Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #83, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Maximal green sequences for triangle products

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Abstract. The existence of maximal green sequences is an important property of a cluster algebra. We construct explicit maximal green sequences for triangle products of an acylic quiver with a Dynkin quiver. As an application we deduce from the work of Gross-Hacking-Keel-Kontsevich the full Fock–Goncharov conjecture for big double Bruhat cells for simply-connected, connected, semisimple groups of simply-laced type.

Keywords: cluster algebras, maximal green sequences, canonical bases

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [8] in their studies on total positivity and canonical bases. A cluster algebra comes with a distinguished set of generators, called cluster variables, which is obtained by recursively mutating an initial seed consisting of a finite quiver without loops and 2-cycles whose vertices are the initial cluster variables.

In most cases one needs to apply an infinite number of mutations to obtain all cluster variables from the initial seed. To be more precise, a finite number of mutations suffices precisely for the cluster algebras obtained from simply-laced Dynkin quivers [9]. There is, however, a second, less restrictive, notion of finiteness in the theory of cluster algebras due to Keller [16] based on work of Gaiotto-Moore-Neitzke [10]. Keller [16] introduces the notion of a maximal green sequence, or slightly more general reddening sequence, for a quiver Q, which is a finite sequence of mutations of Q with favourable properties. A quiver Q may or may not possess a maximal green sequence.

The existence of a maximal green sequence has far-reaching consequences for the cluster algebra associated to Q. In particular, to a maximal green sequence a product of certain quantum dilogarithm series is attached in [16] computing the refined Donaldson–Thomas invariants [19] of Q. Furthermore, as predicted by the Fock–Goncharov conjecture [7] Gross-Hacking-Keel-Kontsevich [14] produce a certain canonical basis, called

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theta basis, of the cluster algebra associated to Q relying on the existence of a maximal green sequence. In Physics, maximal green sequences appear in the computation of spectra of BPS states [1].

In [18] Keller definded the triangle product $Q \boxtimes R$ of two quivers Q and R. In this paper we construct explicit maximal green sequences for $Q \boxtimes R$ in the case that Q is an acyclic quiver and R a Dynkin quiver. The key here is that Q possesses a source maximal green sequence and R a sink maximal green sequence, which can be combined to obtain a maximal green sequence for $Q \boxtimes R$.

As an application we consider the cluster algebra structure introduced by Berenstein-Fomin-Zelevinsky [2] on the coordinate rings of the large double Bruhat cells $G^{\mathrm{e},\mathrm{w}_0}$, $G^{\mathrm{w}_0,\mathrm{e}}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$ of a simply connected, connected, semisimple complex algebraic group G of simply-laced type. We verify the conditions under which the construction of theta-bases due to Gross-Hacking-Keel-Kontsevich yields bases for those cluster algebras satisfying the full Fock–Goncharov conjecture [7]. In particular, in addition to maximal green sequences we construct optimization sequences for the frozen vertices.

This paper originated in [12], where optimization sequences for $G^{\mathrm{w}_0,\mathrm{e}}$ were introduced as an ingredient relating the Landau-Ginzburg potential function introduced by Gross-Hacking-Keel-Kontsevich [14] in their work on theta bases to the decoration function introduced by Berenstein-Kazhdan [3] in their work on geometric crystals.

2 Background and Notations

Throughout this work all quivers are assumed to be finite and without loops and 2-cycles. Let $Q = (Q_0, Q_1)$ be a quiver with vertices Q_0 and edges Q_1. The quiver Q is determined by the skew-symmetric matrix $B = B_Q = (b_{v,w})_{v,w \in Q_0}$, where $b_{v,w}$ is the difference of the number of arrows from v to w and the number of arrows from w to v.

Following [17, sec. 3.3] we also consider *valued quivers*, which are quivers Q with at most one arrow between any two vertices together with a pair $(\mathrm{val}(\alpha)_1, \mathrm{val}(\alpha)_2)$ of natural numbers associated to any arrow $\alpha : v \to w$ of Q such that there exists a function $d : Q_0 \to \mathbb{Z}_{>0}$ with

$$d(v)\,\mathrm{val}(\alpha)_1 = \mathrm{val}(\alpha)_2 d(w).$$

Valued quivers correspond to skew-symmetrizable matrices $B = B_Q = (b_{v,w}) \in \mathrm{Mat}_{n,n}(\mathbb{Z})$ via

$$b_{v,w} = \begin{cases} \mathrm{val}(\alpha)_1 & \text{if there is an arrow from } \alpha : v \to w, \\ -\mathrm{val}(\alpha)_2 & \text{if there is an arrow from } \alpha : w \to v, \\ 0 & \text{else.} \end{cases}$$

