Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #65, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Chromatic symmetric functions of Dyck paths and q-rook theory

Laura Colmenarejo*[1], Alejandro H. Morales†[1], and Greta Panova‡[2]

[1] *Department of Mathematics and Statistics, University of Massachusetts, Amherst (USA)*
[2] *Department of Mathematics, University of Southern California (USA)*

Abstract. The chromatic symmetric function (CSF) of Dyck paths of Stanley and its Shareshian–Wachs q-analogue have important connections to Hessenberg varieties, diagonal harmonics and LLT polynomials. In the case of, so called, abelian Dyck paths they are also curiously related to placements of non-attacking rooks by results of Stanley–Stembridge (1993) and Guay-Paquet (2013). For the q-analogue, these results have been generalized by Abreu–Nigro (2020) and Guay-Paquet (private communication), using q-hit numbers, which are a variant of the ones introduced by Garsia and Remmel. Among our main results is a new proof of Guay-Paquet's elegant identity expressing the q-CSFs in a CSF basis with q-hit coefficients. We further show its equivalence to the Abreu–Nigro identity expanding the q-CSF in the elementary symmetric functions.

Keywords: chromatic symmetric functions, abelian Dyck paths, q-hit numbers, q-rook numbers

1 Introduction

Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$ that are totally ordered $v_1 < v_2 < \cdots < v_n$. In [15], Stanley defined the chromatic symmetric function $X_G(\mathbf{x})$ of G as

$$X_G(\mathbf{x}) = \sum_{\kappa:V\to\mathbb{P},\ \text{proper}} \mathbf{x}^{\kappa} = \sum_{\kappa:V\to\mathbb{P},\ \text{proper}} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots, \tag{1.1}$$

where $\mathbb{P} = \{1, 2, 3, \ldots\}$, $\mathbf{x} = (x_1, x_2, \ldots)$, and the sum is over the proper colorings of the vertices of G.

Stanley and Stembridge [16] conjectured that the chromatic symmetric functions expand with positive coefficients in the basis $\{e_\lambda\}$ of elementary symmetric functions for the following graphs. Given a Dyck path d from $(0,0)$ to (n,n), let $G(d)$ be the graph

*lcolmenarejo@umass.edu. L. Colmenarejo was partially supported by the AMS-Simons Travel Grant.
†ahmorales@math.umass.edu. A. H. Morales was partially supported by the NSF grant DMS-1855536.
‡gpanova@usc.edu. G. Panova was partially supported by the NSF grant DMS-1939717.

with vertices $\{1\dots n\}$ and edges (i,j), $i<j$ if and only if the cell (i,j) is below the path d. These are also the incomparability graphs of *unit interval orders* or graphs obtained from *Hessenberg sequences*.

Shareshian–Wachs [14] introduced a quasisymmetric version of $X_G(\mathbf{x})$ defined by

$$X_G(\mathbf{x},q) = \sum_{\kappa: V\to\mathbb{P},\text{ proper}} q^{\mathrm{asc}(\kappa)}\mathbf{x}^{\kappa},$$

where $\mathrm{asc}(\kappa)$ is the number of edges $\{v_i,v_j\}$ of G with $i<j$ and $\kappa(v_i)<\kappa(v_j)$.

For the graphs $G(d)$ coming from Dyck paths, the quasisymmetric function $X_{G(d)}(\mathbf{x},q)$ is actually symmetric and Shareshian–Wachs gave a refinement of the Stanley–Stembridge conjecture for this Catalan family of graphs.

Conjecture 1.1 (Stanley–Stembridge, Shareshian–Wachs)**.** *Let d be a Dyck path then the coefficients of $X_{G(d)}(\mathbf{x},q)$ in the elementary basis are in $\mathbb{N}[q]$.*

The symmetric functions $X_{G(d)}(\mathbf{x},q)$ are very actively studied thanks to their connections to *Hessenberg varieties* [14], *diagonal harmonics* [4], and *Macdonald polynomials* [2].

Conjecture 1.1 has been verified independently and by different techniques by Cho–Huh [5], Hamada–Precup [12] , and Abreu–Nigro [1] for the case of *abelian* Dyck paths: paths d of size $m+n$ of the form $\mathsf{n}^m w(\lambda)\mathsf{e}^n$ where $w(\lambda)$ is the encoding in North (n) and East (e) steps of the partition $\lambda\subset n\times m$ (see Figure 2). We denote the associated graph by $G(\lambda)$ and the chromatic symmetric function by $X_\lambda(\mathbf{x},q)=X_{G(\lambda)}(\mathbf{x},q)$.

