Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #74, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

P-strict promotion and piecewise-linear rowmotion, with applications to flagged tableaux

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Abstract. In this paper, we define P-strict labelings, as a generalization of semistandard Young tableaux, and show that promotion on these objects is in equivariant bijection with a piecewise-linear toggle action on an associated poset, that in many cases is conjugate to rowmotion. We apply this result to obtain new results and conjectures on flagged tableaux. We also show that in certain cases, P-strict promotion can be equivalently defined using Bender–Knuth and jeu de taquin perspectives.

Keywords: promotion, rowmotion, flagged tableaux, toggles

1 Introduction

This paper is the fourth in a series of papers [17, 3, 4] investigating ever more general domains in which promotion on tableaux (or tableaux-like objects) and rowmotion on order ideals (or generalizations of order ideals) correspond. In [17], N. Williams and the second author proved a general result about rowmotion and toggles which yielded an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rowmotion on order ideals of the triangular poset $\Phi^+(A_{n-1})$ (by reinterpreting the Type A case of a result of D. Armstrong, C. Stump, and H. Thomas [1] as a special case of a general theorem they showed about toggles). In [3], the second author, with K. Dilks and O. Pechenik, found a correspondence between $a \times b$ increasing tableaux with entries at most $a + b + c - 1$ under K-promotion and order ideals of $[a] \times [b] \times [c]$ under rowmotion. In [4], the second and third authors with Dilks broadened this correspondence to generalized promotion on increasing labelings of any finite poset P with restriction function R on the labels and rowmotion on order ideals of a corresponding poset $\Gamma(P, R)$.

In this paper, we generalize from rowmotion on $J(\Gamma(P, R))$ to the piecewise-linear setting and determine the corresponding promotion action on tableaux-like objects we call *P-strict labelings* (named in analogy to column-strict tableaux). This general theorem

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includes all of the previously known correspondences between promotion and rowmotion and gives a new corollary relating promotion on flagged tableaux to rowmotion on P-partitions of posets of interest. This also specializes to include a result of Kirillov and Berenstein [11] that Bender–Knuth involutions on semistandard Young tableaux correspond to piecewise-linear toggles on the corresponding Gelfand–Tsetlin pattern. We discuss this in more detail in the arXiv version.

The paper is structured as follows. We begin by defining our new objects, P-strict labelings, and promotion acting on them in Section 2. In Section 3, we give the definition of piecewise-linear toggles and rowmotion on P-partitions. Section 4 gives our main theorems relating P-strict promotion and toggles, as well as a jeu de taquin characterization of promotion for special P-strict labelings. Finally, Section 5 applies our main theorem to flagged tableaux, obtaining results on enumeration and order of promotion, as well as new cyclic sieving and homomesy conjectures. See Figure 1 for an example.

This is an extended abstract; for proofs and other applications, see the arXiv version.

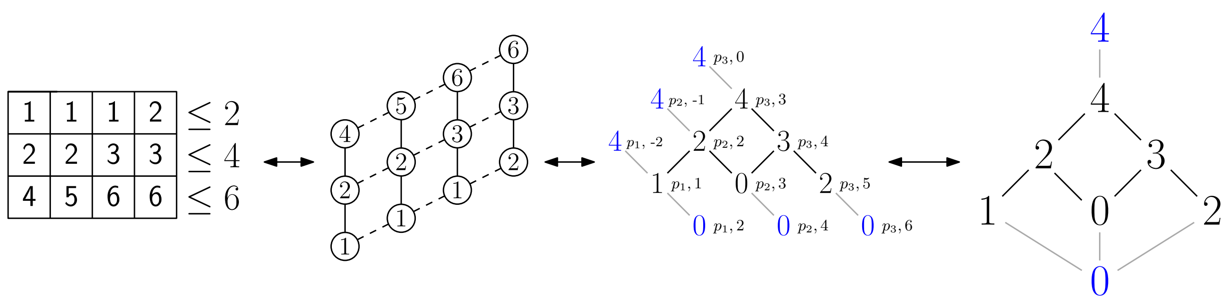


Figure 1: A motivating example of the bijection of this paper relating flagged tableaux to P-partitions on the Type A root poset. See Corollaries 5.5 and 5.7, which imply the order of promotion on these flagged tableaux in this case is 14.

2 Promotion on P-strict labelings

Below, we define *P-strict labelings*, which generalize both semistandard Young tableaux and increasing labelings. We extend the definition of promotion in terms of Bender–Knuth involutions to this setting. We show in Theorem 2.10 in which cases promotion may be equivalently defined using jeu de taquin.

Definition 2.1. In this paper, P represents a finite poset with partial order $\leq_P$, $\lessdot$ indicates a covering relation in a poset, ℓ and q are positive integers, $[\ell]$ denotes a chain poset (total order) of ℓ elements (whose elements will be named as indicated in context), and $P \times [\ell] = \{(p,i) \mid p \in P, i \in \mathbb{N}, \text{ and } 1 \leq i \leq \ell\}$.

