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Young row-strict quasisymmetric Schur functions and 0-Hecke modules

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Abstract. We give a representation-theoretic interpretation of the Young row-strict quasisymmetric Schur basis of quasisymmetric functions by constructing modules of the 0-Hecke algebra whose quasisymmetric characteristics are the Young row-strict quasisymmetric Schur functions. Additionally, we classify when these modules are indecomposable.

Keywords: Quasisymmetric functions, 0-Hecke algebras, Young row-strict quasisymmetric Schur functions

1 Introduction

The Schur functions form a basis of the algebra Sym of symmetric functions that is of particular interest in algebraic combinatorics. One feature among many of these functions is their realization as the images of irreducible characters of the symmetric group under the Frobenius characteristic map.

The algebra QSym of quasisymmetric functions contains Sym as a subalgebra. It is therefore of interest to find bases of QSym that are *Schur-like*, i.e., that reflect properties of the Schur functions. Several such bases have been introduced and studied including the fundamental quasisymmetric functions [8], the dual immaculate quasisymmetric functions [3], the quasisymmetric Schur functions [9], the row-strict quasisymmetric Schur functions [12], and the extended Schur functions [1].

The algebra QSym has an interpretation in terms of representations of 0-*Hecke algebras*, a certain deformation of group algebras of symmetric groups. In [7] an isomorphism of algebras between the Grothendieck group of representations of 0-Hecke algebras and QSym was established. This isomorphism is known as the *quasisymmetric characteristic*. Under the quasisymmetric characteristic map, the images of the irreducible representations of 0-Hecke algebras are exactly the fundamental quasisymmetric functions. This is analogous to the role Schur functions play for the representation theory of symmetric groups, illustrating one sense in which the fundamental quasisymmetric basis is Schur-like.

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The row-strict quasisymmetric Schur functions are the images of the well-studied quasiymmetric Schur functions under an extension of the famous ω involution on Sym to QSym. Similarly to the quasisymmetric Schur functions, they provide a particularly nice refinement of Schur functions. These functions expand positively in the fundamental basis of QSym and thus it is natural to ask for an interpretation in terms of 0-Hecke representations. Indeed, dual immaculate, quasisymmetric Schur and extended Schur functions have all been realized as quasisymmetric characteristics of certain 0-Hecke modules ([4], [16], [15] respectively), and Mason and Niese raised the question of doing so for (Young) row-strict quasisymmetric Schur functions in [13]. Moreover, there has been significant recent interest in the structure of such 0-Hecke modules, especially the modules whose quasisymmetric characteristics are the quasisymmetric Schur functions or closely related functions, and particularly surrounding indecomposability. See, for example, [11], [17], [6], [5].

In this extended abstract, we provide an outline of how in [2] we answer the question of Mason and Niese by constructing 0-Hecke modules whose quasisymmetric characteristics are the Young row-strict quasisymmetric Schur functions. We also classify when these modules are indecomposable, which turns out to be more involved that the indecomposability classifications for modules for dual immaculate, quasisymmetric Schur and extended Schur functions in [4], [16], [15] respectively.

2 Background

2.1 Young row-strict quasisymmetric Schur functions

A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n is a finite sequence of positive integers that sum to n; we write $\alpha \vDash n$. For each composition α, define a subset $\mathbb{S}(\alpha)$ of $\{1, \ldots n-1\}$ by $\mathbb{S}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. The map $\alpha \mapsto \mathbb{S}(\alpha)$ is a bijection between compositions of n and subsets of $\{1, \ldots, n-1\}$. The inverse is denoted by comp_n, i.e., $\mathrm{comp}_n(\mathbb{S}(\alpha)) = \alpha$.

Let $\mathbb{C}[[x_1, x_2, \ldots]]$ denote the algebra of formal power series of bounded degree in commuting variables $x_1, x_2, \ldots$. The algebra QSym of *quasisymmetric functions* is a subalgebra of $\mathbb{C}[[x_1, x_2, \ldots]]$. Bases of QSym are indexed by compositions; two important bases are the *monomial quasisymmetric functions* $\{M_\alpha\}$ and the *fundamental quasisymmetric functions* $\{F_\alpha\}$ [8]. These are defined by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \qquad \text{and} \qquad F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta,$$

where β *refines* α if α can be obtained from β by summing consecutive entries.

