Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #67, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Principal specializations of Schubert polynomials in classical types

Eric Marberg[1] and Brendan Pawlowski[2]

[1]*Department of Mathematics, Hong Kong University of Science and Technology*
[2]*Department of Mathematics, University of Southern California*

Abstract. There is a remarkable formula for the principal specialization of a type A Schubert polynomial as a weighted sum over reduced words. Taking appropriate limits transforms this to an identity for the backstable Schubert polynomials recently introduced by Lam, Lee, and Shimozono. We prove some analogues of the latter formula for principal specializations of Schubert polynomials in classical types B, C, and D. We also derive some more general identities for Grothendieck polynomials.

Keywords: Schubert polynomials, Grothendieck polynomials, Coxeter systems, reduced words

1 Introduction

There is a remarkable formula for the principal specialization $\mathfrak{S}_w(1, q, q^2, \ldots, q^{n-1})$ of a (type A) Schubert polynomial as a weighted sum over reduced words. Originally a conjecture of Macdonald [9], this identity was first proved algebraically by Fomin and Stanley [6]. Billey, Holroyd, and Young [2, 13] have recently found the first bijective proof of Macdonald's conjecture.

Here, we identify some apparently new analogues of Macdonald's identity for the principal specializations of Schubert polynomials in other classical types. Our methods are based on the algebraic techniques of Fomin and Stanley [6].

To state our main theorems we need to recall a few definitions. Throughout, we let x_i for $i \in \mathbb{Z}$ be commuting indeterminates. We use the term *word* to mean a finite sequence $a_1 a_2 \cdots a_p$ whose letters belong to some totally ordered alphabet. This alphabet will usually consist of the integers $\mathbb{Z}$ with their usual ordering.

Definition 1.1. A *bounded compatible sequence* for a word $a = a_1 a_2 \cdots a_p$ is a weakly increasing sequence of integers $\mathbf{i} = (i_1 \leq i_2 \leq \cdots \leq i_p)$ with the property that

$$i_j < i_{j+1} \text{ whenever } a_j \leq a_{j+1} \qquad \text{and} \qquad i_j \leq a_j \text{ whenever } 0 < i_j.$$

eric.marberg@gmail.com. Eric Marberg was partially supported by RGC Grant ECS 26305218.
br.pawlowski@gmail.com.

Let $\mathsf{Compatible}(a)$ denote the set of all such sequences. Given $\mathbf{i} = (i_1 \leq \cdots \leq i_p) \in \mathsf{Compatible}(a)$, define $x_\mathbf{i} = x_{i_1} \cdots x_{i_p}$ and write $0 < \mathbf{i}$ if $i_1, \ldots, i_p$ are all positive.

Let $s_i = (i, i+1)$ denote the permutation of $\mathbb{Z}$ interchanging i and $i+1$. Fix a positive integer n and let $S_n := \langle s_1, s_2, \ldots, s_{n-1} \rangle \subset S_\mathbb{Z} := \langle s_i : i \in \mathbb{Z} \rangle$. Both S_n and $S_\mathbb{Z}$ are Coxeter groups with respect to the generating sets just given. A *reduced word* for $w \in S_\mathbb{Z}$ is a word $a_1 a_2 \cdots a_p$ of shortest possible length such that $w = s_{a_1} s_{a_2} \cdots s_{a_p}$. Let $\mathsf{Reduced}(w)$ denote the set of all such words.

Definition 1.2. The *Schubert polynomial* of $w \in S_n$ is

$$\mathfrak{S}_w := \sum_{a \in \mathsf{Reduced}(w)} \sum_{0 < \mathbf{i} \in \mathsf{Compatible}(a)} x_\mathbf{i} \in \mathbb{Z}[x_1, x_2, \ldots, x_{n-1}].$$

Schubert polynomials are often defined inductively using divided difference operators, following the approach of Lascoux and Schützenberger. The formula that we have given is [3, Thm. 1.1]. The identity of Macdonald [9] mentioned above is as follows.

Theorem 1.3 (Fomin and Stanley [6, Thm. 2.4])**.** If $w \in S_n$ then

$$\mathfrak{S}_w(1, q, q^2, \ldots, q^{n-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathsf{Reduced}(w)} \tfrac{[a_1]_q [a_2]_q \cdots [a_p]_q}{[p]_q!} q^{\mathsf{comaj}(a)}.$$

where $\mathsf{comaj}(a) := \sum_{a_i < a_{i+1}} i$ and $[a]_q := \frac{1-q^a}{1-q}$ and $[p]_q! := [p]_q \cdots [2]_q [1]_q$.

Taking appropriate limits transforms the preceding formula into an identity for the *backstable Schubert polynomials*, which may be defined as follows.

Definition 1.4. The *backstable Schubert polynomial* of $w \in S_n$ is

$$\overleftarrow{\mathfrak{S}}_w := \sum_{a \in \mathsf{Reduced}(w)} \sum_{\mathbf{i} \in \mathsf{Compatible}(a)} x_\mathbf{i} \in \mathbb{Z}[[\ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}]].$$

This is the same as the formula for $\mathfrak{S}_w$ except now $\mathbf{i} = (i_1 \leq i_2 \cdots \leq i_p)$ may contain non-positive integers. If $w \in S_n$ then $\overleftarrow{\mathfrak{S}}_w(\ldots, 0, 0, x_1, x_2, \ldots, x_{n-1}) = \mathfrak{S}_w$, while $\overleftarrow{\mathfrak{S}}_w(\ldots, x_{-2}, x_{-1}, x_0, 0, 0, \ldots, 0)$ is the *Stanley symmetric function* of w in the variables x_i for $i \leq 0$ [8, Thm. 3.2].