Below, we define P-strict labelings on $P \times [\ell]$, and, more generally, on *convex subposets* of $P \times [\ell]$. This level of generality is necessary to, for instance, capture the case of promotion on semistandard Young tableaux of non-rectangular shape.

Definition 2.2. Given S a convex subposet of $P \times [\ell]$, let $L_i = \{(p,i) \in S \mid p \in P\}$ be the ith **layer** of S and $F_p = \{(p,i) \in S \mid 1 \leq i \leq \ell\}$ be the pth **fiber** of S.

Convex subposets of $P \times [\ell]$ have a predictable structure, as we note below.

Definition 2.3. Let $u : P \to \{0, 1, \ldots, \ell\}$ and $v : P \to \{0, 1, \ldots, \ell\}$ with $u(p) + v(p) \leq \ell$ for all $p \in P$ and $v(p_1) \leq v(p_2)$ and $u(p_1) \geq u(p_2)$ whenever $p_1 \leq_P p_2$. Then define $P \times [\ell]_u^v$ as the subposet of $P \times [\ell]$ given by $\{(p,i) \in P \times [\ell] \mid u(p) < i < \ell + 1 - v(p)\}$.

Proposition 2.4. *Let S be a convex subposet of $P \times [\ell]$. Then there exist unique u, v such that $S = P \times [\ell]_u^v$.*

Definition 2.5. Let $\mathcal{P}(\mathbb{Z})$ represent the set of all nonempty, finite subsets of $\mathbb{Z}$. A **restriction function on** P is a map $R : P \to \mathcal{P}(\mathbb{Z})$. In this paper, R will always represent a restriction function.

Definition 2.6. We say that a function $f : P \times [\ell]_u^v \to \mathbb{Z}$ is a P**-strict labeling of** $P \times [\ell]_u^v$ **with restriction function** R if f satisfies the following on $P \times [\ell]_u^v$:

1. $f(p_1, i) < f(p_2, i)$ whenever $p_1 <_P p_2$,
2. $f(p, i_1) \leq f(p, i_2)$ whenever $i_1 \leq i_2$,
3. $f(p, i) \in R(p)$.

That is, f is strictly increasing inside each copy of P (layer), weakly increasing along each copy of the chain $[\ell]$ (fiber), and such that the labels come from the restriction function R.

Let $\mathcal{L}_{P \times [\ell]}(u, v, R)$ denote the set of all P-strict labelings on $P \times [\ell]_u^v$ with restriction function R. If the convex subposet is $P \times [\ell]$ itself, i.e. $u(p) = v(p) = 0$ for all $p \in P$, we use the notation $\mathcal{L}_{P \times [\ell]}(R)$. We denote the restriction function induced by (either global or local) upper and lower bounds as R_a^b, where $a, b : P \to \mathbb{Z}$. In the case of a global upper bound q, our restriction function will be R_1^q, that is, we take a to be the constant function 1 and b to be the constant function q. Since a lower bound of 1 is used frequently, we suppress the subscript 1; that is, if no subscript appears, we take it to be 1.

Definition 2.7. Let $R : P \mapsto \mathcal{P}(\mathbb{Z})$. We say R is **consistent** with respect to $P \times [\ell]_u^v$ if for each $p \in P$ and $k \in R(p)$ there exists some P-strict labeling $f \in \mathcal{L}_{P \times [\ell]}(u, v, R)$ and $u(p) < i < \ell + 1 - v(p)$ such that $f(p, i) = k$.

Let $R(p)_{>k}$ denote the smallest label of $R(p)$ that is larger than k, and let $R(p)_{<k}$ denote the largest label of $R(p)$ less than k.

Say that a label $f(p, i)$ in a P-strict labeling $f \in \mathcal{L}_{P \times [\ell]}(u, v, R)$ is **raisable (lowerable)** if there exists another P-strict labeling $g \in \mathcal{L}_{P \times [\ell]}(u, v, R)$ where $f(p, i) < g(p, i)$ ($f(p, i) > g(p, i)$), and $f(p', i') = g(p', i')$ for all $(p', i') \in P \times [\ell]_u^v$, $p' \neq p$.

Definition 2.8. Let the action of the k**th Bender–Knuth involution** ρ_k on a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$ be as follows: identify all raisable labels $f(p,i) = k$ and all lowerable labels $f(p,i) = R(p)_{>k}$ (if $k = \max R(p)$, then there are no such labels on the fiber F_p). Call these labels 'free'. Suppose the labels $f(F_p)$ include a free k labels followed by b free $R(p)_{>k}$ labels; ρ_k changes these labels to b copies of k followed by a copies of $R(p)_{>k}$. **Promotion** on P-strict labelings is defined as the composition of these involutions: $\mathrm{Pro}(f) = \cdots \circ \rho_3 \circ \rho_2 \circ \rho_1 \circ \cdots (f)$. Note that since R induces upper and lower bounds on the labels, only a finite number of Bender–Knuth involutions act nontrivially.