Example 2.1. Let $\alpha = (1,2,2)$. We have

$$M_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k$$

and

$$F_{(2,2,1)} = M_{(2,2,1)} + M_{(1,1,2,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}.$$

The *diagram* $D(\alpha)$ of a composition α is the array of boxes having α_i boxes in row i, left-justified. We use French notation for composition diagrams, i.e., the rows are numbered from bottom to top. Let (c,r) denote the box in row r and column c. We say the box $(c+1,r) \in D(\alpha)$ is *right-adjacent* to the box (c,r), and that (c,r) is *left-adjacent* to $(c+1,r)$.

Example 2.2. Let $\alpha = (2,4)$. Then $D(\alpha) = \begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$.

We now describe the combinatorial objects needed to define Young row-strict quasisymmetric Schur functions. Let $\alpha \vDash n$. A *standard Young row-strict composition tableau* [13] of shape α is a filling T of $D(\alpha)$ with the integers $1, 2, \ldots, n$, each used once, satisfying the following conditions:

(R1) Entries increase from left to right along every row

(R2) Entries increase from bottom to top in the leftmost column

(R3) If boxes (c,r) and $(c+1,r')$ for $r' < r$ are in $D(\alpha)$ and $T(c,r) < T(c+1,r')$, then $T(c+1,r) < T(c+1,r')$, where we define $T(c+1,r) = \infty$ if $(c+1,r) \notin D(\alpha)$.

Condition (R3), sometimes referred to as the *triple condition*, states that for any three boxes arranged as below, if $a < c$ then $b < c$.

$$\begin{array}{cc} \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} \\ \hline & \vdots \\ \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \cline{2-2} \end{array}$$

Denote by $\mathrm{SYRT}(\alpha)$ the set of all standard Young row-strict composition tableaux of shape α. Given $T \in \mathrm{SYRT}(\alpha)$, if a box with entry i is left-adjacent to a box with entry j in T, we say i is left-adjacent to j (and j is right-adjacent to i).

Example 2.3. Illustrating (R3), the two tableaux of shape $(3,2,2)$ below fail (R3), with entries from a triple that cause the failure underlined. In the tableau on the right we have $a = 3$, $c = 5$ and $b = \infty$.

$$\begin{array}{|c|c|c} \cline{1-2} \underline{3} & \underline{6} & \\ \cline{1-2} 2 & 7 & \\ \hline 1 & \underline{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \cline{1-2} 6 & 7 & \\ \cline{1-2} 2 & \underline{3} & \\ \hline 1 & 4 & \multicolumn{1}{c|}{\underline{5}} \\ \hline \end{array}$$

Define the *descent set* of $T \in \mathrm{SYRT}(\alpha)$, denoted $\mathrm{Des}(T)$, to be the set of entries i such that $i+1$ appears strictly to the right of i in T.

Example 2.4. Let $\alpha = (2,4)$. The tableaux in $\mathrm{SYRT}(\alpha)$, along with their descent sets, are shown below.

$$\begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 6 \\ \hline 1 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 6 \\ \hline 1 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline 1 & 6 \\ \cline{1-2}\end{array}$$

$$\{1,3,4,5\} \qquad \{2,4,5\} \qquad \{2,3,5\} \qquad \{2,3,4\}$$

For $\alpha \vDash n$, the Young row-strict quasisymmetric Schur function $\mathcal{R}_\alpha$ [13] is defined by

$$\mathcal{R}_\alpha = \sum_{T \in \mathrm{SYRT}(\alpha)} F_{\mathrm{comp}_n(\mathrm{Des}(T))}.$$

Example 2.5. By Example 2.4 above, we have

$$\mathcal{R}_{(2,4)} = F_{(1,2,1,1,1)} + F_{(2,2,1,1)} + F_{(2,1,2,1)} + F_{(2,1,1,2)}.$$

Remark 2.6. The term *row-strict* comes from a semistandard variant of the SYRTs, which index the monomial expansion of $\mathcal{R}_\alpha$ as opposed to the fundamental quasisymmetric expansion. In a semistandard Young row-strict tableau, entries are required to strictly increase along rows, but may be repeated in columns; see [13]. We do not need the semistandard variant for our purposes.