Following [8] the *mutation of B_Q at a vertex $x \in Q_0$* is the matrix $B' = (b'_{v,w})_{v,w \in B_0}$ defined by

$$b'_{v,w} = \begin{cases} -b_{v,w} & \text{if } v = x \text{ or } w = x, \\ b_{v,w} + \text{sgn}(b_{v,x}) \max(0, b_{v,x} b_{x,w}) & \text{else.} \end{cases}$$

If $B = B_Q$ is skew-symmetric then $B' = B_{Q'}$, where the quiver Q' is obtained from Q by

1. adding an arrow $v \to w$ for any pair $v \to x$, $x \to w$,
2. reversing arrows with source or target x
3. removing 2-cycles.

For a sequence $\mathcal{S} = (\mathcal{S}_i)_{i=1}^n \subset Q_0$ of vertices of Q we denote by $Q\mathcal{S}$ the quiver obtained by successively mutating at $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$.

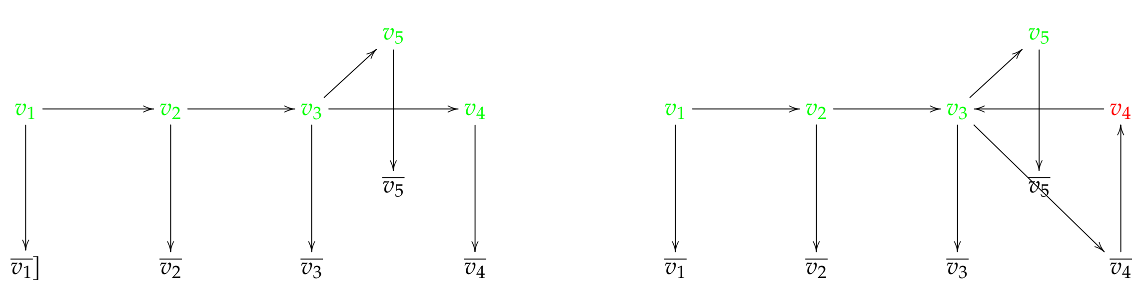
In the following it is convenient to restrict mutations to a subset of vertices called *mutable vertices*. Vertices which are not mutable are called *frozen*. By an *ice quiver* $\widetilde{Q}$ we mean a (possibly valued) quiver $\widetilde{Q}$ together with a set $Q_0 \subset \widetilde{Q}_0$ of mutable vertices such that there are no arrows between frozen vertices. The full subquiver $Q \subset \widetilde{Q}$ supported on the mutable vertices Q_0 is called the *mutable part of* $\widetilde{Q}$.

Following [5] we introduce

Definition 2.1. *The* framed quiver $\widehat{Q}$ *associated to a quiver Q is obtained by adding for each vertex $v \in Q_0$ a frozen vertex $\widehat{v}$ and an arrow $v \to \overline{v}$. Dually, we define the* co-framed quiver $\check{Q}$ *by adding arrows $\overline{v} \to v$ instead.*

Definition 2.2. *A vertex $v \in Q_0$ is called* green *if no arrow departing from a frozen vertex $w \in \widetilde{Q}_0 \backslash Q_0$ targets v. Similarly, $v \in Q_0$ is called* red *if no arrow departing at v targets a frozen vertex $w \in \widetilde{Q}_0 \backslash Q_0$.*

Example 2.3. *We depict the framing $\widehat{D}$ of an oriented Dynkin diagram D of type D_5 as well as $\widehat{D}v_4$:*



Let $\mathcal{S} \subset Q_0$ be a sequence of mutable vertices. By [6] any vertex in $\widehat{Q}\mathcal{S}$ has precisely one of the two colours red and green. Keller [16] introduced certain sequences relating $\widehat{Q}$ and $\check{Q}$:

Definition 2.4. *A sequence $\mathcal{S} = (\mathcal{S}_i)_{i=1}^n \subset Q$ is called* green sequence *if the vertices $\mathcal{S}_i \in \widehat{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_{i-1})$ are green for all $i < n$. A sequence $\mathcal{S} \subset Q$ is called* reddening sequence *if all vertices are red in $\widehat{Q}\mathcal{S}$. A green reddening sequence is called a* maximal green sequence.