Remark 2.9. In the case $\ell = 1$, $\mathcal{L}_{P\times[\ell]}(R)$ equals $\mathrm{Inc}^R(P)$, the set of increasing labelings of P with restriction function R. So the above definition specializes to *generalized Bender–Knuth involutions* and *increasing labeling promotion* IncPro, as studied in [4]. If, in addition, P is (skew-) partition shaped, these increasing labelings are equivalent to *(skew-) increasing tableaux*, and the above definition specializes to *K-Bender–Knuth involutions* and *K-Promotion*, as in [3]. If we restrict our attention to *linear extensions* of P, the above definition specializes to usual Bender–Knuth involutions and promotion, as studied in [15]. If $P = [n]$, then $\mathcal{L}_{P\times[\ell]}(u,v,R^q)$ is a set of (skew-)semistandard Young tableaux and the above definition specializes to usual Bender–Knuth involutions and promotion. Since Definition 2.8 specializes to the right thing in each of these cases, we will not use the prefixes K-, increasing labeling, or generalized, but rather say 'Bender–Knuth involutions' and 'promotion', letting the object acted upon specify the context.

The following theorem shows promotion on P-strict labelings $\mathcal{L}_{P\times[\ell]}(u,v,R^q)$ via jeu de taquin is equivalent to our Definition 2.8 via Bender–Knuth involutions. See the arXiv version for the proof and definition of JdtPro.

Theorem 2.10. *For* $f \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$, $\mathrm{JdtPro}(f) = \mathrm{Pro}(f)$.

3 Rowmotion on P-partitions

Rowmotion is an intriguing action that has recently generated significant interest as a prototypical action in dynamical algebraic combinatorics; see, for example, the survey articles [12, 16]. Rowmotion was generalized to piecewise-linear and birational domains by D. Einstein and J. Propp [5]. In this paper, we discuss toggling and rowmotion on P-partitions, as a rescaling of the piecewise-linear version.

Definition 3.1. A P**-partition** is a map $\sigma : P \to \mathbb{N}_{\geq 0}$ such that if $p \leq_P p'$, then $\sigma(p) \leq \sigma(p')$. Let $\hat{P}$ denote P with $\hat{0}$ added below all elements and $\hat{1}$ added above all elements. Let $\mathcal{A}^\ell(P)$ denote the set of all $\hat{P}$-partitions σ with $\sigma(\hat{0}) = 0$ and $\sigma(\hat{1}) = \ell$.

In Definition 3.2, we generalize Definition 3.1 by specifying bounds element-wise. Then in Definition 3.4, we define our main objects of study: B-bounded P-partitions. Note these correspond to rational points in a certain *marked order polytope*.

Definition 3.2. Let $\delta, \epsilon \in \mathcal{A}^\ell(P)$. Let $\mathcal{A}^\delta_\epsilon(P)$ denote the set of all P-partitions $\sigma \in \mathcal{A}^\ell(P)$ with $\epsilon(p) \leq \sigma(p) \leq \delta(p)$.

Remark 3.3. If $\delta(p) = \ell$ and $\epsilon(p) = 0$ for all $p \in P$, then $\mathcal{A}^\delta_\epsilon(P) = \mathcal{A}^\ell(P)$.

Definition 3.4. Let $B \in \mathcal{A}^\ell(X)$ where X is a subset of P that includes all maximal and minimal elements. Let $\mathcal{A}^B(P)$ denote the set of all P-partitions $\sigma \in \mathcal{A}^\ell(P)$ with $\sigma(p) = B(p)$ for all $p \in X$. Call these B**-bounded** P**-partitions**. We refer to X as $\operatorname{dom}(B)$.

Remark 3.5. If B is defined as $B(\hat{0}) = 0$, $B(\hat{1}) = \ell$, then $\mathcal{A}^B(\hat{P}) = \mathcal{A}^\ell(P)$.

Remark 3.6. Let P' be the poset P with two additional elements added for each $p \in P$: a minimal element $\hat{0}_p$ covered by p and a maximal element $\hat{1}_p$ covering p. If B is defined as $B(\hat{0}_p) = \epsilon(p)$, $B(\hat{1}_p) = \delta(p)$, then $\mathcal{A}^B(P') = \mathcal{A}^\delta_\epsilon(P)$.

In Definitions 3.7 and 3.8 below, we define toggles and rowmotion. In the case of $\mathcal{A}^\ell(P)$, these definitions are equivalent (by rescaling) to those first given by Einstein and Propp on the order polytope [5].