2.2 0-Hecke algebras and quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$-algebra having $n-1$ generators $T_1, \ldots, T_{n-1}$ subject to the relations

$$\begin{aligned} T_i^2 &= T_i \quad \text{for all } 1 \le i \le n-1 \\ T_iT_j &= T_jT_i \quad \text{for all } 1 \le i,j \le n-1 \text{ such that } |i-j| \ge 2 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for all } 1 \le i \le n-2. \end{aligned}$$

Given a permutation $\sigma \in S_n$, one can define $T_\sigma \in H_n(0)$ by $T_\sigma = T_{i_1}T_{i_2}\cdots T_{i_r}$ where $s_{i_1}s_{i_2}\cdots s_{i_r}$ is any reduced word for σ. This is well-defined since the T_i satisfy the same braid and commutativity relations as the generators s_i of S_n. Then $\{T_\sigma : \sigma \in S_n\}$ is an additive basis for $H_n(0)$.

The *Grothendieck group* $\mathcal{G}_0(H_n(0))$ is the linear span of the isomorphism classes of the finite-dimensional representations of $H_n(0)$, subject to the relation $[Y] = [X] + [Z]$ whenever there is a short exact sequence $0 \to X \to Y \to Z \to 0$ of $H_n(0)$-representations X, Y, Z.

The irreducible representations of $H_n(0)$ can be indexed by the 2^{n-1} compositions of n. Given a composition α, let $\mathbf{F}_\alpha$ denote the corresponding irreducible representation. By [14], $\mathbf{F}_\alpha$ is one-dimensional; let $\{v_\alpha\}$ be a basis of $\mathbf{F}_\alpha$. The structure of $\mathbf{F}_\alpha$ as a $H_n(0)$-representation is given by the following action of the T_i:

$$T_i(v_\alpha) = \begin{cases} v_\alpha & \text{if } i \notin \mathbb{S}(\alpha) \\ 0 & \text{if } i \in \mathbb{S}(\alpha). \end{cases} \tag{2.1}$$

Now define

$$\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0)).$$

The set $\{[\mathbf{F}_\alpha]\}$ as α ranges over all compositions is a basis of $\mathcal{G}$. There is an isomorphism of algebras $ch : \mathcal{G} \to \mathrm{QSym}$, given by setting $ch([\mathbf{F}_\alpha]) = F_\alpha$ [7]. For X an $H_n(0)$-module, the image $ch([X])$ is called the *quasisymmetric characteristic* of X.

3 Modules for Young row-strict quasisymmetric Schur functions

In this section, for any composition α of n we construct an $H_n(0)$-module $\mathbf{R}_\alpha$ whose quasisymmetric characteristic is the Young row-strict quasisymmetric Schur function $\mathcal{R}_\alpha$. This answers a question of Mason and Niese [13].

Let $\alpha \vDash n$. Given $T \in \mathrm{SYRT}(\alpha)$, let $s_i(T)$ denote the filling of $D(\alpha)$ obtained by swapping the entries i and $i+1$ of T. Then for any $T \in \mathrm{SYRT}(\alpha)$ and any $1 \leq i \leq n-1$, we define

$$\pi_i(T) = \begin{cases} T & \text{if i+1 is weakly left of i in T} \\ 0 & \text{if i+1 is right-adjacent of i in T} \\ s_i(T) & \text{otherwise.} \end{cases}$$

Example 3.1. Let $\alpha = (2,4)$, and let

$$T = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 6 \\ \hline 1 & 5 \\ \cline{1-2} \end{array} \in \mathrm{SYRT}(\alpha).$$

Then $\pi_1(T) = \pi_4(T) = T$, $\pi_2(T) = \pi_3(T) = 0$ and

$$\pi_5(T) = s_5(T) = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline 1 & 6 \\ \cline{1-2} \end{array} \in \mathrm{SYRT}(\alpha).$$

Let $\mathbf{R}_\alpha$ denote the complex vector space with basis $\mathrm{SYRT}(\alpha)$. The following theorem is proved via case-checking.