The symmetric functions of abelian Dyck paths are deeply related to the *q-rook theory* of Garsia–Remmel [8] as was unveiled in the Abreu–Nigro expansion, itself a q-analogue of a result of Stanley–Stembridge [16]. The following statements use the standard notation $[n]_k=[n][n-1]\cdots[n-k+1]$, $[n]!=[n]_n$, $\left[{n \atop k}\right]=[n]_k/[k]!$, where $[x]=(1-q^x)/(1-q)$. Also $H_j^n(\lambda)$ denotes q-hit numbers [7] which are equal to the Garsia–Remmel q-hit numbers [8] up to a power of q. Moreover, the $H_j^n(\lambda)$ are symmetric polynomials in $\mathbb{N}[q]$ that at $q=1$ give the number of permutations of size n with permutation matrix having support of size j in the board of λ.

Theorem 1.2 (Abreu–Nigro [1])**.** *Let λ be partition inside an $n\times m$ board with $\ell(\lambda)=k\le\lambda_1$. Then*

$$X_\lambda(\mathbf{x},q) = [k]!H_k^{n+m-k}(\lambda)\cdot e_{m+n-k,k} + \sum_{j=0}^{k-1} q^j\,[j]!\,[n+m-2j]\,H_j^{m+n-j-1}(\lambda)\cdot e_{m+n-j,j}.$$

Central to this paper is a new identity of Guay-Paquet (private communication [9]) that expands $X_\lambda(\mathbf{x},q)$ in terms of chromatic symmetric functions for rectangular shape with coefficients given by *q-hit numbers* of *rectangular boards* of size $n\times m$ that we denote by $H_j^{m,n}(\lambda)$ and satisfy $\sum_{j=0}^n H_j^{m,n}(\lambda)=[m]_n$.

Theorem 1.3 (Guay-Paquet [9]). *Let λ be partition inside an $n \times m$ board ($n \leq m$). Then*

$$X_\lambda(\mathbf{x}, q) = \frac{1}{[m]_n} \sum_{j=0}^{n} H_j^{m,n}(\lambda) \cdot X_{m^j}(\mathbf{x}, q).$$

The original proofs of the two statements above use a linear relation satisfied by $X_{G(d)}(\mathbf{x}, q)$ called the *modular relation* [1, 3, 10]. Our ***first main result*** is an elementary proof of Theorem 1.3 using a desymmetrizing recursive relation and q-rook theory (Section 3) and ***our second main result*** is the equivalence of this result and Theorem 1.2 (Section 4). As a by-product of our arguments, we obtain a new proof of the Abreu–Nigro expansion, a new recurrence to compute $X_\lambda(\mathbf{x}, q)$ (Lemma 3.2), and new relations of *q-rook numbers* and *q-hit numbers* (Lemma 3.4 and 4.3) that develop further the q-rook theory of rectangular boards [13].

The full version of this paper is available at [6].

2 Background on q-rook theory

For the rest of the paper, we assume m and n are non-negative integers with $m \geq n$.

2.1 q-rook numbers

Definition 2.1 (q-rook numbers [8]). *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ inside an $n \times m$ board, the* Garsia-Remmel q-rook numbers *are defined as $R_k(\lambda) = \sum_p q^{\mathrm{inv}(p)}$, where the sum is over all placements p of k non-attacking rooks on λ and $\mathrm{inv}(p)$ is the number of cells of λ left after each rook cancels its cell, the cells North in its column and the cells West in its row (see Figure 1).*

Proposition 2.2 (Garsia-Remmel [8]). *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ we have that*

$$F(x; \lambda) := \sum_{k=0}^{\ell} R_k(\lambda)[x]_{\ell-k} = \prod_{i=1}^{\ell} [x + \lambda_{\ell-i+1} - i + 1]. \tag{2.1}$$

Lemma 2.3. *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ we have that*

$$q^{\lambda_1}[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x - \ell + \lambda_1] - \sum_{j=1}^{\lambda_1} q^{\lambda_1 - j} \prod_{t=1}^{\lambda'_j} \frac{[x + \lambda_t - 1 - \ell + t]}{[x + \lambda_t - \ell + t]}.$$

Proof. We use induction on $\ell(\lambda)$ and apply Proposition 2.2. For $\ell(\lambda) = 1$, we have

$$[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x-1] + \sum_{j=1}^{\lambda_1} q^{-j} - q^{-j}\frac{[x+\lambda_1-1]}{[x+\lambda_1]} = q^{-\lambda_1}\frac{[x+\lambda_1-1]}{[x+\lambda_1]}([x+\lambda_1] - [\lambda_1]).$$

Next, expanding the RHS of the above identity and doing standard manipulations gives

$$[x-\ell+\lambda_1] - \sum_{j=1}^{\lambda_1} q^{\lambda_1-j} \prod_{t=1}^{\lambda'_j} \frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]} =$$

$$= q^{\lambda_1-\lambda_2} \frac{[x+\lambda_1-\ell]}{[x+\lambda_1-\ell+1]} \left([x+\lambda_2-(\ell-1)] - \sum_{j=1}^{\lambda_2} q^{\lambda_2-j} \prod_{t=2}^{\lambda'_j} \frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]} \right).$$