Note that $\overleftarrow{\mathfrak{S}}_w$ is usually not a polynomial. These power series were introduced by Lam, Lee, and Shimozono [8] in connection with Schubert calculus on infinite flag varieties. They also arise as cohomology classes of degeneracy loci in products of flag varieties [12].

If $F \in \mathbb{Z}[[\ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}]]$ is homogeneous then the formal power series $F(x_i \mapsto q^{i-1})$ obtained by setting $x_i = q^{i-1}$ for all integers $i < n$ is well-defined. The following result is easy to derive from Theorem 1.3 and is also a special case of Theorem 3.3. In this statement, for a word $a = a_1 a_2 \cdots a_p$ we write $\sum a := \sum_{i=1}^p a_i$ and $\ell(a) := p$.

Theorem 1.5. If $w \in S_n$ then $\overleftarrow{\mathfrak{S}}_w(x_i \mapsto q^{i-1}) = \sum_{a\in\mathsf{Reduced}(w)} \frac{q^{\sum a + \mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)}$ where the right hand expression is interpreted as a Laurent series in q^{-1}.

Example 1.6. Setting $x_i = q^{i-1}$ in $\overleftarrow{\mathfrak{S}}_w$ gives another formula for $\overleftarrow{\mathfrak{S}}_w(x_i \mapsto q^{i-1})$ as a sum over the reduced words for w. The corresponding terms in these two summations need not agree, however: for a given word $a = a_1a_2\cdots a_p \in \mathsf{Reduced}(w)$, it can happen that

$$\sum_{\mathbf{i}\in\mathsf{Compatible}(a)} q^{(i_1-1)+(i_2-1)+\cdots+(i_p-1)} \neq \frac{q^{\sum a + \mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^p-1)}.$$

If $w = (1,2)(3,4)$ and $a = a_1a_2 = 1,3$ then $\sum_{\mathbf{i}\in\mathsf{Compatible}(a)} q^{(i_1-1)+\cdots+(i_p-1)}$ is

$$\sum_{1\geq i_1<i_2\leq 3} q^{(i_1-1)+(i_2-1)} \in q^2+2q+2+q^{-1}\mathbb{Z}[[q^{-1}]]$$

while $\frac{q^{\sum a + \mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^p-1)} = \frac{q^5}{(q-1)(q^2-1)}$ expands into the Laurent series

$$q^5(q^{-1}+q^{-2}+q^{-3}+\dots)(q^{-2}+q^{-4}+\dots) \in q^2+q+2+q^{-1}\mathbb{Z}[[q^{-1}]].$$

For $w = (1,2)(3,4)$ there are only two reduced words and one has

$$\overleftarrow{\mathfrak{S}}_{(1,2)(3,4)} = \overleftarrow{e}_1^2 + (2x_1+x_2+x_3)\overleftarrow{e}_1 + x_1^2 + x_1x_2 + x_1x_3$$

where $\overleftarrow{e}_d$ is the symmetric function $\sum_{i_1<i_2<\cdots<i_d\leq 0} x_{i_1}x_{i_2}\cdots x_{i_d}$. One computes

$$\overleftarrow{\mathfrak{S}}_{(1,2)(3,4)}(x_i \mapsto q^{i-1}) = \frac{q^4}{(q-1)^2} = \cdots + 7q^{-4} + 6q^{-3} + 5q^{-2} + 4q^{-1} + 3 + 2q + q^2$$

using either Theorem 1.5 or the formula $\overleftarrow{e}_d(q^{-1}, q^{-2}, \dots) = \frac{1}{(q-1)(q^2-1)\cdots(q^d-1)}$.

Our first new results are versions of the preceding theorem for other classical types. We begin with type B/C. Given $0 < i < n$, define $t_i = t_{-i} := (i, i+1)(-i, -i-1)$ and $t_0 := (-1, 1)$. Define $W_n^{\mathsf{BC}} := \langle t_0, t_1, \dots, t_{n-1}\rangle$ to be the Coxeter group consisting of the permutations w of $\mathbb{Z}$ with $w(i) = i$ for $|i| > n$ and $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$.

A *signed reduced word of type B* for $w \in W_n^{\mathsf{BC}}$ is a word $a_1a_2\cdots a_p$ with letters in the set $\{-n+1, \dots, -1, 0, 1, \dots, n-1\}$ of shortest possible length such that $w = t_{a_1}t_{a_2}\cdots t_{a_p}$. Let -0 denote a formal symbol distinct from 0 that satisfies $-1 < -0 < 0 < 1$ and set $t_{-0} := t_0$. A *signed reduced word of type* C for $w \in W_n^{\mathsf{BC}}$ is a word $a_1a_2\cdots a_p$ with letters in $\{-n+1, \dots, -1, -0, 0, 1, \dots, n-1\}$ of shortest possible length such that $w = t_{a_1}t_{a_2}\cdots t_{a_p}$. Let $\mathsf{Reduced}_B^{\pm}(w)$ and $\mathsf{Reduced}_C^{\pm}(w)$ denote these sets of signed reduced words for w.

Definition 1.7. The *type B/C Schubert polynomials* of $w \in W_n^{\mathsf{BC}}$ are

$$\mathfrak{S}_w^{\mathsf{B}} := \sum_{\substack{a \in \mathsf{Reduced}_B^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} x_{\mathbf{i}} \qquad \text{and} \qquad \mathfrak{S}_w^{\mathsf{C}} := \sum_{\substack{a \in \mathsf{Reduced}_C^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} x_{\mathbf{i}} = 2^{\ell_0(w)} \mathfrak{S}_w^{\mathsf{B}}$$

where $\ell_0(w) := |\{i \in \mathbb{Z} : w(i) < 0 < i\}|$.