Theorem 3.2. *The operators π_i define an $H_n(0)$-action on $\mathbf{R}_\alpha$.*

Thus $\mathbf{R}_\alpha$ is an $H_n(0)$-module. To show it has quasisymmetric characteristic $\mathcal{R}_\alpha$, we define a relation $\preceq$ on $\mathrm{SYRT}(\alpha)$ by $T \preceq S$ if S can be obtained from T via applying a (possibly empty) sequence of the π_i operators.

Lemma 3.3. *The relation $\preceq$ defines a partial order on* $\mathrm{SYRT}(\alpha)$.

Fix a total order $\preceq^\star$ on $\mathrm{SYRT}(\alpha)$ that extends the partial order $\preceq$. We may assume the elements of $\mathrm{SYRT}(\alpha)$ are ordered $T_m \preceq^\star T_{m-1} \preceq^\star \cdots \preceq^\star T_1$. For each $1 \le j \le m$, define $X_j = \mathrm{span}\{T_1, \dots, T_j\}$. Then for all $1 \le j \le m$, X_j is a $H_n(0)$-submodule of $\mathbf{R}_\alpha$, and we have the filtration

$$0 := X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathbf{R}_\alpha$$

of $\mathbf{R}_\alpha$. Each quotient module X_j/X_{j-1} is, by definition, one-dimensional with basis $\{T_j\}$.

Lemma 3.4. *For each $1 \le i \le n-1$ and each $1 \le j \le m$, in X_j/X_{j-1} we have*

$$\pi_i(T_j) = \begin{cases} T_j & \text{if } i+1 \text{ is weakly left of } i \text{ in } T_j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.5. *Let $\alpha \vDash n$. Then $ch([\mathbf{R}_\alpha]) = \mathcal{R}_\alpha$.*

Proof. Each of the $H_n(0)$-modules X_j/X_{j-1} is one-dimensional and thus irreducible. By Lemma 3.4 we have

$$\pi_i(T_j) = \begin{cases} T_j & \text{if } i \notin \mathrm{Des}(T_j) \\ 0 & \text{if } i \in \mathrm{Des}(T_j). \end{cases}$$

By (2.1), this implies that X_j/X_{j-1} is isomorphic as $H_n(0)$-modules to $\mathbf{F}_{\mathrm{comp}_n(\mathrm{Des}(T_j))}$, hence $[X_j/X_{j-1}] = [\mathbf{F}_{\mathrm{comp}_n(\mathrm{Des}(T_j))}]$. It follows that

$$ch([\mathbf{R}_\alpha]) = \sum_{j=1}^{m} ch([X_j/X_{j-1}]) = \sum_{j=1}^{m} ch([\mathbf{F}_{\mathrm{comp}_n(\mathrm{Des}(T_j))}]) = \sum_{T \in \mathrm{SYRT}(\alpha)} F_{\mathrm{comp}_n(\mathrm{Des}(T))} = \mathcal{R}_\alpha. \quad \square$$

4 Structure of the modules

The goal for the remainder of this abstract is to classify for which α the $H_n(0)$-module $\mathbf{R}_\alpha$ is indecomposable. In this section we give a formula decomposing $\mathbf{R}_\alpha$ into a direct sum of nonzero submodules, show that each of these submodules is generated by a single SYRT, and classify for which α this formula has only a single term. The results in this section, which will be needed for the classification of indecomposability, are similar to those obtained in [16] for the modules for quasisymmetric Schur functions.

Define an equivalence relation $\sim$ on $\mathrm{SYRT}(\alpha)$ by declaring $T \sim T'$ if for every $1 \le k \le \max(\alpha)$, the relative order of the entries of the kth column of T is the same as the relative order of the entries in the kth column of T'.

Example 4.1. In Example 2.4, the first SYRT forms one equivalence class, and the remaining three SYRTs form a second equivalence class.

Suppose that $\sim$ decomposes $\mathrm{SYRT}(\alpha)$ into equivalence classes $E_0, E_1, \ldots, E_r$, where E_0 denotes the class of all $T \in \mathrm{SYRT}(\alpha)$ such that entries increase from bottom to top in every column of T.