Definition 3.7. For $\sigma \in \mathcal{A}^B(P)$ and $p \in P \setminus \operatorname{dom}(B)$, let $\alpha_\sigma(p) = \min\{\sigma(x) \mid x \in P \text{ covers } p\}$ and $\beta_\sigma(p) = \max\{\sigma(y) \mid y \in P \text{ is covered by } p\}$. Define the **toggle**, $\tau_p : \mathcal{A}^B(P) \to \mathcal{A}^B(P)$ by

$$\tau_p(\sigma)(p') := \begin{cases} \sigma(p') & p \neq p' \\ \alpha_\sigma(p') + \beta_\sigma(p') - \sigma(p') & p = p'. \end{cases}$$

Definition 3.8. Rowmotion on $\mathcal{A}^B(P)$ is defined as the toggle composition $\text{Row} := \tau_{p_1} \circ \tau_{p_2} \circ \cdots \circ \tau_{p_m}$ where $p_1, p_2, \ldots, p_m$ is any linear extension of $P \setminus \operatorname{dom}(B)$.

Remark 3.9. It may be argued that we should call these actions piecewise-linear toggles and piecewise-linear rowmotion, but as in the case of promotion on tableaux/labelings, unless clarification is needed, we choose to leave the names of these actions adjective-free, allowing the objects acted upon to indicate the context.

4 Main theorems: *P*-strict promotion and rowmotion

In this section, we give our main theorems relating promotion on P-strict labelings with restriction function R and toggle-promotion / rowmotion on B-bounded Q-partitions, where Q is the poset $\Gamma(P, \hat{R})$ constructed below.

Definition 4.1 ([4])**.** Let P be a poset and $R : P \to \mathcal{P}(\mathbb{Z})$ a (not necessarily consistent) map of possible labels. For $p \in P$, let $R(p)^*$ denote $R(p)$ with its largest element removed. Then define $\Gamma(P, R)$ to be the poset whose elements are (p, k) with $p \in P$ and $k \in R(p)^*$, and covering relations given by $(p_1, k_1) \lessdot (p_2, k_2)$ if and only if either

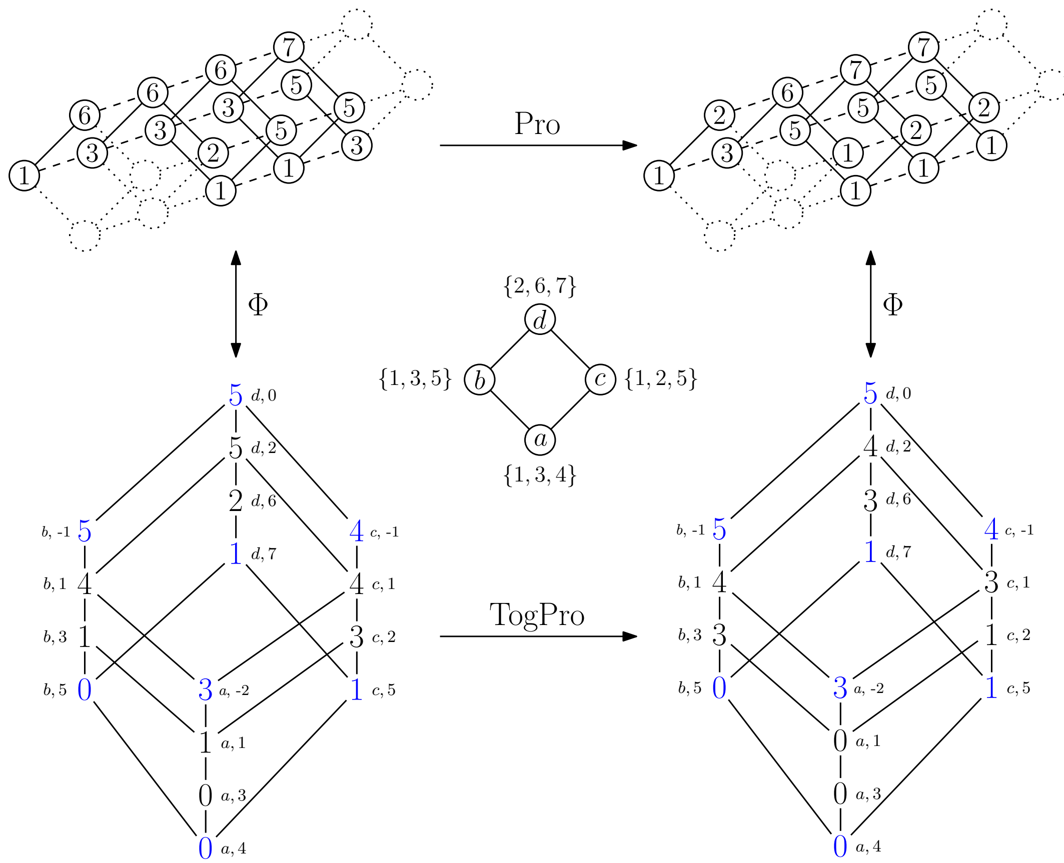


Figure 2: An illustration of Theorem 4.6. Promotion on the P-strict labeling corresponds to toggle-promotion on the $\hat{B}$-bounded $\Gamma(P,\hat{R})$-partition. The poset $P = \{a,b,c,d\}$ along with the restriction function R are shown in the center.