Both $\mathfrak{S}_w^{\mathsf{B}}$ and $\mathfrak{S}_w^{\mathsf{C}}$ are formal power series in $\mathbb{Z}[[\dots, x_{-1}, x_0, x_1, \dots, x_{n-1}]]$. If we substitute $x_i \mapsto z_i$ for $i > 0$ and $x_i \mapsto x_{1-i}$ for $i \leq 0$, then $\mathfrak{S}_w^{\mathsf{B}}$ and $\mathfrak{S}_w^{\mathsf{C}}$ specialize to the Schubert polynomials of types B and C defined by Billey and Haiman in [1]; compare our definition with [1, Thm. 3].

Let $\mathsf{Reduced}_C(w)$ for $w \in W_n^{\mathsf{BC}}$ denote the subset of words in $\mathsf{Reduced}_C^{\pm}(w)$ whose letters all belong to $\{0, 1, \dots, n-1\}$. In Section 2.2 we sketch a proof the following:

Theorem 1.8. If $w \in W_n^{\mathsf{BC}}$ then

$$\mathfrak{S}_w^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathsf{Reduced}_C(w)} \tfrac{(q^{a_1}+1)(q^{a_2}+1)\cdots(q^{a_p}+1)}{(q-1)(q^2-1)\cdots(q^p-1)} q^{\mathrm{comaj}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

Example 1.9. If $w = (1, -2)(2, -1) \in W_n^{\mathsf{BC}}$ then the set $\mathsf{Reduced}_C^{\pm}(w)$ has 8 elements, formed by adding arbitrary signs to the letters in $a_1 a_2 a_3 = 0, 1, 0$. One can compute that

$$\mathfrak{S}_{(1,-2)(2,-1)}^{\mathsf{C}} = 4 \overleftarrow{e}_2 \overleftarrow{e}_1 - 4 \overleftarrow{e}_3$$

where $\overleftarrow{e}_d := \sum_{i_1 < i_2 < \cdots < i_d \leq 0} x_{i_1} x_{i_2} \cdots x_{i_d}$ as in Example 1.6. It follows that

$$\mathfrak{S}_{(1,-2)(2,-1)}^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \tfrac{4q}{(q-1)^2(q^3-1)} = \cdots + 28q^{-8} + 20q^{-7} + 12q^{-6} + 8q^{-5} + 4q^{-4}.$$

We turn to type D. For $1 < i < n$, let $r_i = r_{-i} := (i, i+1)(-i, -i-1) = t_i$ but define

$$r_1 := (1,2)(-1,-2) = t_1 \qquad \text{and} \qquad r_{-1} := (1,-2)(-1,2) = t_0 t_1 t_0.$$

Let $W_n^{\mathsf{D}} := \langle r_{-1}, r_1, r_2, \dots, r_{n-1} \rangle$ be the Coxeter group of permutations $w \in W_n^{\mathsf{BC}}$ for which the number of integers $i > 0$ with $w(i) < 0$ is even. A *signed reduced word* for $w \in W_n^{\mathsf{D}}$ is a word $a_1 a_2 \cdots a_p$ with letters in $\{-n+1, \dots, -2, -1, 1, 2, \dots, n-1\}$ of shortest possible length with $w = r_{a_1} r_{a_2} \cdots r_{a_p}$. Let $\mathsf{Reduced}_D^{\pm}(w)$ be the set of such words.

Definition 1.10. The *type D Schubert polynomial* of $w \in W_n^{\mathsf{D}}$ is

$$\mathfrak{S}_w^{\mathsf{D}} = \sum_{\substack{a \in \mathsf{Reduced}_D^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} x_{\mathbf{i}} \in \mathbb{Z}[[\dots, x_{-1}, x_0, x_1, \dots, x_{n-1}]].$$

If we again substitute $x_i \mapsto z_i$ for $i > 0$ and $x_i \mapsto x_{1-i}$ for $i \leq 0$, then our definition of the power series $\mathfrak{S}^{\mathsf{D}}_w$ specializes to Billey and Haiman's formula for the Schubert polynomial of type D given in [1, Thm. 4].

Suppose $a = a_1 a_2 \cdots a_p$ is a sequence where $a_i \in \{\pm 1, \pm 2, \pm 3, \ldots, \pm(n-1)\}$. Define

$$\mathrm{comaj}_{\mathsf{D}}(a) := |\{i : a_i > 0\}| + \sum_{a_i \prec a_{i+1}} 2i \tag{1.1}$$

where $\prec$ is the order $-1 \prec -2 \prec \cdots \prec -n \prec 1 \prec 2 \prec \cdots \prec n$. For example, if $a = a_1 a_2 a_3 a_4 = -1, -2, 3, 1$ then $\mathrm{comaj}_{\mathsf{D}}(a) = 2 + (2+4) = 8$.

Theorem 1.11. If $w \in W^{\mathsf{D}}_n$ then

$$\mathfrak{S}^{\mathsf{D}}_w(x_i \mapsto q^{i-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathsf{Reduced}^{\pm}_D(w)} \frac{(q^{|a_1|+1})(q^{|a_2|}+1)\cdots(q^{|a_p|}+1)}{(q^2-1)(q^4-1)\cdots(q^{2p}-1)} q^{\mathrm{comaj}_{\mathsf{D}}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