Denote by $\mathbf{R}_\alpha^{E_j}$ the subspace of $\mathbf{R}_\alpha$ given by the complex span of E_j.

Proposition 4.2. *Let $\alpha \vDash n$. Then for each j, the vector space $\mathbf{R}_\alpha^{E_j}$ is an $H_n(0)$-submodule of $\mathbf{R}_\alpha$.*

As a result, we have

Corollary 4.3. *Let $\alpha \vDash n$. Then $\mathbf{R}_\alpha$ is isomorphic as $H_n(0)$-modules to $\bigoplus_{j=0}^{r} \mathbf{R}_\alpha^{E_j}$.*

We now show that each E_j contains a unique element that generates $\mathbf{R}_\alpha^{E_j}$. Following the nomenclature of [16, 11], we call $T \in E_j$ a *source tableau* if there is no $T' \in E_j$ such that $T' \neq T$ and $\pi_i(T') = T$ for some i. Existence of source tableaux is immediate from the partial order (Lemma 3.3) on $\mathrm{SYRT}(\alpha)$ restricted to E_j.

Lemma 4.4. *Let $\alpha \vDash n$. There is at least one source tableau in any equivalence class $E_j \subset \mathrm{SYRT}(\alpha)$.*

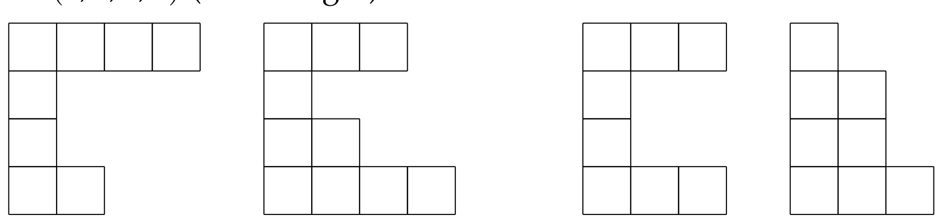
In fact, each equivalence class E_j contains *exactly* one source tableau.

Proposition 4.5. *Let $\alpha \vDash n$. The submodule $\mathbf{R}_\alpha^{E_j}$ is cyclic, generated by the unique source tableau in E_j.*

The fact that $\mathbf{R}_\alpha^{E_j}$ is cyclic follows from the uniqueness of the source tableaux: every tableau in E_j can be obtained by applying a sequence of 0-Hecke operators to the source tableau. The proof of uniqueness of source tableaux is somewhat technical, and involves showing that any two source tableaux in E_j must have the entry n in the same box, then arguing by induction that their entries must agree in every box.

We now characterise the compositions that give rise to only a single equivalence class of SYRT's. Following nomenclature of [16], we define a composition α to be *simple* if whenever $\alpha_j \geq \alpha_i \geq 2$ for some $1 \leq i < j \leq \alpha$, there is some k such that $i \leq k \leq j$ and $\alpha_k = \alpha_i - 1$. Pictorially, given a pair of rows in $D(\alpha)$ where the lower row is weakly shorter (and of length at least 2), there is another row between this pair of rows that is one box shorter than the lower one.

Example 4.6. The compositions $(4,1,1,2)$ and $(3,2,1,4)$ (on the left) are simple, whereas $(3,1,1,3)$ and $(3,2,2,1)$ (on the right) are not.



Proposition 4.7. *Let $\alpha \vDash n$ be a composition. Then α is simple if and only if for every $T \in \mathrm{SYRT}(\alpha)$, entries increase from bottom to top in each column of T.*

Remark 4.8. This definition of simple composition is the same (up to reversal) of that in [16]. This is due to similarity between the definition (R1) – (R3) of SYRTs and the definition of the standard composition tableaux that index the fundamental expansion of quasisymmetric Schur functions. However, these two families of tableaux have quite a different notion of descent, which leads to the difference in the 0-Hecke action and the arguments needed to classify indecomposability for the corresponding modules.

5 Classification of indecomposability

We devote this section to sketching the proof of the following classification of indecomposability of the 0-Hecke modules for Young row-strict quasisymmetric Schur functions.