1. $p_1 = p_2$ and $R(p_1)_{>k_2} = k_1$ (i.e., k_1 is the next largest possible label after k_2), or

2. $p_1 \lessdot p_2$ (in P), $k_1 = R(p_1)_{<k_2} \neq \max(R(p_1))$, and no greater k in $R(p_2)$ has $k_1 = R(p_1)_{<k}$. That is to say, k_1 is the largest label of $R(p_1)$ less than k_2 ($k_1 \neq \max(R(p_1))$), and there is no greater $k \in R(p_2)$ having k_1 as the largest label of $R(p_1)$ less than k.

In [4] it is shown that, if R consistent on P, increasing labelings on P under *increasing labeling promotion* are in equivariant bijection with order ideals of $\Gamma(P,R)$ under *toggle-promotion*. This correspondence drives our first main theorem.

The next definition constructs a useful restriction function $\hat{R}$ on P from a given restriction function R. We use the structure of $\Gamma(P,\hat{R})$ in our main result.

Definition 4.2. Suppose R is a consistent restriction function on $P \times [\ell]_u^v$. Denote the number of elements less than or equal to p in a maximum length chain containing p as $h(p)$ and the number of elements greater than or equal to p in a maximum length chain containing p as $\tilde{h}(p)$. Define a new restriction function $\hat{R}$ on P where, for all $p \in P$, $\hat{R}(p) := R(p) \cup \left\{\min \bigcup_{q\in P} R(q) - \tilde{h}(p),\ \max \bigcup_{q \in P} R(q) + h(p)\right\}$.

Proposition 4.3. *If R is a consistent restriction function on $P \times [\ell]_u^v$, then $\hat{R}$ is consistent on P.*

Below, we give our first main theorem, Theorem 4.6, relating P-strict promotion and toggle-promotion. First, we define an action on B-bounded $\Gamma(P, \hat{R})$-partitions:

Definition 4.4. Toggle-promotion on $\mathcal{A}^B(\Gamma(P, \hat{R}))$ is defined as the toggle composition $\text{TogPro} := \cdots \circ \tau_2 \circ \tau_1 \circ \tau_0 \circ \tau_{-1} \circ \tau_{-2} \circ \cdots$, where τ_k denotes the composition of all the $\tau_{(p,k)}$ over all $p \in P$ such that $(p, k) \notin \text{dom}(B)$.

Definition 4.5. Define $\hat{B}$ as $\hat{B}(p, \min \hat{R}(p)^*) = \ell - u(p)$ and $\hat{B}(p, \max \hat{R}(p)^*) = v(p)$.

Theorem 4.6. *$\mathcal{L}_{P\times[\ell]}(u, v, R)$ under* Pro *is in equivariant bijection with $\mathcal{A}^{\hat{B}}(\Gamma(P, \hat{R}))$ under* TogPro*. More specifically, for $f \in \mathcal{L}_{P\times[\ell]}(u, v, R)$, $\Phi(\text{Pro}(f)) = \text{TogPro}(\Phi(f))$, where Φ is given in Definition 4.7.*

See Figures 2 and 3 for an example illustrating Theorem 4.6 and an example of Φ.

Definition 4.7. We define the map $\Phi : \mathcal{L}_{P\times[\ell]}(u, v, R) \to \mathcal{A}^{\hat{B}}(\Gamma(P, \hat{R}))$ as the composition of three intermediate maps ϕ_1, ϕ_2, and ϕ_3. Start with a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u, v, R)$. Let $\phi_1(f) = \hat{f} \in \mathcal{L}_{P\times[\ell]}(\hat{R})$ where $\hat{f}$ is given by

$$\hat{f}(p,i) = \begin{cases} \min \hat{R}(p) & i \leq u(p) \\ f(p,i) & u(p) < i < \ell + 1 - v(p) \\ \max \hat{R}(p) & \ell + 1 - v(p) \leq i \end{cases}$$

Next, ϕ_2 sends $\hat{f}$ to the multichain $\mathcal{O}_\ell \leq \mathcal{O}_{\ell-1} \leq \cdots \leq \mathcal{O}_1$ in $J(\Gamma(P, \hat{R}))$ where, for $1 \leq i \leq \ell$ and L_i the ith layer of $P \times [\ell]_u^v$, ϕ_2 sends $\hat{f}(L_i)$ to its associated order ideal $\mathcal{O}_i \in J(\Gamma(P, \hat{R}))$ via the map in [4]. Lastly, ϕ_3 maps the above multichain to a $\Gamma(P, \hat{R})$-partition σ as seen in [14, p. 11], where $\sigma(p, k) = \#\{i \mid (p, k) \notin \mathcal{O}_i\}$, the number of order ideals not including (p, k). Let $\Phi = \phi_3 \circ \phi_2 \circ \phi_1$.