By induction hypothesis the parenthetical on the RHS above is $q^{\lambda_2}[x]F(x-1;\tilde{\lambda})/F(x;\tilde{\lambda})$ where $\tilde{\lambda} = (\lambda_2, \ldots, \lambda_\ell)$. Using $\tilde{\lambda}_t = \lambda_{t+1}$ for the reindexing we obtain the result. □

2.2 q-hit numbers

The q-hit numbers are defined in terms of the q-rook numbers by a change of basis. Let $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$ denote the *q-Pochhammer symbol*.

Definition 2.4 ([13, Def. 3.1, Prop. 3.5]). *For λ inside an $n \times m$ board, we define the q-*hit polynomial *of λ by*

$$\sum_{i=0}^{n} H_i^{m,n}(\lambda) x^i := \frac{q^{-|\lambda|}}{[m-n]!} \sum_{i=0}^{n} R_i(\lambda)\, [m-i]! (-1)^i q^{mi-\binom{i}{2}} (x;q)_i, \tag{2.2}$$

*where the coefficients $H_i^{m,n}(\lambda)$ are the q-*hit numbers *associated to λ.*

Notation 2.5. *For square boards with $m = n$, we denote the q-hit number by $H_j^m(\lambda)$.*

Remark 2.6. *For the case $n = m$, Garsia–Remmel defined q-hit numbers $\widetilde{H}_k^n(\lambda)$ by the relation*

$$\sum_{i=0}^{n} \widetilde{H}_i^n(\lambda) x^i = \sum_{i=0}^{n} R_i(\lambda)[n-i]! \prod_{k=n-i+1}^{n} (x-q^k). \tag{2.3}$$

One can show that the Garsia–Remmel q-hit numbers and our q-hit numbers differ by a power of q, namely $\widetilde{H}_k^n(\lambda) = q^{|\lambda|-kn} H_k^n(\lambda)$ (see [6, Appendix]).

For the case of square boards, Garsia and Remmel showed that $\widetilde{H}_k^n(\lambda)$ are in $\mathbb{N}[q]$. Later, Dworkin [7] and Haglund [11] found different Mahonian statistics on rook placements that realize the polynomials $\widetilde{H}_j(\lambda)$. Guay-Paquet [9] defined the rectangular q-hit numbers using a statistic similar to Dworkin's statistic in [7] that we define next.

Definition 2.7 (Statistic for the q-hit numbers). *Let λ be a partition inside an $n \times m$ board. Given a placements p of n nonattacking rooks on a $n \times m$ board, with exactly j inside λ, let stat(p) be the number of cells c in the $n \times m$ board such that (i) there is no rook in c, (ii) there is no rook above c on the same column, and either, (iii) if c is in λ then the rook on the same row of c is in λ and to the right of c or (iv) if c is not in λ then the rook on same row of c is either in λ or to the right of c.*

Remark 2.8. *Intuitively, Dworkin's statistic* $\mathrm{stat}(p)$ *is the number of remaining cells in the* $n \times m$ *board after: wrapping this board on a vertical cylinder and each rook of* p *cancels the cells South in its column and the cells East in its row until the border of* λ.

Theorem 2.9. *Let* λ *be a partition inside an* $n \times m$ *board and* $j = 0, \ldots, n$ *then* $H_j^{m,n}(\lambda) = \sum_p q^{stat(p)}$, *where the sum is over all placements* p *of* n *non-attacking rooks on a* $n \times m$ *board, with exactly* j *rooks inside* λ.

The proof of this result is included in [6, Appendix]. Moreover, for each partition λ, the statistic $\mathrm{stat}(\cdot)$ is *Mahonian*. This results follows readily from Definition 2.4.

Corollary 2.10. *Let* λ *be a partition inside an* $n \times m$ *board, then* $\sum_{j=0}^{n} H_j^{m,n}(\lambda) = [m]_n$.

Example 2.11. *Consider the partition* $\lambda = (6,3,3,1)$ *inside a* 6×8 *board. In Figure 1, we present an example of a placement* p *of two rooks on* λ *with* $\mathrm{inv}(p) = 7$ *and an example of a placement* p' *of six rooks on the* 6×8 *board with two hits on* λ *and* $\mathrm{stat}(p') = 13$.