Theorem 5.1. *Let $\alpha \vDash n$. Then the $H_n(0)$-module $\mathbf{R}_\alpha$ is indecomposable if and only if α is simple.*

By Corollary 4.3, $\mathbf{R}_\alpha$ is decomposable whenever $\mathrm{SYRT}(\alpha)$ has more than one equivalence class. It is also straightforward to confirm that the class E_0 of SYRTs that increase up each column is nonempty for every α. Therefore by Proposition 4.7, if α is not simple, then $\mathbf{R}_\alpha$ is decomposable.

For the converse, the approach is to show that the submodule $\mathbf{R}_\alpha^{E_0}$ of $\mathbf{R}_\alpha$ is indecomposable. Then by Proposition 4.7, it follows that when α is simple, $\mathbf{R}_\alpha = \mathbf{R}_\alpha^{E_0}$ is indecomposable.

We start by establishing a concrete description of the source tableau for E_0, which we will need in order to prove that $\mathbf{R}_\alpha^{E_0}$ is indecomposable. We call a box in $D(\alpha)$ a *boundary box* if it is in the first column, or if it has no box strictly above it in the same column or in the column immediately to the left. Order the boundary boxes by $(a, b) < (c, d)$ if either $a = c = 1$ and $b < d$, or $a < c$.

To each boundary box we associate a collection of boxes in $D(\alpha)$ called a *thread*. The thread of the first boundary box $(1, 1)$ is the box $(1, 1)$ itself. Assuming threads have been associated to the first $k - 1$ boundary boxes, the thread of the *kth* boundary box consists of the *kth* boundary box $\mathfrak{b}$, the highest unthreaded box strictly below $\mathfrak{b}$ in the column immediately to the right of $\mathfrak{b}$, the highest unthreaded box strictly below that in the next column to the right, etc. The thread terminates when there is no unthreaded box strictly below in the next column to the right. See Example 5.4 below.

In this way, each thread is a sequence of boxes in consecutive columns, proceeding strictly northwest to southeast in $D(\alpha)$, and each box belongs to at most one thread. It is not hard to show that every box in $D(\alpha)$ belongs to some thread, thus $D(\alpha)$ is partitioned by the threads.

We use the thread decomposition to define a standard filling T_{sup} of $D(\alpha)$. Suppose $D(\alpha)$ has threads $L_1, \ldots, L_m$, in order. Define T_{sup} by filling thread L_k with integers $|L_1| + \ldots + |L_k - 1| + 1, |L_1| + \ldots + |L_k - 1| + 2, \ldots |L_1| + \ldots + |L_k - 1| + |L_k|$ consecutively from right to left.

Proposition 5.2. *Let $\alpha \vDash n$. Then T_{sup} is the source tableau of E_0.*

The proof of Proposition 5.2, and of later results, depend on the following lemma.

Lemma 5.3. *Let $\alpha \vDash n$. During the threading process, there is never an unthreaded box weakly southwest of a threaded box.*

In particular, Lemma 5.3 establishes that entries increase along each row and up each column of T_{sup}.

Example 5.4. Let $\alpha = (2, 4, 1, 2)$. The thread decomposition of $D(\alpha)$ is on the left, where the boxes are labelled by the number of the thread they belong to. The boundary boxes are the leftmost box in each thread. On the right is T_{sup}.

4	5		
3			
2	3	5	6
1	2		

6	8		
5			
3	4	7	9
1	2		

It is well-known (see e.g. [10]) that a module is indecomposable if and only if the only idempotent endomorphisms of that module are 0 and 1.

Any $H_n(0)$-module morphism $f : \mathbf{R}_\alpha^{E_0} \to \mathbf{R}_\alpha^{E_0}$ is determined by $f(T_{\text{sup}})$, since T_{sup} generates $\mathbf{R}_\alpha^{E_0}$. Let

$$f(T_{\text{sup}}) = \sum_{T \in E_0} a_T T.$$

The goal is to show that $a_T = 0$ for all $T \neq T_{\text{sup}}$. (It then follows from idempotence of f that $a_{T_{\text{sup}}}$ is either 0 or 1, and thus $f = 0$ or $f = 1$). The following lemma, whose proof is short and straightforward, eliminates a large family of SYRTs in E_0 from consideration.