By [4, Prop. 2.10] for each reddening sequence $\mathcal{S} \subset Q$ there exists a unique isomorphism of ice quivers

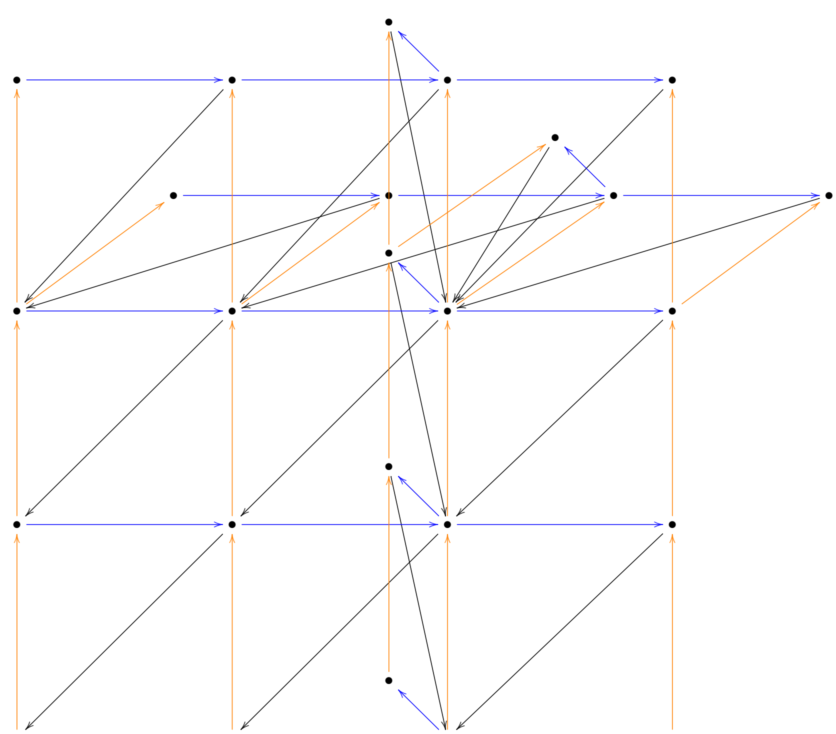
$$\sigma_{\mathcal{S}} : \hat{Q}\mathcal{S} \xrightarrow{\sim} \check{Q}.$$

3 Maximal green sequences for triangle products of an acyclic quiver with a Dynkin quiver

Following [18] we introduce:

Definition 3.1. *Let Q be a (valued) quiver and R a quiver. The* triangle product $Q \boxtimes R$ *of Q and R is obtained from their product $Q \times R$ by adding an arrow with value $(\lambda'\lambda_2, \lambda'\lambda_1)$ from $(q', r') \in Q_0 \times R_0$ to $(q, r) \in Q_0 \times R_0$ if Q contains an arrow from q to q' with value (λ_1, λ_2) and R contains λ' arrows from r to r'.*

Example 3.2. *Continuing example 2.3 we depict $D \boxtimes D$:*



We refer to a quiver R as a *Dynkin quiver* if it emerges as an orientation of a simply laced Dynkin diagram. In this section we provide explicit maximal green sequence for the triangle product $Q \boxtimes R$ for a (possibly valued) acyclic quiver Q and a Dynkin quiver R. More precisely, we merely assume that Q admits a *source maximal green sequence*, i.e. a maximal green sequences consisting solely of sources, and that R admits a *sink maximal green sequence*. By [4, Lemma 2.20] a quiver Q admits a source maximal green sequence precisely if it is acyclic. Furthermore, by [4, Proof of Theorem 4.4] any Dynkin quiver R admits a sink maximal green sequence. We conjecture