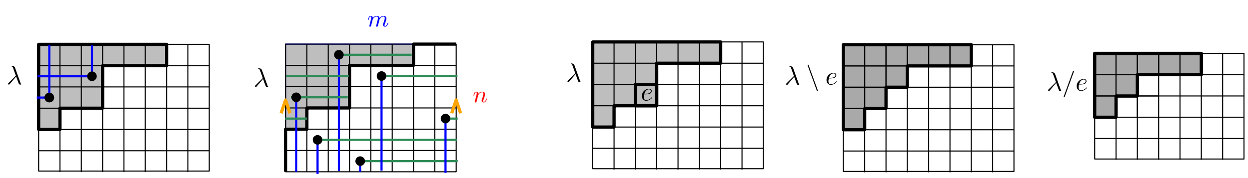

Figure 1: Left: Example of the statistics of a q-rook number and a q-hit number. Right: Example of the deletion and contraction of the board of a partition λ.

We finish this section with some results for q-hit numbers. First, we give a deletion/contraction relation. Given a shape λ and a corner cell e in λ, $\lambda \backslash e$ denotes the shape obtained after deleting the cell e in λ, and λ/e denotes the shape obtained after deleting in λ the row and column containing e. See Figure 1 for an example.

Lemma 2.12. *We have the following deletion/contraction relation:*

$$H_j^{m,n}(\lambda) = H_j^{m,n}(\lambda \backslash e) + q^{|\lambda/e|-|\lambda|+j+m-1}\left(H_{j-1}^{m-1,n-1}(\lambda/e) - qH_j^{m-1,n-1}(\lambda/e)\right). \quad (2.4)$$

The next results show the relation between the q-hit numbers when we change the dimensions of the board.

Lemma 2.13. *Let* λ *be a partition inside an* $n \times m$ *board. Then,* $H_j^{m,n}(\lambda) = \frac{1}{[m-n]!} H_j^{m,m}(\lambda)$.

Lemma 2.14. *Let* λ *be a partition inside an* $(n-1) \times m$ *board. Then,*

$$H_j^{m,n}(\lambda) = [m+1-n]\, H_j^{m,n-1}(\lambda). \quad (2.5)$$

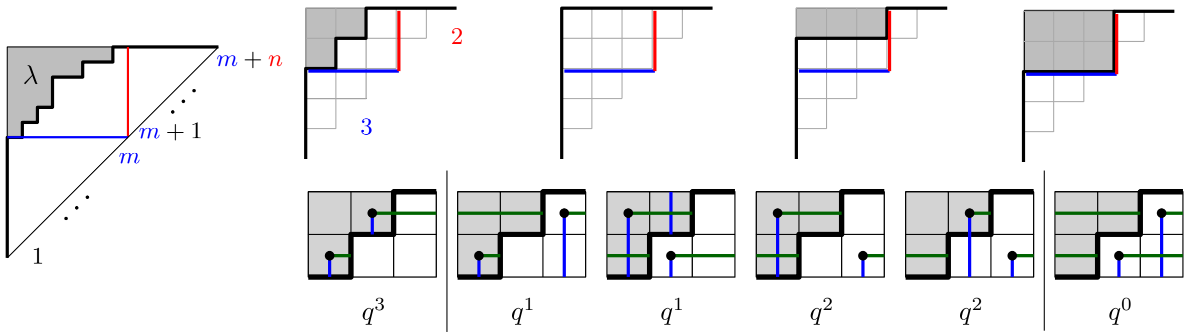

Figure 2: Left: an abelian Dyck path λ inside an $n \times m$ board. Top right: the paths for $\lambda = (2,1)$, and for the rectangles $3^0, 3^1, 3^2$ inside a 2×3 board. Bottom right: the six placements of 2 rooks in 2×3 divided by how many rooks "hit" $(2,1)$ (in gray) and the associated statistic to each rook placement.

Finally, we give formulas for q-hit numbers for rectangular shapes and omit their standard proofs.

Proposition 2.15. $H_k^N(m^j) = q^{(N-j-m+k)k}\,[m]_k\,[N-j]!\dfrac{[N-m]_{j-k}\,[j]_{j-k}}{[j-k]!}$.

Proposition 2.16. *$H_r^{m,n}((m-1)^k) = 0$ for $0 \le r \le k-2$, $H_{k-1}^{m,n}((m-1)^k) = [k][m-1]_{n-1}$, and $H_k^{m,n}((m-1)^k) = q^k[m-k][m-1]_{n-1}$.*

3 The Guay-Paquet q-hit identity

In this section we sketch our main result, a proof of Theorem 1.3 using q-rook theory. We start by giving an example of this elegant identity.