Lemma 4.8. *The map Φ is invertible and equivariantly takes the generalized Bender–Knuth involution ρ_k to the toggle operator τ_k.*

Proof of Theorem 4.6. By Lemma 4.8, Φ is a bijection and $\Phi(\text{Pro}(f)) = \Phi(\cdots \circ \rho_2 \circ \rho_1 \circ \rho_0 \circ \rho_{-1} \circ \rho_{-2} \circ \cdots (f)) = \cdots \circ \tau_2 \circ \tau_1 \circ \tau_0 \circ \tau_{-1} \circ \tau_{-2} \circ \cdots (\Phi(f)) = \text{TogPro}(\Phi(f))$. □

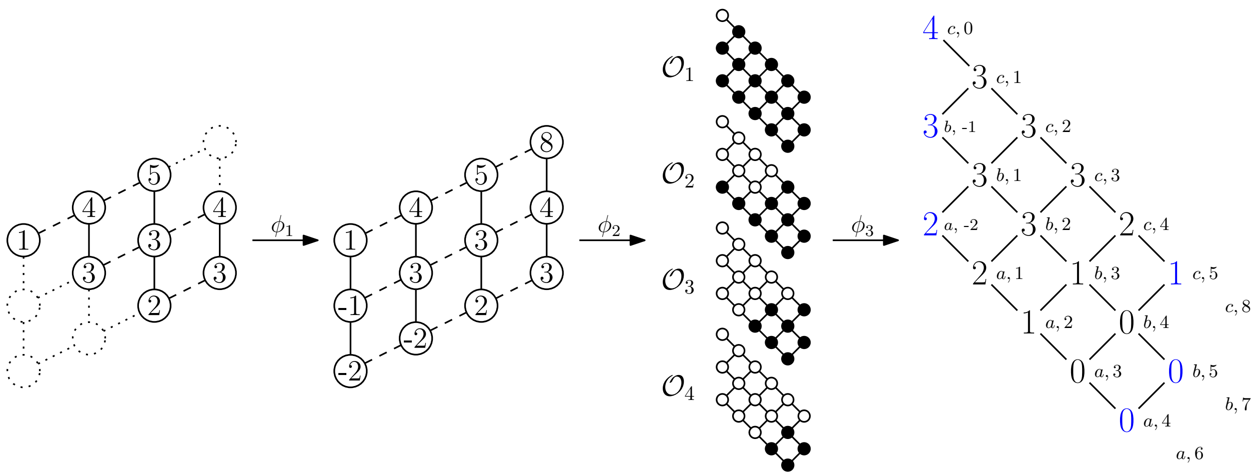


Figure 3: An example of the bijection map $\Phi = \phi_3 \circ \phi_2 \circ \phi_1$ of Theorem 4.6, beginning with $f \in \mathcal{L}_{P\times[4]}(u, v, R^5)$ on the left and ending with $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P, \widehat{R^5}))$ on the right, where P is the chain $a \lessdot b \lessdot c$, $u(a, b, c) = (2, 1, 0)$, and $v(a, b, c) = (0, 0, 1)$.

Our next main result, Theorem 4.10, says that for certain kinds of restriction functions, promotion on P-strict labelings of $P \times [\ell]_u^v$ with restriction function R is equivariant with rowmotion on B-bounded $\Gamma(P, \hat{R})$-partitions.

Definition 4.9. We call an element $p \in P$ **fixed in** $\mathcal{A}^B(P)$ if there exists some value x such that $\sigma(p) = x$ for all $\sigma \in \mathcal{A}^B(P)$.

We say that $\mathcal{A}^B(\Gamma(P, R))$ is **column-adjacent** if whenever $(p_1, k_1) \lessdot (p_2, k_2)$ in $\Gamma(P, R)$ and neither of (p_1, k_1) nor (p_2, k_2) are fixed in $\mathcal{A}^B(\Gamma(P, R))$, then $|k_2 - k_1| = 1$.

Theorem 4.10. *If $\mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$ is column-adjacent, then $\mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$ under* Row *is in equivariant bijection with $\mathcal{L}_{P\times[\ell]}(u, v, R)$ under* Pro*.*

For the case where our restriction function is induced by upper and lower bounds for each element (this includes the case of a global bound q), we have the column-adjacent property, so Theorem 4.10 yields Corollary 4.11.

Corollary 4.11. *$\mathcal{A}^{\widehat{B}}(\Gamma(P, \widehat{R_a^b}))$ under* Row *is in equivariant bijection with $\mathcal{L}_{P\times[\ell]}(u, v, R_a^b)$ under* Pro*.*

5 Application of the main theorems to flagged tableaux

In this section, we first specialize Theorem 4.6 to flagged tableaux and use this correspondence to enumerate the corresponding set of B-bounded $\Gamma(P, \hat{R})$-partitions. Then, we state some recent cyclic sieving and new homomesy conjectures and use Theorem 4.6 to translate these conjectures between the two domains.