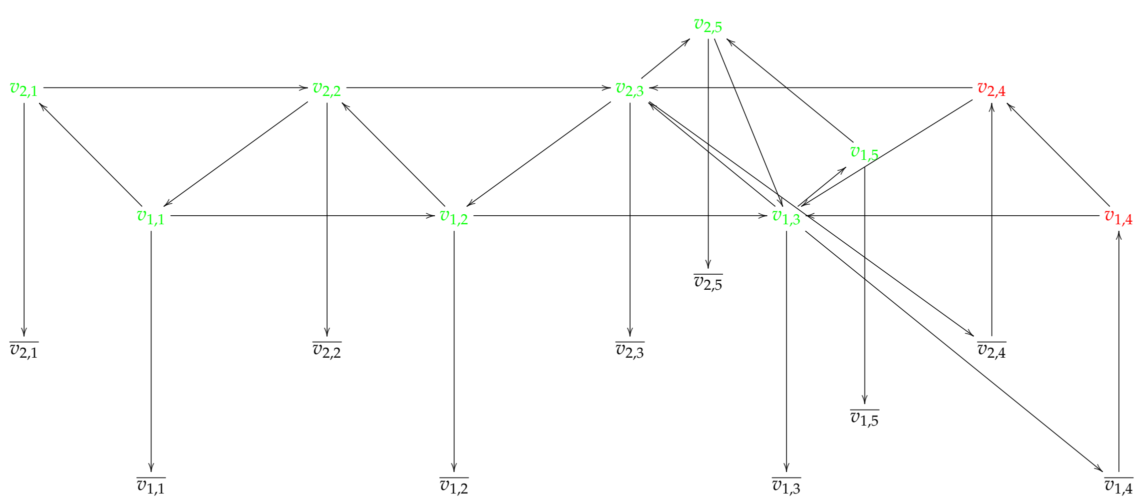
Conjecture 3.3. *A quiver R has a sink maximal green sequence precisely if it is a Dynkin quiver.*

For a quiver Q and an ice quiver $\widetilde{R}$ we introduce the following variant of the triangle product:

Definition 3.4. *Let $\widetilde{R}$ be an ice quiver with mutable part R and Q be a quiver. The vertex set of the* frozen triangle product $Q \boxtimes^{\mathrm{f}} \widetilde{R}$ *of Q and $\widetilde{R}$ is $(Q \boxtimes^{\mathrm{f}} \widetilde{R})_0 = Q_0 \times \widetilde{R}_0$. The mutable part of $Q \boxtimes^{\mathrm{f}} \widetilde{R}$ is $Q \boxtimes R$. The arrows between the frozen vertices $Q_0 \times (\widetilde{R}_0 \backslash R_0)$ and the mutable vertices $Q_0 \times R_0$ are given by the images of the arrows between $(\widetilde{R}_0 \backslash R_0)$ and R_0 under the maps*

$$\widetilde{R} \hookrightarrow Q \boxtimes^{\mathrm{f}} \widetilde{R}, \quad r \mapsto (q, r) \qquad (q \in Q_0). \tag{3.1}$$

Example 3.5. *We continue Example 2.3 and depict $A_2 \boxtimes^{\mathrm{f}} (\widehat{D}v_4)$:*



We assume that t is a sink of R and that Q is acyclic and thus possesses a source maximal green sequence $\mathcal{S}$. Then the frozen triangle product $Q \boxtimes^{\mathrm{f}} \widetilde{R}$ is well-behaved under mutation at $\mathcal{S} \times \{t\}$ in the following sense (compare [18, Lemma 7.2] for a similar statement).

Proposition 3.6. *Let $\widetilde{R}$ be an ice quiver with mutable part R and $t \in R$ a green sink. If Q is an acyclic quiver with source maximal green sequence $\mathcal{S}$ then*

$$Q\mathcal{S} \boxtimes^{\mathrm{f}} \widetilde{R}t = \left(Q \boxtimes^{\mathrm{f}} \widetilde{R}\right)(\mathcal{S} \times \{t\}). \tag{3.2}$$

Proof. Since both triangle products and source mutations commute with unions it suffices to show the claim for $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$. Since $\mathcal{S}_1$ is a source in Q and $t \in R$ a sink we have

$$Q\mathcal{S} \boxtimes Rt = (Q \boxtimes R)\,(\mathcal{S}_1, t)(\mathcal{S}_2, t). \tag{3.3}$$

We say a vertex is of level i if it is in $\{\mathcal{S}_i\} \times \widetilde{R}$. We need to show that there are no arrows between frozen vertices of level 1 and mutable vertices of level 2 in $(Q \boxtimes^{\mathrm{f}} \widetilde{R})(\mathcal{S}_1, t)(\mathcal{S}_2, t)$ and vice versa. Since $\mathcal{S}_1 \in Q$ is a source and $t \in R$ is a sink, there is no arrow from a vertex of level 2 targeting $(\mathcal{S}_1, t) \in Q \boxtimes^{\mathrm{f}} \widetilde{R}$ and no arrow from a vertex of level 2 targeting $(\mathcal{S}_2, t) \in (Q \boxtimes^{\mathrm{f}} \widetilde{R})(\mathcal{S}_1, t)$. The claim thus follows since there are no arrows departing from a frozen vertex of level k targeting $(\mathcal{S}_k, t) \in Q \boxtimes^{\mathrm{f}} \widetilde{R}$ due to $t \in \widetilde{R}$ being green. □

Our maximal green sequence on $Q \boxtimes R$ are composed of maximal green sequences of Q and R utilizing the following construction.