Example 3.1. *For $\lambda = (2,1)$ inside a 2×3 board, looking at Figure 2, we see that $H_0^{3,2}(\lambda) = q^0 = 1$, $H_1^{3,2}(\lambda) = 2q + 2q^2$, $H_2^{3,2}(\lambda) = q^3$. One can verify that*

$$X_{21}(\mathbf{x}, q) = \frac{1}{[3][2]}\left(X_{3^0}(\mathbf{x}, q) + (2q^2 + 2q)X_{3^1}(\mathbf{x}, q) + q^3 X_{3^2}(\mathbf{x}, q)\right),$$

We consider the chromatic symmetric functions in variables $x_1, \ldots, x_M$ and each monomial appearing as a particular assignment of the variables (i.e. colors) to the vertices. That is, the vertices $1, \ldots, N = m+n$ are colored $\{1, \ldots, M\}$. For simplicity, we denote by $X_\lambda^N(M)$ the chromatic symmetric polynomial $X_{G(\lambda)}(x_1, \ldots, x_M; q)$ where the graph $G(\lambda)$ has N vertices. We will use induction on both M and n, m when necessary, driven by the following recursion.

Lemma 3.2. *For $\lambda \subset m \times n$ we have the following recursion*

$$X_\lambda^{m+n}(M) = X_\lambda^{m+n-1}(M-1) + x_M \sum_{i=1}^{m+n} q^{m+n-i-\lambda'_i} X_{\lambda/i}^{m+n-1}(M-1) + x_M^2 \sum_{(i,j)\in\lambda} q^{i-1+(m+n-j-\lambda'_j)} X_{\lambda/(i,j)}^{m+n-2}(M-1),$$

where $\lambda/(i,j)$ is the partition obtained by removing row i and column j from λ, and λ/i means we remove from λ column i, for $i = 1,\dots,m$ or row $m+n-i+1$, for $i = m+1,\dots,m+n$.

Proof. In the abelian case, the graph $G(\lambda)$ consists of a clique with vertices $\{1,\dots m\}$, a clique with vertices $\{m+1,\dots,m+n\}$ and a bipartite graph in between with edges $(i, m+j)$ for each (i,j) in $\overline{\lambda}$. So a coloring of this graph has at most two vertices of the same color. If the colors used are in $\{1,\dots,M\}$, there are three cases for the color M:

1. No vertex is colored M, this term contributes $X_\lambda^{m+n-1}(M-1)$ to $X_\lambda^{m+n}(M)$.
2. Only one vertex is colored M. Suppose this vertex is in column j (from left) and row $i = N-j$ (from top to bottom). It creates ascents with all vertices above it but not in λ, giving $N-j-\lambda'_j$ ascents. Deleting this vertex corresponds to deleting its row and column (only one would be a row/column of λ) and we get a graph on $N-1$ vertices with shape λ/j (deleting row $N-j$, column j or row j, column $N-j$). These terms contribute $x_M \sum_j q^{N-j-\lambda'_j} X_{\lambda/j}(M-1)$.
3. Two vertices are colored M. Suppose that one is in column j and the other one is in row i, necessarily with $(i,j) \in \lambda$. The ascents they contribute are $N-j-\lambda'_j+i-1$. We can remove these two vertices, by removing row i and column j from λ and decreasing N by 2. These terms contribute $x_M^2 \sum_{(i,j)} q^{N-j-\lambda'_j+i-1} X_{\lambda/(i,j)}^{N-2}(M-1)$.

□

For rectangular shapes $\lambda = (m^k)$, Lemma 3.2 gives the following recursive expansion.

Lemma 3.3.

$$X_{m^k}^{m+n}(M) = X_{m^k}^{m+n}(M-1) + x_M \Big(q^{n-k}[m] X_{(m-1)^k}^{m+n-1}(M-1) + [k] X_{m^{k-1}}^{m+n-1}(M-1) + q^k[n-k] X_{m^k}^{m+n-1}(M-1) \Big) + x_M^2 q^{n-k}[k][m] X_{(m-1)^{k-1}}^{m+n-2}(M-1).$$

Proof. This follows by carefully applying Lemma 3.2 to the shape $\lambda = m^k$, since λ/i is either $(m-1)^k$ or m^{k-1} and $\lambda/(i,j) = (m-1)^{k-1}$. □

Proof of Theorem 1.3. Translating Theorem 1.3 into chromatic symmetric polynomials, we want to prove that for every M we have

$$X_\lambda(M) = \frac{1}{[m]_n} \sum_{j=0}^{n} H_j^{m,n}(\lambda) \cdot X_{m^j}(M). \tag{3.1}$$