Definition 5.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions with $\mu \subset \lambda$ and let $b = (b_1, \ldots, b_n)$ where $b_i \in \mathbb{Z}^+$ and $b_1 \leq \ldots \leq b_n$. A **flagged tableau** of shape λ/μ and flag b is a skew semistandard Young tableau of shape λ/μ whose entries in row i do not exceed b_i. Let $FT(\lambda/\mu, b)$ denote the set of flagged tableaux of shape λ/μ and flag b.

Note that, depending on context, b represents either the increasing sequence of positive integers $(b_1, \ldots, b_n)$ or the function $b : [n] \to \mathbb{Z}^+$ with $b(p_i) = b_i$.

Proposition 5.2. *$FT(\lambda/\mu, b)$ is equivalent to $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^b)$ where $u(p_i) = \mu_i$ and $v(p_i) = \lambda_1 - \lambda_i$ for all $1 \leq i \leq n$.*

We now specify the $\hat{B}$-bounded $\Gamma(P, \hat{R})$-partitions in bijection with $FT(\lambda/\mu, b)$.

Corollary 5.3. *$FT(\lambda/\mu, b)$ under* Pro *is in equivariant bijection with $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ under* Row*, where $\hat{B}$ is as in Definition 4.5, with $u(p_i) = \mu_i$ and $v(p_i) = \lambda_1 - \lambda_i$ for all $1 \leq i \leq n$.*

Flagged tableaux are enumerated by an analogue of the Jacobi-Trudi formula. This is due to I. Gessel and X. Viennot [7] with an alternative proof by M. Wachs [18]. We translate this to enumerate $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$.

Corollary 5.4. $|\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))| = \det\left[\binom{b_i + \ell - v(p_i) - u(p_j) - i + j - 1}{\ell - v(p_i) - u(p_j) - i + j}\right]_{i,j=1}^{n}.$

We obtain the following for flagged tableaux of shape ℓ^n and flag $b = (2, 4, \ldots, 2n)$.

Corollary 5.5. *$FT(\ell^n, (2, 4, \ldots, 2n))$ under* Pro *is in equivariant bijection with $\mathcal{A}^{\ell}(\Phi^+(A_n))$ under* Row*, where $\Phi^+(A_n)$ is the positive root poset of type A_n.*

See Figure 1 for an example. Also, D. Grinberg and T. Roby proved the following result on the order of birational rowmotion on $\Phi^+(A_n)$, which implies Corollary 5.7.

Theorem 5.6 ([8, Corollary 66])**.** Row *on $\mathcal{A}^{\ell}(\Phi^+(A_n))$ has order dividing $2(n+1)$.*

Corollary 5.7. Pro *on $FT(\ell^n, (2, 4, \ldots, 2n))$ has order dividing $2(n+1)$.*

Note, the order does not depend on ℓ. Therefore, the order of promotion in this case is independent of the number of columns.

J. Propp conjectured the following instance of the cyclic sieving phenomenon on $(\mathcal{A}^{\ell}(\Phi^+(A_n))$ under rowmotion with a polynomial analogue of the Catalan numbers. S. Hopkins recently extended this conjecture to positive root posets of all coincidental types (see [10, Remark 5.5]).

Conjecture 5.8. $\left(\mathcal{A}^{\ell}(\Phi^{+}(A_n)), \langle \mathrm{Row} \rangle, Cat_{\ell}(x)\right)$ *exhibits the cyclic sieving phenomenon, where* $Cat_{\ell}(x) := \prod_{j=0}^{\ell-1} \prod_{i=1}^{n} \frac{1 - x^{n+1+i+2j}}{1 - x^{i+2j}}$.

Thus, Corollary 5.5 implies the equivalence this conjecture and the following.

Conjecture 5.9. $(FT(\ell^n, (2, 4, \ldots, 2n)), \langle \mathrm{Pro} \rangle, Cat_{\ell}(x))$ *exhibits the cyclic sieving phenomenon.*

Though we do not have a proof of this cyclic sieving conjecture, we have evidence to conjecture the following homomesy statement, which was proved for $\ell = 1$ by S. Haddadan [9].

Conjecture 5.10. $\left(\mathcal{A}^{\ell}(\Phi^{+}(A_n)), \mathrm{TogPro}, \mathcal{R}\right)$ *is* 0*-mesic for* n *even and* $\frac{\ell}{2}$*-mesic for* n *odd, where* $\mathcal{R}(\sigma) = \sum_{p \in P} (-1)^{\mathrm{rk}(p)} \sigma(p)$ *is the rank-alternating label sum statistic.*

We have checked this conjecture using Sage for $n \leq 6$ and $\ell \leq 3$. Using Sage, we have also verified that a similar statement fails to hold for the Type B/C case when $n = 2$ and $\ell = 1$, and the Type D case when $n = 4$ and $\ell = 1$.