Lemma 5.5. *Let $T \in \mathrm{SYRT}(\alpha)$ such that there exists an i satisfying $i \in \mathrm{Des}(T)$ but $i \notin \mathrm{Des}(T_{\text{sup}})$. Then $a_T = 0$.*

Remark 5.6. The appropriate analogues of Lemma 5.5 actually immediately characterise indecomposability for the modules associated to dual immaculate quasisymmetric functions [3], quasisymmetric Schur functions [16], and extended Schur functions [15]. In all of these cases, the source tableau of the relevant cyclic 0-Hecke (sub)module has an especially simple form, and in fact every non-source tableau has some descent that is not a descent of the source tableau. However for $\mathbf{R}_\alpha^{E_0}$ the source tableau is more complicated, and moreover not every $T \in E_0$ has a descent that is not a descent of the source tableau, even if α is simple. Therefore establishing indecomposability of $\mathbf{R}_\alpha^{E_0}$ requires more analysis.

From now on, let $\hat{T}$ denote an element of E_0 such that $\hat{T} \neq T_{\text{sup}}$ and $\text{Des}(\hat{T}) \subseteq \text{Des}(T_{\text{sup}})$. It remains to show that $a_{\hat{T}} = 0$. To achieve this we make use of a technique of König, applied in [11] to determine indecomposability for certain submodules of the 0-Hecke modules for quasisymmetric Schur functions. This requires establishing the existence of a sequence of operators that sends T_{sup} to 0 but does not send $\hat{T}$ to 0, such that each operator in the sequence applied to $\hat{T}$ exchanges entries of the SYRT it is acting on (Proposition 5.9 below). We sketch the necessary arguments, which require further structural results on $\hat{T}$ and T_{sup}.

Since T_{sup} generates $\mathbf{R}_\alpha^{E_0}$, we have $\hat{T} = \pi_\sigma(T_{\text{sup}})$ for some (nonidentity) $\sigma \in S_n$. Fix a reduced word $s_{i_1} \cdots s_{i_r}$ for σ; we have $\pi_{i_1} \cdots \pi_{i_r}(T_{\text{sup}}) = s_{i_1} \cdots s_{i_r}(T_{\text{sup}}) = \hat{T}$. Let ε denote the smallest integer that occupies a different box in $\hat{T}$ to that it occupies in T_{sup}.

Example 5.10 can be used as a running example for the following results.

Lemma 5.7. *The box containing ε in T_{sup} is the rightmost box in its thread.*

Lemma 5.7 is proved as follows. First, one can show that if $\pi_{i_1} \cdots \pi_{i_r}(T_{\text{sup}}) = s_{i_1} \cdots s_{i_r}(T_{\text{sup}}) = \hat{T}$, then all of $i_1, \ldots, i_r$ are strictly larger than ε. In particular, $\pi_{\varepsilon-1}$ is never applied, so ε is never moved leftwards by this sequence of operators. Then, since ε is in a different box in $\hat{T}$ than it is in T_{sup}, π_ε must be applied at least once in the sequence, and the box containing ε in $\hat{T}$ is strictly right of the box containing ε in T_{sup}.

Now we can argue by contradiction. If ε did not occupy the rightmost box in its thread, then $\varepsilon - 1$ would be in the column immediately right of the column of ε. It can then be shown that the first time π_ε is applied, it moves ε at least two columns rightwards (and thus strictly right of $\varepsilon - 1$). However then $\varepsilon - 1$ is a descent in $\hat{T}$ but not in T_{sup}, contradicting that $\text{Des}(\hat{T}) \subseteq \text{Des}(T_{\text{sup}})$.

With Lemma 5.7 established, one can argue from the thread structure and the fact that $\text{Des}(\hat{T}) \subseteq \text{Des}(T_{\text{sup}})$ that $\varepsilon - 1$ must be strictly left of ε in T_{sup}. In particular, ε is not in the leftmost column of T_{sup}. Therefore, we may define x to be the entry left-adjacent to ε in T_{sup}. By (R1), we have $x < \varepsilon$.