We apply Lemma 3.3 to each term $X_{m^j}(M)$ appearing in the right hand side of the formula in (3.1). We also apply Lemma 3.2 and the induction hypothesis to the left hand side of the formula in (3.1), i.e. to $X_\lambda(M)$. Then, we obtain an expression where both sides are written in terms of $X^{m+n}_{m^k}(M-1)$, $x_M X^{m+n-1}_{m^k}(M-1)$, $x_M X^{m+n-1}_{(m-1)^k}(M-1)$ and $x_M^2 X^{m+n-2}_{(m-1)^k}(M-1)$. The term at x_M^0 corresponds to the coefficient of $X^{m+n}_{m^k}(M-1)$, which is $\frac{1}{[m]_n} H^{m,n}_k(\lambda)$ on both sides by induction on M. For the linear term, matching the coefficients of $x_M X^{m+n-1}_{(m)^k}(M-1)$ and $x_M X^{m+n-1}_{(m-1)^k}(M-1)$ separately, is equivalent to two q-hit identities we need to prove:

$$\sum_{i=1}^{m} q^{m+n-i-\lambda'_i} H^{m-1,n}_k(\lambda/(m+n-i,i)) = [m-n] H^{m,n}_k(\lambda) q^{n-k}, \tag{3.2}$$

$$[m-n+1] \sum_{i=m+1}^{m+n} q^{m+n-i-\lambda'_i} H^{m,n-1}_k(\lambda/(m+n-i,i)) = H^{m,n}_k(\lambda) q^k [n-k] + H^{m,n}_{k+1}(\lambda)[k+1]. \tag{3.3}$$

Finally, the quadratic term $x_M^2 X^{m+n-2}_{(m-1)^k}(M-1)$ corresponds to the q-hit identity:

$$q^k \sum_{(i,j)\in\lambda} q^{i+(m-j-\lambda'_j)} H^{m-1,n-1}_k(\lambda/(i,j)) = [k+1] H^{m,n}_{k+1}(\lambda). \tag{3.4}$$

By Definition 2.4, we translate these three identities involving q-hit numbers into three identities involving q-rook numbers that are in Lemma 3.4. These identities complete the proof of Theorem 1.3. □

Lemma 3.4. *The q-hit identities* (3.2), (3.3)*, and* (3.4) *are equivalent, in that order, to:*

$$\sum_{j=1}^{m} q^{m-j} R_k(\lambda/j) = R_k(\lambda)[m-k] - R_{k+1}(\lambda)(q^m - q^{m-k-1}), \tag{3.5}$$

$$\sum_{i=1}^{n} q^{i-1+\lambda_i} R_k(\lambda/i) = ([n]-[k]) R_k(\lambda), \tag{3.6}$$

$$\sum_{(i,j)\in\lambda} q^{i-j+\lambda_i} R_k(\lambda/(i,j)) = q[k+1] R_{k+1}(\lambda). \tag{3.7}$$

We give the proof of the first relation (3.5). The arguments for the other two relations have a similar flavor.

Proof of (3.5). Let $\ell = \ell(\lambda)$. Multiplying on both sides by $[x]_{\ell-i}$ and summing over $i = 0, \ldots, \ell$, the claimed relation is equivalent to the generating function identity:

$$\begin{aligned}\sum_{j=1}^{m} q^{m-j} F(x;\lambda/j) &= \sum_i R_i(\lambda) \Big([m-i][x]_{\ell-i} - (q^m - q^{m-i})[x]_{\ell-i+1} \Big) \\ &= [m+x-\ell] F(x;\lambda) - q^m [x] F(x-1;\lambda),\end{aligned}$$

where $F(x;\lambda)$ is as in (2.1) and we used the observation that

$$[m-i][x]_{\ell-i}-(q^m-q^{m-i})[x]_{\ell-i+1}=[x]_{\ell-i}[x+m-\ell]-q^m[x]_{\ell-i+1}.$$

We have that

$$F(x;\lambda/j)=\prod_{i=1}^{\lambda'_j}[x+\lambda_i-1-\ell+i]\prod_{i=\lambda'_j+1}^{\ell}[x+\lambda_i-\ell+i]=F(x;\lambda)\prod_{i=1}^{\lambda'_j}\frac{[x-1+\lambda_i-\ell+i]}{[x+\lambda_i-\ell+i]}.$$

Using Lemma 2.3, we have that since $\lambda/j=\lambda$ if $j>\lambda_1$,

$$\begin{aligned}\sum_{j=1}^{m}q^{m-j}F(x;\lambda/j)&=F(x;\lambda)\left([m-\lambda_1]+q^{m-\lambda_1}\sum_{j=1}^{m}q^{\lambda_1-j}\prod_{i=1}^{\lambda'_j}\frac{[x-1+\lambda_i-\ell+i]}{[x+\lambda_i-\ell+i]}\right)\\&=F(x;\lambda)\left([m-\lambda_1]+q^{m-\lambda_1}[x-\ell+\lambda_1]-q^{m-\lambda_1}q^{\lambda_1}[x]\frac{F(x-1;\lambda)}{F(x;\lambda)}\right)\\&=F(x;\lambda)[x-\ell+\lambda_1+m-\lambda_1]-q^m[x]F(x-1;\lambda),\end{aligned}$$

as desired. □

4 The Abreu–Nigro expansion in the elementary basis

In this section we show that Guay-Paquet's identity (Theorem 1.3) is equivalent to Abreu–Nigro's identity presented (Theorem 1.2). We start by giving a proof of Abreu–Nigro's identity for rectangular shapes.