We use our main results to translate this to a conjecture on flagged tableaux.

Conjecture 5.11. $(FT(\ell^n, (2, 4, \ldots, 2n)), \langle \mathrm{Pro} \rangle, \sum |R_O \cap E| - \sum |R_E \cap O|)$ *is* 0*-mesic when* n *is even and* $\frac{\ell}{2}$*-mesic when* n *is odd, where* $\sum |R_O \cap E| - \sum |R_E \cap O|$ *denotes the difference between the number of boxes in odd rows of* T *that contain an even integer and the number of boxes in even rows of* T *that contain an odd integer.*

Theorem 5.12. *Conjecture 5.10 and Conjecture 5.11, imply each other.*

Another set of flagged tableaux of interest in the literature is that of staircase shape $sc_n = (n, n-1, \ldots, 2, 1)$ with flag $b = (\ell+1, \ell+2, \ldots, \ell+n)$. The Type A case of a result of C. Ceballos, J.-P. Labbé, and C. Stump on multi-cluster complexes along with a bijection of L. Serrano and Stump yields the following result on the order of promotion on these flagged tableaux.

Theorem 5.13 ([2, Theorem 8.8], [13, Theorem 4.7])**.** *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$. Pro *on* $FT(sc_n, b)$ *is of order dividing* $n + 1 + 2\ell$.

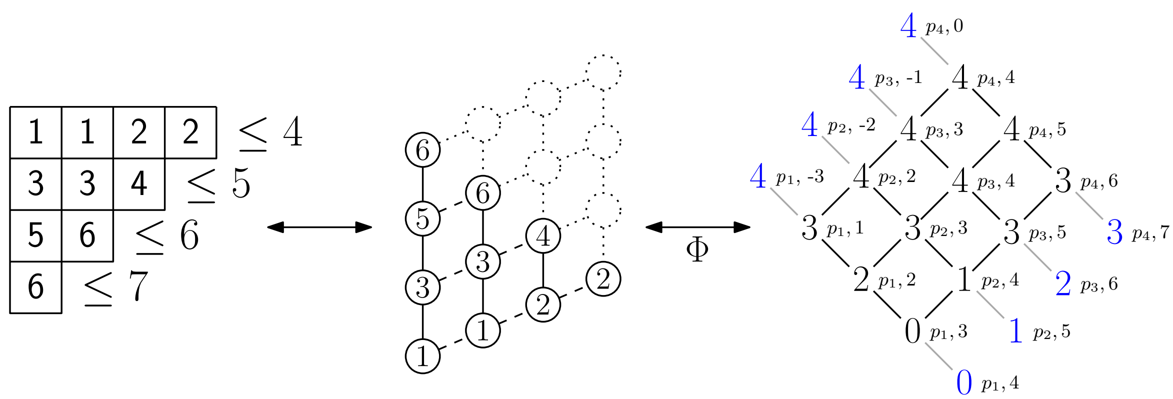
The following conjecture is given in terms of flagged tableaux in [13] and in terms of multi-cluster complexes in [2].

Conjecture 5.14 ([13, Conjecture 1.7],[2, Open Problem 9.2])**.** *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$ *and* $Cat_{\ell}(x)$ *be as in Conjecture 5.8.* $(FT(sc_n, b), \langle \mathrm{Pro} \rangle, Cat_{\ell}(x))$ *exhibits the cyclic sieving phenomenon.*

Note this is a set of flagged tableaux with different shape and flag but the same cardinality as the flagged tableaux in Corollary 5.5, the same conjectured cyclic sieving polynomial, and a different order of promotion. The case $\ell = 1$ follows from a result of S.P. Eu and T.S. Fu [6] on cyclic sieving of faces of generalized cluster complexes, but for $\ell > 1$ this conjecture is still open.

We can translate this conjecture to rowmotion on P-partitions with the following corollary of Theorem 4.6. Here and below, specify the following notation for the elements of the product of two chains poset $[n] \times [\ell] = \{(i,j) \mid 1 \le i \le n, 1 \le j \le \ell\}$.

Corollary 5.15. *Let $b = (\ell+1, \ell+2, \dots, \ell+n)$. There is an equivariant bijection between $FT(sc_n, b)$ under* Pro *and $\mathcal{A}^{\delta}_{\epsilon}([n] \times [\ell])$ under* Row*, where for $(i,j) \in [n] \times [\ell]$, $\delta(i,j) = n$ and $\epsilon(i,j) = i-1$ for all i.*



Corollary 5.15 implies the equivalence of this and the following new conjecture.

Conjecture 5.16. $\left(\mathcal{A}^{\delta}_{\epsilon}([n] \times [\ell])\right), \langle \text{Row} \rangle, Cat_{\ell}(x)\big)$ *exhibits the cyclic sieving phenomenon, where $\delta(i,j) = n$ and $\epsilon(i,j) = i-1$ for all i.*

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