Lemma 5.8. *The entries $x, x+1, \ldots \varepsilon - 2, \varepsilon - 1$ all reside strictly left of ε in T_{sup}. Moreover, the entries $x, x+1, \ldots \varepsilon - 2, \varepsilon - 1$ all reside strictly left of ε in $\hat{T}$, and the entry left-adjacent to ε in $\hat{T}$ is strictly smaller than x.*

The first part is proved using Lemma 5.7 and properties of the thread structure of T_{sup}, in particular Lemma 5.3. The second part follows from the first part and the fact that each of $x, x+1, \ldots \varepsilon - 2, \varepsilon - 1$ occupy the same box in $\hat{T}$ as they do in T_{sup}, and the fact that the box containing ε in $\hat{T}$ is strictly right of the box containing ε in T_{sup}.

Then, immediately from Lemma 5.8 and the definition of the entry x, we obtain

Proposition 5.9. *The operator $\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}$ satisfies*

1. $\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(T_{\text{sup}}) = 0$; *and*

2. $\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(\hat{T}) = s_x s_{x+1} \dots s_{\varepsilon-2}s_{\varepsilon-1}(\hat{T}) \neq 0$.

Example 5.10. Let $\alpha = (6, 4, 5, 2, 3)$. Below are T_{sup} and a $\hat{T} \in E_0$ with $\text{Des}(\hat{T}) = \{1, 3, 6, 10, 15, 16, 18\} = \text{Des}(T_{\text{sup}})$.

$$T_{\text{sup}} = \begin{array}{|c|c|c|c|c|c|} \hline 15 & 16 & 18 \\ \cline{1-3} 10 & 14 \\ \cline{1-5} 6 & 9 & 13 & 17 & 20 \\ \cline{1-5} 3 & 5 & 8 & 12 \\ \hline 1 & 2 & 4 & 7 & 11 & 19 \\ \hline \end{array} \qquad \hat{T} = \begin{array}{|c|c|c|c|c|c|} \hline 15 & 16 & 20 \\ \cline{1-3} 10 & 14 \\ \cline{1-5} 6 & 9 & 13 & 18 & 19 \\ \cline{1-5} 3 & 5 & 8 & 12 \\ \hline 1 & 2 & 4 & 7 & 11 & 17 \\ \hline \end{array}$$

In this example we have $\hat{T} = \pi_{19}\pi_{17}\pi_{18}(T_{\text{sup}})$. We have $\varepsilon = 17$ and $x = 13$. We observe that $\pi_{13}\pi_{14}\pi_{15}\pi_{16}(T_{\text{sup}}) = 0$ while $\pi_{13}\pi_{14}\pi_{15}\pi_{16}(\hat{T}) = s_{13}s_{14}s_{15}s_{16}(\hat{T}) \neq 0$.

Recall the partial ordering on $\text{SYRT}(\alpha)$ given in Lemma 3.3. Restrict this ordering to E_0, and define the rank of $T \in E_0$ to be the smallest number of operators π_i needed for a sequence that sends T_{sup} to T. This is well-defined since in a minimal sequence of operators all π_i act as s_i, and all reduced words in the symmetric group S_n have the same length. In this way, the poset $(E_0, \preceq)$ is graded.

Theorem 5.1 can now be proved by contradiction. Suppose some $\hat{T} \neq T_{\text{sup}}$ appears in $f(T_{\text{sup}})$ with nonzero coefficient. We may assume $\hat{T}$ has maximal rank among such SYRTs. By Lemma 5.5 we have $\text{Des}(\hat{T}) \subseteq \text{Des}(T_{\text{sup}})$. Let $T' = \pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(\hat{T}) = s_x s_{x+1} \dots s_{\varepsilon-2}s_{\varepsilon-1}(\hat{T}) \neq 0$, by Proposition 5.9. The maximality of the rank of $\hat{T}$ ensures that the coefficient of T' in

$$\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(f(T_{\text{sup}})) = \sum_{T \in E_0} a_T \pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(T)$$

is exactly $a_{\hat{T}}$. However,

$$\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(f(T_{\text{sup}})) = f(\pi_x \pi_{x+1} \dots \pi_{\varepsilon-2}\pi_{\varepsilon-1}(T_{\text{sup}})) = f(0) = 0.$$

This implies $a_{\hat{T}} = 0$, which is the desired contradiction.

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