Example 1.12. If $w = (1,-1)(4,-4) \in W^{\mathsf{D}}_n$ then the set $\mathsf{Reduced}^{\pm}_D(w)$ has 32 elements, formed by adding signs to the letters in $a_1 a_2 a_3 a_4 a_5 a_6 = 3, 2, 1, 1, 2, 3$ in all ways that give opposite signs to the two entries with absolute value one. One can compute that

$$\mathfrak{S}^{\mathsf{D}}_{(1,-1)(4,-4)} = x_1 x_2 x_3 \overleftarrow{P}_3 + (x_1 x_2 + x_1 x_3 + x_2 x_3) \overleftarrow{P}_4 + (x_1 + x_2 + x_3) \overleftarrow{P}_5 + \overleftarrow{P}_6$$

where $\overleftarrow{P}_d$ for $d > 0$ is the Schur P-function $\frac{1}{2} \sum_{a=0}^{d} e_a(x_0, x_{-1}, \ldots) h_{d-a}(x_0, x_{-1}, \ldots)$. Using the formula $\overleftarrow{P}_d(q^{-1}, q^{-2}, \ldots) = \frac{(q+1)(q^2+1)\cdots(q^{d-1}+1)}{(q-1)(q^2-1)\cdots(q^d-1)}$ one can check that

$$\mathfrak{S}^{\mathsf{D}}_{(1,-1)(4,-4)}(x_i \mapsto q^{i-1}) = \frac{q^{12}(q^2+1)}{(q-1)^3(q^3-1)(q^5-1)} = \cdots + 27q^{-4} + 15q^{-3} + 7q^{-2} + 3q^{-1} + 1,$$

which agrees with Theorem 1.11.

Setting $q = 1$ in Theorem 1.5 leads to surprising enumerative formulas involving reduced words, compatible sequences, and plane partitions [5]. By contrast, the power series $\overleftarrow{\mathfrak{S}}_w$, $\mathfrak{S}^{\mathsf{B}}_w$, $\mathfrak{S}^{\mathsf{C}}_w$, and $\mathfrak{S}^{\mathsf{D}}_w$ do not converge upon specializing $x_i \mapsto 1$ for all i. It would be interesting to find variations of our formulas with clearer enumerative content.

The second half of this abstract contains a few other related results. In Section 3, we extend Theorems 1.5, 1.8, and 1.11 to *Grothendieck polynomials*. We originally derived these formulas by adapting the algebraic methods in [6, 7]. It would be interesting to find bijective proofs of these identities along the lines of [2].

This extended abstract is an abridged version of [11]. To save space, we have omitted the proofs of most propositions, while retaining proof sketches for the main theorems.

Acknowledgements

We thank Sergey Fomin for suggesting the problem of finding analogues of Macdonald's formulas for Schubert polynomials outside type A.

2 Principal specializations of Schubert polynomials

This section contains our proofs of Theorems 1.8 and 1.11. Throughout, we fix a positive integer n and let R be a commutative ring containing $\mathbb{Z}[[x_i : i < n]]$.

2.1 Nil-Coxeter algebras

The *nil-Coxeter algebra* introduced in this section figures prominently in [6]. Let (W, S) be a Coxeter system with length function ℓ. Let $\mathsf{NilCox} = \mathsf{NilCox}(W)$ be the R-module of all formal R-linear combinations of the symbols u_w for $w \in W$. This module has a unique R-algebra structure with bilinear multiplication satisfying

$$u_v u_w = \begin{cases} u_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{if } \ell(vw) < \ell(v) + \ell(w) \end{cases} \qquad \text{for } v, w \in W.$$

Choose $x, y \in R$. Given $s \in S$, define $h_s(x) := 1 + xu_s \in \mathsf{NilCox}$. One checks that if $s, t \in S$ and $st = ts$ then $h_s(x)h_s(y) = h_s(x+y)$ and $h_s(x)h_t(y) = h_t(y)h_s(x)$.

We will refer back several times to the following general identity, which is equivalent to [6, Lem. 5.4] after some minor changes of variables:

Lemma 2.1 ([6, Lem. 5.4])**.** Let $\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_N$ be some elements of an R-algebra with identity 1, and suppose q, z_1, z_2, $\ldots z_N$ are formal variables. Then

$$\prod_{j=-\infty}^{0} \prod_{i=1}^{N} (1 + q^{j-1} z_i \mathfrak{t}_i) = \sum_{p \geq 0} \sum_{a_1, a_2, \ldots, a_p} \frac{z_{a_1} z_{a_2} \cdots z_{a_p}}{(q-1)(q^2-1)\cdots(q^p-1)} q^{\operatorname{comaj}(a)} \mathfrak{t}_{a_1} \mathfrak{t}_{a_2} \cdots \mathfrak{t}_{a_p}$$

where $\operatorname{comaj}(a) := \sum_{a_i < a_{i+1}} i$ and the coefficients on the right are viewed as Laurent series in q^{-1}.

2.2 Type B/C

Here, let $\mathsf{NilCox} = \mathsf{NilCox}(W_n^{\mathsf{BC}})$ denote the nil-Coxeter algebra of type B/C Coxeter system $(W, S) = (W_n^{\mathsf{BC}}, \{t_0, t_1, \ldots, t_{n-1}\})$ and define $h_i(x) := 1 + xu_{t_i} \in \mathsf{NilCox}$ for integers $-n < i < n$ and $x \in R$. Recall that $t_i = t_{-i}$ so we always have $h_i(x) = h_{-i}(x)$. Let

$$\begin{aligned} A_i(x) &:= h_{n-1}(x)h_{n-2}(x) \cdots h_i(x), \\ B(x) &:= h_{n-1}(x) \cdots h_1(x)h_0(x)h_{-1}(x) \cdots h_{-n+1}(x), \\ C(x) &:= h_{n-1}(x) \cdots h_1(x)h_0(x)h_0(x)h_{-1}(x) \cdots h_{-n+1}(x), \end{aligned} \tag{2.1}$$

so that $h_0(x)h_0(x) = h_0(2x)$. Finally consider the infinite products in NilCox given by

$$\mathfrak{S}^{\mathsf{B}} := \prod_{i=-\infty}^{0} B(x_i) \prod_{i=1}^{n-1} A_i(x_i) \qquad \text{and} \qquad \mathfrak{S}^{\mathsf{C}} := \prod_{i=-\infty}^{0} C(x_i) \prod_{i=1}^{n-1} A_i(x_i). \tag{2.2}$$

It is straightforward to see that $\mathfrak{S}^{\mathsf{B}} = \sum_{w\in W_n^{\mathsf{BC}}} \mathfrak{S}^{\mathsf{B}}_w \cdot u_w$ and $\mathfrak{S}^{\mathsf{C}} = \sum_{w\in W_n^{\mathsf{BC}}} \mathfrak{S}^{\mathsf{C}}_w \cdot u_w$.