Lemma 4.1 (Abreu–Nigro's formula for rectangles)**.**

$$X_{m^k}(\mathbf{x},q)=[k]!H_k^{m+n-k}(m^k)\cdot e_{m+n-k,k}+\sum_{r=0}^{k-1}q^r\,[r]!\,[n+m-2r]\,H_r^{m+n-r-1}(m^k)\cdot e_{m+n-r,r}.$$

In order to prove this case of the Abreu–Nigro identity we need the following result.

Lemma 4.2 (Guay-Paquet formula for rectangles)**.**

$$[m]X^{m+n-1}_{(m-1)^k}=q^k[m-k]X_{m^k}+[k]X_{m^{k-1}}. \tag{4.1}$$

Proof. By Theorem 1.3 for the shape $\lambda=(m-1)^k\subset n\times m$ and the formula for the q-hit numbers $H_r^{m,n}((m-1)^k)$ from Proposition 2.16 we obtain

$$\begin{aligned}X_{(m-1)^k}&=\frac{1}{[m]_n}q^k[m-1]_k[m-k]_{n-k}X_{m^k}+\frac{1}{[m]_n}[k][m-1]_{k-1}[m-k]_{n-k}X_{m^{k-1}},\\ [m]X_{(m-1)^k}&=q^k[m-k]X_{m^k}+[k]X_{m^{k-1}}.\end{aligned}$$

□

Proof sketch of Lemma 4.1. We use induction on m and k. For the base case, note that $X_{m^0} = [m+n]!e_{m+n} = H_0^{m+n}(m^0)e_{m+n}$. By Lemma 4.2 we have that

$$X_{m^k} = \frac{1}{q^k[m-k]}\left([m]X_{(m-1)^k} - [k]X_{m^{k-1}}\right).$$

Next, we use the induction hypothesis on $X_{(m-1)^k}$ and $X_{m^{k-1}}$, Proposition 2.15 for the hit numbers of rectangles, and routine simplifications to verify the desired formula for X_{m^k}. □

We are now ready to prove that the Guay-Paquet's identity and Abreu–Nigro's follow from each other. As a corollary, we obtain a new proof of the latter.

Proof of Theorem 1.2. Applying Lemma 4.1 to the RHS of the formula in Theorem 1.3, we obtain that

$$\frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\cdot X_{m^j}(\mathbf{x},q) = \frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\left([j]!H_j^{n+m-j}(m^j)\cdot e_{m+n-j,j}\right)$$
$$+\frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\left(q^r\sum_{r=0}^{j-1}[r]!\,[n+m-2r]\,H_r^{m+n-r-1}(m^j)\cdot e_{m+n-r,r}\right).$$

Now, switching the summation order, we have that

$$\frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\cdot X_{m^j}(\mathbf{x},q) = \sum_{r=0}^{n} e_{m+n-r,r}\frac{1}{[m]_n}\,[r]!H_r^{m+n-r}(m^r)H_r^{m,n}(\lambda)$$
$$+\sum_{r=0}^{n-1} e_{m+n-r,r}\frac{1}{[m]_n}\left(q^r\sum_{j=r+1}^{n}[r]!\,[n+m-2r]\,H_r^{m+n-r-1}(m^j)H_j^{m,n}(\lambda)\right).$$

Thus, we need to show that for $r = k = \ell(\lambda)$,

$$[m]_n\,H_k^{m+n-k}(\lambda) = H_k^{m+n-k}(m^k)H_k^{m,n}(\lambda) + q^k\sum_{j=k+1}^{n}[n+m-2k]\,H_k^{m+n-k-1}(m^j)H_j^{m,n}(\lambda),$$

and for $r < k = \ell(\lambda)$,

$$[m]_n\,q^r\,[n+m-2r]\,H_r^{m+n-r-1}(\lambda) = H_r^{m+n-r}(m^r)H_r^{m,n}(\lambda)$$
$$+q^r\sum_{j=r+1}^{n}[n+m-2r]\,H_r^{m+n-r-1}(m^j)H_j^{m,n}(\lambda).$$

After using Proposition 2.15, these two relations are equivalent to the following identities relating q-hit numbers of λ in square boards and rectangular boards. Finally, the Abreu-Nigro expansion for $X_\lambda(\mathbf{x},q)$ follows now from Lemma 4.3. □

Lemma 4.3. *Let λ be a partition inside an $n \times m$ board and $k = \ell(\lambda)$, then*

$$\begin{bmatrix} m-k \\ n-k \end{bmatrix} H_k^{m+n-k}(\lambda) = q^{k(n-k)} [m+n-2k]_{m-k} H_k^{m,n}(\lambda). \tag{4.2}$$

