad\;\, \text{If no such index exists, then set } \mathsf{mtr}(M) = \dim(M).
\end{aligned}$$

Corollary 3.2. *There is a bijection between $\mathcal{F}_n$ and itself such that the following symmetric distribution holds:*

$$(\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{mtr})M = (\mathsf{rowsum}_1, \mathsf{mtr}, \mathsf{ne})(\phi \circ \Phi \circ \phi^{-1})M,$$

where Φ is the bijection in Theorem 2.1 and $\phi : \mathcal{A}_n \to \mathcal{F}_n$ is the bijection from [6, Theorem 3.6].

Remark 3.3. The three Stirling statistics $\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{mtr}$ are pairwise symmetric on $\mathcal{F}_n$. The fact that the pair $(\mathsf{ne}, \mathsf{mtr})$ is symmetric on $\mathcal{F}_n$ is a direct consequence of Corollary 3.2 and it is known from [6, 13, 18] that the other two pairs $(\mathsf{rowsum}_1, \mathsf{ne})$ and $(\mathsf{rowsum}_1, \mathsf{mtr})$ are also symmetric.

3.2 A conjecture about a symmetric quintuple equidistribution on inversion sequences

We pose a conjecture on a symmetric equidistribution of Euler–Stirling statistics on inversion sequences, which is analogous to Theorem 2.1 but with the two statistics $\mathsf{ealm}, \mathsf{rpos}$ (only defined on ascent sequences) being removed, and $\mathcal{A}_n$ (the set of ascent sequences) being replaced by $\mathcal{I}_n$ (the set of inversion sequences).

Conjecture 3.4. *There is a bijection $\Omega : \mathcal{I}_n \to \mathcal{I}_n$ such that for all $s \in \mathcal{I}_n$,*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{rmin})s = (\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{rmin}, \mathsf{max})\Omega(s).$$

Consequently for all $\pi \in \mathfrak{S}_n$,

$$(\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin}, \mathsf{rmax})\pi = (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{rmax}, \mathsf{lmin})(b^{-1} \circ \Omega \circ b)(\pi),$$

where $b : \mathfrak{S}_n \to \mathcal{I}_n$ is a bijection due to Baril and Vajnovszki (see Theorem 1 of [3]).

The validity of Conjecture 3.4 has been verified by Maple up to $n = 10$. See [19, Section 7] for a discussion on a possible approach to prove the conjecture.

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