Proposition 2.2. It holds that

$$\mathfrak{S}^{\mathsf{B}} = \prod_{j=-\infty}^{0} \left(h_0(x_j) \prod_{i=1}^{n-1} h_i(x_{i+j} + x_j) \right) \quad \text{and} \quad \mathfrak{S}^{\mathsf{C}} = \prod_{j=-\infty}^{0} \prod_{i=0}^{n-1} h_i(x_{i+j} + x_j).$$

Proof of Theorem 1.8. Set $x_i = q^{i-1}$ in Proposition 2.2, apply Lemma 2.1 with $N = n$, $z_i = 1 + q^{i-1}$, and $\mathrm{t}_i = u_{t_{i-1}}$, and then extract the coefficient of u_w. □

2.3 Type D

Let $\mathsf{NilCox} = \mathsf{NilCox}(W_n^{\mathsf{D}})$ be the nil-Coxeter algebra of $(W, S) = (W_n^{\mathsf{D}}, \{r_{-1}, r_1, \dots, r_{n-1}\})$ and define $h_i(x) := 1 + xu_{t_i} \in \mathsf{NilCox}$ for all $i \in \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ and $x \in R$. Let

$$\begin{aligned} A_i(x) &:= h_{n-1}(x)h_{n-2}(x)\cdots h_i(x),\\ \widetilde{A}_i(x) &:= h_i(x)h_{i+1}(x)\cdots h_{n-1}(x),\\ D(x) &:= h_{n-1}(x)\cdots h_1(x)h_{-1}(x)\cdots h_{-n+1}(x). \end{aligned} \tag{2.3}$$

The Coxeter group W_n^{D} has a unique automorphism $w \mapsto w^*$ that maps $r_i \mapsto r_{-i}$ for $1 \le i < n$. This map extends by linearity to an R-algebra automorphism of NilCox with $u_w^* := u_{w^*}$. We have $A_i(x)^* = A_i(x)$ for $1 < i < n$ and $D(x)^* = D(x)$, while $A_1(x)^* = h_{n-1}(x)h_{n-2}(x)\cdots h_2(x)h_{-1}(x)$. Consider the infinite products

$$\mathfrak{S}^{\mathsf{D}} := \prod_{i=-\infty}^{0} D(x_i) \prod_{i=1}^{n-1} A_i(x_i) \qquad \text{and} \qquad (\mathfrak{S}^{\mathsf{D}})^* := \prod_{i=-\infty}^{0} D(x_i) \prod_{i=1}^{n-1} A_i(x_i)^*. \tag{2.4}$$

It is easy to see that $\mathfrak{S}^{\mathsf{D}} = \sum_{w\in W_n^{\mathsf{D}}} \mathfrak{S}^{\mathsf{D}}_w \cdot u_w$ and $(\mathfrak{S}^{\mathsf{D}})^* = \sum_{w\in W_n^{\mathsf{D}}} \mathfrak{S}^{\mathsf{D}}_w \cdot u_{w^*}$. In addition:

Proposition 2.3. One has

$$\mathfrak{S}^{\mathsf{D}} = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} h_{-i}(x_{i+2j-1} + x_{2j-1}) \prod_{i=1}^{n-1} h_i(x_{i+2j} + x_{2j}) \right).$$

Proof of Theorem 1.11. By Proposition 2.3 we have

$$\mathfrak{S}^{\mathsf{D}}(x_i \mapsto q^{i-1}) = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} \left(1 + q^{2(j-1)} \cdot (1+q^i) \cdot u_{r_{-i}}\right) \cdot \prod_{i=1}^{n-1} \left(1 + q^{2(j-1)} \cdot q(1+q^i) \cdot u_{r_i}\right) \right).$$

Apply Lemma 2.1 with q replaced by q^2 and $N = 2n-2$ to the right side of the preceding identity, using the parameters $z_i = 1 + q^i$, $z_{n-1+i} = q(1+q^i)$, $\mathrm{t}_i = u_{r_{-i}}$, and $\mathrm{t}_{n-1+i} = u_{r_i}$ for $1 \le i < n$. Then extract the coefficient of u_w. □

3 Principal specializations of Grothendieck polynomials

In this section we describe some extensions of Theorems 1.5, 1.8, and 1.11 for *Grothendieck polynomials* in classical types.

3.1 Id-Coxeter algebras

Again let (W, S) be an arbitrary Coxeter system with length function ℓ. We will work in a generalization of the algebra $\mathsf{NilCox}(W)$. Fix an element $\beta \in R$. Let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W)$ be the R-module of all formal R-linear combinations of the symbols π_w for $w \in W$. This module has a unique R-algebra structure with bilinear multiplication satisfying

$$\pi_v \pi_w = \pi_{vw} \text{ if } \ell(vw) = \ell(v) + \ell(w) \qquad \text{and} \qquad \pi_s^2 = \beta\pi_s$$

for $v, w \in W$, $s \in S$ [7, Def. 1], which we call the *id-Coxeter algebra* of (W, S). For $x, y \in R$ and $s \in S$, define $x \oplus y := x + y + \beta xy$ and $h_s^{(\beta)}(x) := 1 + x\pi_s$. Then $h_s^{(\beta)}(x)h_s^{(\beta)}(y) = h_s^{(\beta)}(x \oplus y)$, and if $st = ts$ then $h_s^{(\beta)}(x)h_t^{(\beta)}(y) = h_t^{(\beta)}(y)h_s^{(\beta)}(x)$ [7, Lem. 1].