Definition 3.7. *Given sequences $\mathcal{S} = (\mathcal{S}_i)_{i=1}^n \subset Q_0$ and $\mathcal{T} = (\mathcal{T}_j)_{j=1}^m \subset R_0$ of vertices of Q and R, respectively, we write*

$$\mathcal{S} \boxtimes \mathcal{T} \subset (Q \boxtimes R)_0 = \{(q, r) \mid q \in Q, r \in R\}$$

for the sequence

$$\prod_{j=1}^{m} \prod_{i=1}^{n} (\mathcal{S}_i, \mathcal{T}_j) := (\mathcal{S}_1, \mathcal{T}_1), \ldots, (\mathcal{S}_n, \mathcal{T}_1), \ldots, (\mathcal{S}_1, \mathcal{T}_m), \ldots, (\mathcal{S}_n, \mathcal{T}_m).$$

We show

Theorem 3.8. *Let $\mathcal{S}$ and $\mathcal{T}$ be maximal green sequences of Q and R, respectively. If $\mathcal{S}$ is a source sequence and $\mathcal{T}$ is a sink sequence, then $\mathcal{S} \boxtimes \mathcal{T}$ is a maximal green sequence for $Q \boxtimes R$. Furthermore:*

$$\sigma_{\mathcal{S} \boxtimes \mathcal{T}} = \sigma_{\mathcal{S}} \times \sigma_{\mathcal{T}} = \mathrm{id} \times \sigma_{\mathcal{T}}$$

Proof. We write $\mathcal{T}^k := (\mathcal{T}_i)_{i=1}^k$ for $k = 0, \ldots, m$ and prove by induction on k

$$\left(Q \boxtimes^{\mathrm{f}} \widehat{R}\right)\left(\mathcal{S} \boxtimes \mathcal{T}^k\right) = (Q\mathcal{S}) \boxtimes^{\mathrm{f}} \left(\widehat{R}\mathcal{T}^k\right) \tag{3.4}$$

as follows:

$$\begin{aligned}
\left(Q \boxtimes^{\mathrm{f}} \widehat{R}\right)\left(\mathcal{S} \boxtimes \mathcal{T}^{k+1}\right) &= \left(Q \boxtimes^{\mathrm{f}} \widehat{R}\right)\left(\mathcal{S} \boxtimes \mathcal{T}^k\right)(\mathcal{S} \times \{\mathcal{T}_{k+1}\}) \\
&= \left((Q\mathcal{S}) \boxtimes^{\mathrm{f}} \left(\widehat{R}\mathcal{T}^k\right)\right)(\mathcal{S} \times \{\mathcal{T}_{k+1}\}) \\
&= \left(Q \boxtimes^{\mathrm{f}} \left(\widehat{R}\mathcal{T}^k\right)\right)(\mathcal{S} \times \{\mathcal{T}_{k+1}\}) \\
&= (Q\mathcal{S}) \boxtimes^{\mathrm{f}} \left(\widehat{R}\mathcal{T}^{k+1}\right)
\end{aligned}$$

The first equality follows from Definition 3.7. The second equality holds by the induction hypothesis. The third equality holds since for any source maximal green sequence $\mathcal{S}$ we have $\sigma_{\mathcal{S}} = \mathrm{id}$. Finally, the fourth equality is due to Proposition 3.6.

Since $\widehat{Q \boxtimes R} \simeq Q \boxtimes^{\mathrm{f}} \widehat{R}$ we obtain from (3.4) with $k = n$ that $\mathcal{S} \boxtimes \mathcal{T}$ is a reddening sequence for $Q \boxtimes R$ with $\sigma_{\mathcal{S}\boxtimes\mathcal{T}} = \sigma_{\mathcal{S}} \times \sigma_{\mathcal{T}}$. Since $\mathcal{S}$ is a source sequence we have $\sigma_{\mathcal{S}} = \mathrm{id}$.