For $0 \le r < k$, we have

$$\begin{bmatrix} m-r \\ n-r \end{bmatrix} H_r^{m+n-r-1}(\lambda) = q^{r(n-r-1)} [m+n-2r-1]_{m-r-1} H_r^{m,n}(\lambda) + \sum_{j=r+1}^{n} q^{r(n-1-j)} \begin{bmatrix} j \\ r \end{bmatrix} \frac{[m+n-r-j-1]_{m-r}}{[n-r]} H_j^{m,n}(\lambda). \tag{4.3}$$

Proof sketch of Lemma 4.3. Each identity follows by using Definition 2.4 to rewrite both the LHS and RHS in terms of q-rook numbers $R_j(\lambda)$ and showing the resulting expressions are equal via routine q-factorial manipulations.

We remark that there is a more interesting proof using the deletion/contraction formula in Lemma 2.12 on both sides of each relation, the rectangle-resizing identity in Lemma 2.14, and induction. □

5 Open problems

Since our proof of Theorem 1.3 uses q-rook theory, it would be interesting to find a bijective proof of this result relating colorings with rook placements.

There are other rules for the elementary basis expansion of $X_\lambda(\mathbf{x}, q)$. In particular, Cho–Huh [5] give an expansion in terms of *P-tableaux* of shape $2^j 1^{m+n-2j}$ such that there is no $s \ge j+2$ such that $(a_{i,1}, a_{s,1}) \in \lambda$ for all $i \in \{\ell+1, \ldots, s-1\}$. Let $c_j^{m,n}(q) := \sum_T q^{\mathrm{inv}_{G(\lambda)}(T)}$, where the sum is over such tableaux (see [14, Sec. 6]). It would be interesting to find a weight-preserving bijection that shows that

$$c_j^{m,n}(q) = \begin{cases} [j]! H_j^{m+n-j}(\lambda) & \text{if } j = \ell(\lambda), \\ q^j [j]! [m+n-2j] H_j^{m+n-j-1}(\lambda) & \text{if } j < \ell(\lambda). \end{cases}$$

Acknowledgements

We thank Mathieu Guay-Paquet for generously sharing the notes [9] with Theorem 1.3 as well as Alex Abreu, Per Alexandersson, and Antonio Nigro for insightful discussions.

References

[1] A. Abreu and A. Nigro. "Chromatic symmetric functions from the modular law". *J. Combin. Theory Ser. A* **180** (2021), p. 105407.

[2] P. Alexandersson and G. Panova. "LLT polynomials, chromatic quasisymmetric functions and graphs with cycles". *Discrete Math.* **341**.12 (2018), pp. 3453–3482.

[3] P. Alexandersson and R. Sulzgruber. "A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions". 2020. arXiv:2004.09198.

[4] E. Carlsson and A. Mellit. "A proof of the shuffle conjecture". *J. Amer. Math. Soc.* **31**.3 (2018), pp. 661–697.

[5] S. Cho and J. Huh. "On e-positivity and e-unimodality of chromatic quasisymmetric functions". *Sém. Lothar. Combin.* **80B** (2018), Art. 59, 12.

[6] L. Colmenarejo, A. H. Morales, and G. Panova. "Chromatic symmetric functions of Dyck paths and q-rook theory". 2021. arXiv:2104.07599.

[7] M. Dworkin. "An interpretation for Garsia and Remmel's q-hit numbers". *J. Combin. Theory Ser. A* **81**.2 (1998), pp. 149–175.

[8] A. M. Garsia and J. Remmel. "q-counting rook configurations and a formula of Frobenius". *J. Combin. Theory Ser. A* **41** (1986), pp. 246–275.

[9] M. Guay-Paquet. "Linear Relations Between q-Chromatic symmetric functions". Non-published, shared by private communication.

[10] M. Guay-Paquet. "A modular relation for the chromatic symmetric functions of (3+1)-free posets". 2013. arXiv:1306.2400.

[11] J. Haglund. "q-rook polynomials and matrices over finite fields". *Adv. in Appl. Math.* **20**.4 (1998), pp. 450–487.

[12] M. Harada and M. E. Precup. "The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture". *Algebr. Comb.* **2**.6 (2019), pp. 1059–1108.

[13] J. B. Lewis and A. H. Morales. "Rook theory of the finite general linear group". *Exp. Math.* (2018), pp. 1–19.

[14] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* **295** (2016), pp. 497–551.

[15] R. P. Stanley. "A symmetric function generalization of the chromatic polynomial of a graph". *Adv. Math.* **111** (1995), pp. 166–194.

[16] R. P. Stanley and J. R. Stembridge. "On immanants of Jacobi-Trudi matrices and permutations with restricted position". *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279.