3.2 Type A

Let $\overleftarrow{S}_n := \langle s_i : i < n \rangle$ be the Coxeter group of permutations $w \in S_\mathbb{Z}$ with $w(i) = i$ for all $i > n$. In this section we write $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(\overleftarrow{S}_n)$ and set $\pi_i := \pi_{s_i} \in \mathsf{IdCox}_\beta$ for integers $i < n$. Define $\mathsf{Hecke}(w)$ for $w \in \overleftarrow{S}_n$ to be the set of words $a_1a_2 \cdots a_N$ such that $\pi_w = \beta^{N-\ell(w)}\pi_{a_1}\pi_{a_2} \cdots \pi_{a_N}$. Recall the set $\mathsf{Compatible}(a)$ from Definition 1.1.

Definition 3.1. The *backstable Grothendieck polynomial* of $w \in S_n \subsetneq \overleftarrow{S}_n$ is

$$\overleftarrow{\mathfrak{G}}_w := \sum_{a \in \mathsf{Hecke}(w)} \sum_{\mathbf{i} \in \mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i})-\ell(w)} x_\mathbf{i} \in \mathbb{Z}[\beta][[\ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}]].$$

The function $\mathfrak{G}_w := \overleftarrow{\mathfrak{G}}_w(\ldots, 0, 0, x_1, x_2, \ldots, x_{n-1})$ is the ordinary *Grothendieck polynomial* of $w \in S_n$. The power series $G_w := \overleftarrow{\mathfrak{G}}_w(\ldots, x_3, x_2, x_1, 0, 0, \ldots, 0)$ given by setting $x_i \mapsto 0$ for $i > 0$ and $x_i \mapsto x_{1-i}$ for $i \leq 0$ is a symmetric function in the x_i variables, which is usually called the *stable Grothendieck polynomial* of $w \in S_n$. Specializing $\beta \mapsto 0$ transforms $\overleftarrow{\mathfrak{G}}_w \mapsto \overleftarrow{\mathfrak{S}}_w$ from Section 1. The Grothendieck polynomials $\mathfrak{G}_w$ are closely related to the K-theory of flag varieties and Grassmannians [4, 10].

For $i < n$, let $h_i^{(\beta)}(x) := 1 + x\pi_i$ and $A_i^{(\beta)}(x) := h_{n-1}^{(\beta)}(x)h_{n-2}^{(\beta)}(x) \cdots h_i^{(\beta)}(x)$. Define

$$\overleftarrow{\mathfrak{G}} := \cdots A_{n-3}^{(\beta)}(x_{n-3})A_{n-2}^{(\beta)}(x_{n-2})A_{n-1}^{(\beta)}(x_{n-1}) = \prod_{i=-\infty}^{n-1} A_i^{(\beta)}(x_i) \in \mathsf{IdCox}_\beta. \tag{3.1}$$

If $w \in S_n$ then the coefficient of π_w in this expression is $\overleftarrow{\mathfrak{G}}_w$.

Proposition 3.2. It holds that $\overleftarrow{\mathfrak{G}} = \prod_{j=-\infty}^{0} \prod_{i=-\infty}^{n-1} h_i^{(\beta)}(x_{i+j})$.

Theorem 3.3. If $w \in S_n \subsetneq \overleftarrow{S}_n$ then

$$\overleftarrow{\mathfrak{G}}_w(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{Hecke}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\sum a + \mathsf{comaj}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

Proof. For $w \in S_n$, the coefficient of π_w in $\overleftarrow{\mathfrak{G}}$ is the same as the coefficient of π_w in the product $\prod_{j=-\infty}^{0} \prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+j})$. This coefficient is $\overleftarrow{\mathfrak{G}}_w$, and the theorem follows by applying Lemma 2.1 with $N = n-1$ and $z_i \mathsf{t}_i = q^i \pi_{s_i}$ to the latter expression. □

There are *Grothendieck polynomials* in the other classical types [7] which generalize $\mathfrak{S}_w^{\mathsf{B}}$, $\mathfrak{S}_w^{\mathsf{C}}$, and $\mathfrak{S}_w^{\mathsf{D}}$. We discuss these formal power series next.

3.3 Type B/C

In this section let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}})$ and write $\pi_i := \pi_{t_i} \in \mathsf{IdCox}_\beta$ for $-n < i < n$. Given a permutation $w \in W_n^{\mathsf{BC}}$, define $\mathsf{Hecke}_B^{\pm}(w)$ and $\mathsf{Hecke}_C^{\pm}(w)$ to be the sets of words $a_1 a_2 \cdots a_N$, with letters in $\{-n+1, \dots, -1, 0, 1, \dots, n-1\}$ and $\{-n+1 < \cdots < -1 < -0 < 0 < 1 < \cdots < n-1\}$, respectively, such that $\pi_w = \beta^{N-\ell(w)} \pi_{a_1} \pi_{a_2} \cdots \pi_{a_N} \in \mathsf{IdCox}_\beta$, where $\pi_{-0} := \pi_0 \in \mathsf{IdCox}_\beta$. Recall that we view -0 as a symbol distinct from 0.

Definition 3.4. The *type B/C Grothendieck polynomials* of $w \in W_n^{\mathsf{BC}}$ are

$$\mathfrak{G}_w^{\mathsf{B}} := \sum_{\substack{a \in \mathsf{Hecke}_B^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}} \quad \text{and} \quad \mathfrak{G}_w^{\mathsf{C}} := \sum_{\substack{a \in \mathsf{Hecke}_C^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}}.$$

We may consider the finite sums

$$\mathfrak{G}^{\mathsf{B}} := \sum_{w \in W_n^{\mathsf{BC}}} \mathfrak{G}_w^{\mathsf{B}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}) \quad \text{and} \quad \mathfrak{G}^{\mathsf{C}} := \sum_{w \in W_n^{\mathsf{BC}}} \mathfrak{G}_w^{\mathsf{C}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}).$$

Define $A_i^{(\beta)}(x)$, $B^{(\beta)}(x)$, and $C^{(\beta)}(x)$ as in (2.1) but with $h_i(x)$ replaced by

$$h_i^{(\beta)}(x) := 1 + x\pi_i \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}) \qquad \text{for } -n < i < n \text{ and } x \in R.$$

Then $\mathfrak{G}^{\mathsf{B}}$ and $\mathfrak{G}^{\mathsf{C}}$ are given by (2.2) with A_i, B, C replaced by $A_i^{(\beta)}$, $B^{(\beta)}$, $C^{(\beta)}$. [7, Def. 9] shows that $\mathfrak{G}_w^{\mathsf{B}}$ and $\mathfrak{G}_w^{\mathsf{C}}$ are obtained from Kirillov and Naruse's *double Grothendieck polynomials* $\mathcal{G}_w^{\mathsf{B}}(a, b; x)$ and $\mathcal{G}_w^{\mathsf{C}}(a, b; x)$ by setting $a_i \mapsto x_i$, $b_i \mapsto 0$, and $x_i \mapsto x_{1-i}$.

Proposition 3.5. It holds that

$$\mathfrak{G}^{\mathsf{B}} = \prod_{j=-\infty}^{0} \left(h_0^{(\beta)}(x_j) \prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+j} \oplus x_j) \right) \quad \text{and} \quad \mathfrak{G}^{\mathsf{C}} = \prod_{j=-\infty}^{0} \prod_{i=0}^{n-1} h_i^{(\beta)}(x_{i+j} \oplus x_j).$$

For a word $a = a_1 a_2 \cdots a_p$ with $a_i \in \{-n+1 < \cdots < -1 < -0 < 0 < 1 < \cdots < n-1\}$, let $I(a)$ be the set of indices $i \in [p]$ with $a_i \in \{1, 2, \dots, n-1\}$ and define

$$\Sigma_{\mathsf{BC}}(a) := \sum_{i \in I(a)} a_i \qquad \text{and} \qquad \mathrm{comaj}_{\mathsf{BC}}(a) := \sum_{a_i \prec a_{i+1}} i \tag{3.2}$$

where $\prec$ is the order $-0 \prec 0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \dots$. For example, if $a = -1, 1, -2, 1$ then $\Sigma_{\mathsf{BC}}(a) = 1 + 1 = 2$ and $\mathrm{comaj}_{\mathsf{BC}}(a) = 1 + 2 = 3$.

Theorem 3.6. If $w \in W_n^{\mathsf{BC}}$ then the following identities hold:

(a) $\mathfrak{G}_w^{\mathsf{B}}(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{Hecke}_B^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\Sigma_{\mathsf{BC}}(a) + \mathrm{comaj}_{\mathsf{BC}}(a)}$.

(b) $\mathfrak{G}_w^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{Hecke}_C^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\Sigma_{\mathsf{BC}}(a) + \mathrm{comaj}_{\mathsf{BC}}(a)}$.

The right hand expressions in both parts are interpreted as Laurent series in q^{-1}.

One can check that the second identity reduces to Theorem 1.8 when $\beta = 0$.

Proof. Part (a) is similar so we just prove (b). As $h_i^{(\beta)}(x_{i+j} \oplus x_j) = h_i^{(\beta)}(x_j) h_i^{(\beta)}(x_{i+j})$, we have $\mathfrak{G}^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \prod_{j=-\infty}^{0} \prod_{i=0}^{n-1} (1 + q^{j-1} \cdot \pi_i)(1 + q^{j-1} \cdot q^i \cdot \pi_i)$ by Proposition 3.5. The identity for $\mathfrak{G}_w^{\mathsf{C}}$ follows by extracting the coefficient of π_w from the right side after applying Lemma 2.1 with $N = 2n$ and with $z_1, z_2, \dots, z_{2n}$ and $\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_{2n}$ replaced by $1, 1, 1, q, 1, q^2, \dots, 1, q^{n-1}$ and $\pi_0, \pi_0, \pi_1, \pi_1, \dots, \pi_{n-1}, \pi_{n-1}$. ☐

3.4 Type D

In this section let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W_n^{\mathsf{D}})$ and $\pi_i := \pi_{r_i} \in \mathsf{IdCox}_\beta$. Given $w \in W_n^{\mathsf{D}}$, let $\mathsf{Hecke}_D^{\pm}(w)$ be the set of words $a_1 a_2 \cdots a_N$ with letters in $[\pm(n-1)] := \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ such that $\pi_w = \beta^{N-\ell(w)} \pi_{a_1} \pi_{a_2} \cdots \pi_{a_N} \in \mathsf{IdCox}_\beta$.

Definition 3.7. The *type D Grothendieck polynomial* of $w \in W_n^{\mathsf{D}}$ is

$$\mathfrak{G}_w^{\mathsf{D}} := \sum_{a \in \mathsf{Hecke}_D^{\pm}(w)} \sum_{\mathbf{i} \in \mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i}) - \ell(w)} x_{\mathbf{i}}.$$

We consider the sum

$$\mathfrak{G}^{\mathsf{D}} := \sum_{w \in W_n^{\mathsf{D}}} \mathfrak{G}_w^{\mathsf{D}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{D}}).$$

If we define $A_i^{(\beta)}(x)$ and $D^{(\beta)}(x)$ as in (2.3) but with $h_i(x)$ replaced by

$$h_i^{(\beta)}(x) := 1 + x\pi_i \in \mathsf{IdCox}_\beta(W_n^{\mathsf{D}}) \qquad \text{for } i \in [\pm(n-1)] \text{ and } x \in R,$$

then $\mathfrak{G}^{\mathsf{D}}$ is given by the formula in (2.4) with A_i and D replaced by $A_i^{(\beta)}$ and $D^{(\beta)}$. Comparing with [7, Def. 9] shows that $\mathfrak{G}_w^{\mathsf{D}}$ is obtained from Kirillov and Naruse's *double Grothendieck polynomial* $\mathcal{G}_w^{\mathsf{D}}(a, b; x)$ by substituting $a_i \mapsto x_i$, $b_i \mapsto 0$, and $x_i \mapsto x_{1-i}$.

Proposition 3.8. It holds that $\mathfrak{G}^{\mathsf{D}} = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} h_{-i}^{(\beta)}(x_{i+2j-1} \oplus x_{2j-1}) \prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+2j} \oplus x_{2j}) \right)$.

To state an analogue of Theorem 1.11 for $\mathfrak{G}_w^{\mathsf{D}}$, we must consider the ordered alphabet $\{-1' \prec -1 \prec -2' \prec -2 \prec \cdots \prec -n' \prec -n \prec 1' \prec 1 \prec 2' \prec 2 \prec \cdots \prec n' \prec n\}$. If $w \in W_n^{\mathsf{D}}$ then let $\mathsf{PrimedHecke}_D^{\pm}(w)$ denote the set of words in this alphabet which become elements of $\mathsf{Hecke}_D^{\pm}(w)$ when all primes are removed. Given such a word $a = a_1 a_2 \cdots a_p$, let $J(a)$ be the set of indices $i \in [p]$ for which a_i is unprimed, and define $\Sigma_{\mathsf{D}}(a) := \sum_{i \in J(a)} |a_i|$ and $\mathrm{comaj}_{\mathsf{D}}(a) := |\{i : a_i \in \{1', 1, 2', 2, \dots\}\}| + \sum_{a_i \prec a_{i+1}} 2i$. For example, if $a = 2', -1', -1, -3, 2$ then $\Sigma_{\mathsf{D}}(a) = 6$ and $\mathrm{comaj}_{\mathsf{D}}(a) = 20$.

Theorem 3.9. If $w \in W_n^{\mathsf{D}}$ then

$$\mathfrak{G}_w^{\mathsf{D}}(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{PrimedHecke}_D^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q^2-1)(q^4-1)\cdots(q^{2\ell(a)}-1)} q^{\Sigma_{\mathsf{D}}(a) + \mathrm{comaj}_{\mathsf{D}}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

As with Theorem 3.6, this identity reduces to Theorem 1.11 when $\beta = 0$.

Proof. Proposition 3.8 implies that $\mathfrak{G}^{\mathsf{D}}(x_i \mapsto q^{i-1})$ is

$$\prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} (1 + q^{2(j-1)} \cdot \pi_{-i})(1 + q^{2(j-1)} \cdot q^i \cdot \pi_{-i}) \cdot \prod_{i=1}^{n-1} (1 + q^{2(j-1)} \cdot q \cdot \pi_i)(1 + q^{2(j-1)} \cdot q^{i+1} \cdot \pi_i) \right).$$

The identity for $\mathfrak{G}_w^{\mathsf{D}}$ follows by extracting the coefficient of π_w from this expression, using Lemma 2.1 with q replaced by q^2 and with $N = 4n - 4$. When applying the lemma, we set $z_1, z_2, \dots, z_{2n-2}$ (respectively, $z_{2n-1}, z_{2n}, \dots, z_{4n-4}$) to $1, q, 1, q^2, 1, q^3 \dots$ (respectively, $q, q^2, q, q^3, q, q^4 \dots$), while taking $\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_{2n-2}$ (respectively, $\mathfrak{t}_{2n-1}, \mathfrak{t}_{2n}, \dots, \mathfrak{t}_{4n-4}$) to be $\pi_{-1}, \pi_{-1}, \pi_{-2}, \pi_{-2}, \dots$ (respectively, $\pi_1, \pi_1, \pi_2, \pi_2, \dots$). □

References

[1] S. C. Billey and M. Haiman. "Schubert polynomials for the classical groups". *J. Amer. Math. Soc.* **8** (1995), pp. 443–482.

[2] S. C. Billey, A. E. Holroyd, and B. Young. "A bijective proof of Macdonald's reduced word formula". *Algebraic Combinatorics* **2** (2019), pp. 217–248.

[3] S. C. Billey, W. Jockusch, and R. P. Stanley. "Some Combinatorial Properties of Schubert Polynomials". *J. Algebr. Combin.* **2** (1993), pp. 345–374.

[4] A. S. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189** (2002), pp. 37–78.

[5] S. Fomin and A. N. Kirillov. "Reduced Words and Plane Partitions". *J. Algebr. Combin.* **6** (1997), pp. 311–319.

[6] S. Fomin and R. P. Stanley. "Schubert Polynomials and the NilCoxeter Algebra". *Adv. Math.* **103** (1994), pp. 196–207.

[7] A. N. Kirillov and H. Naruse. "Construction of Double Grothendieck Polynomials of Classical Types using IdCoxeter Algebras". *Tokyo J. Math.* **39** (2017), pp. 695–728.

[8] T. Lam, S. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compos. Math.* (to appear).

[9] I. G. Macdonald. *Notes on Schubert Polynomials*. Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, Montreal, 1991.

[10] E. Marberg and B. Pawlowski. "K-theory formulas for orthogonal and symplectic orbit closures". *Adv. Math.* **372** (2020), p. 107299.

[11] E. Marberg and B. Pawlowski. "Principal specializations of Schubert polynomials in classical types". *Algebraic Combinatorics* (to appear).

[12] B. Pawlowski. "Universal graph Schubert varieties". *Transform. Groups* (to appear).

[13] B. Young. "A Markov growth process for Macdonald's distribution on reduced words". 2014.