It remains to prove that $\mathcal{S} \boxtimes \mathcal{T}$ is a green sequence for $Q \boxtimes R$. We need to show that for $j \in [m]$ and $i \in [n]$ the vertex $(\mathcal{S}_i, \mathcal{T}_j)$ is green in

$$(\widehat{Q \boxtimes R})(\mathcal{S} \boxtimes \mathcal{T}^{j-1}) \prod_{k=1}^{i-1} (\mathcal{S}_k, \mathcal{T}_j) \simeq (Q \boxtimes^{\mathrm{f}} \widehat{R})(\mathcal{S} \boxtimes \mathcal{T}^{j-1}) \prod_{k=1}^{i-1} (\mathcal{S}_k, \mathcal{T}_j). \tag{3.5}$$

Since $\mathcal{T}$ is a green sequence for R, the vertex $\mathcal{T}_j$ is green in $\widehat{R}\mathcal{T}^{j-1}$ by virtue of an arrow $\mathcal{T}_j \to \widehat{\mathcal{T}}_k$. Thus, by (3.4) the vertex $(\mathcal{S}_i, \mathcal{T}_j)$ is green in $(Q \boxtimes^{\mathrm{f}} \widehat{R})(\mathcal{S} \boxtimes \mathcal{T}^{j-1})$ by virtue of $(\mathcal{S}_i, \mathcal{T}_j) \to \overline{(\mathcal{S}_i, \mathcal{T}_k)}$. Again by (3.4) the full subgraph of $(Q \boxtimes^{\mathrm{f}} \widehat{R})(\mathcal{S} \boxtimes \mathcal{T}^{j-1})$ supported on $\{(\mathcal{S}_1, \mathcal{T}_j), \ldots, (\mathcal{S}_{i-1}, \mathcal{T}_j)\}$ is not connected to $\overline{(\mathcal{S}_i, \mathcal{T}_k)}$. Consequently, $(\mathcal{S}_i, \mathcal{T}_j)$ remains green in (3.5) by virtue of $(\mathcal{S}_i, \mathcal{T}_j) \to \overline{(\mathcal{S}_i, \mathcal{T}_k)}$. □

Remark 3.9. *In [15, Theorem 2] the existence of maximal green sequences for triangle products of a cycle with a Dynkin quiver is proved.*

Dually we obtain

Theorem 3.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be maximal red sequences of Q and R, respectively. If $\mathcal{S}$ is a sink sequence and $\mathcal{T}$ is a source sequence, then $\mathcal{S} \boxtimes \mathcal{T}$ is a maximal red sequence for $Q \boxtimes R$. Furthermore:*

$$\sigma_{\mathcal{S}\boxtimes\mathcal{T}} = \sigma_{\mathcal{S}} \times \sigma_{\mathcal{T}} = \mathrm{id} \times \sigma_{\mathcal{T}}$$

Proof. The statement follows from Theorem 3.8 by dualizing, i.e. turning around all arrows. □

4 The Fock–Goncharov conjecture

Let $\widetilde{Q}$ be an (ice-)quiver without loops and 2-cycles. Fomin and Zelevinsky associated in [8] to $\widetilde{Q}$ a cluster algebra $A = A(\widetilde{Q})$. In [14] a canonical basis, called *theta basis*, which is naturally identified with the tropical points of the corresponding mirror dual cluster algebra as predicted by the full Fock–Goncharov conjecture [7] is constructed for cluster algebras $A(\widetilde{Q})$ satisfying certain conditions. In particular, the full Fock–Goncharov conjecture for the cluster algebra associated to an ice quiver $\widetilde{Q}$ follows from the work of Gross–Hacking–Keel–Kontsevich ([14, Proposition 8.24, Proposition 8.25, Proposition 8.27, Lemma 9.10]) assuming

1. the existence of a reddening sequence for Q,
2. that every frozen vertex has an optimized seed,
3. that the matrix B has full rank.

Here B is the matrix associated to the mutable part Q_B of $\widetilde{Q}$ and we make use of the following

Definition 4.1. *A frozen vertex v of an ice quiver $\widetilde{Q}$ is said to have an optimized seed, if there exists a sequence of mutations S such that in $\widetilde{Q}S$ there is no arrow with source v. We call such a sequence S an* optimization sequence *for $v \in \widetilde{Q}$.*

Example 4.2. *All frozen vertices in Example 3.5 are optimized besides $\overline{v_{1,4}}$ and $\overline{v_{2,4}}$.*

As an immediate application of Theorem 3.8 we obtain property (1) for triangle products $Q \boxtimes R$ of an acyclic quiver Q and a Dynkin quiver R. In general (3) ceases to hold for $Q \boxtimes R$. We are, however, interested in the quivers $Q \boxtimes A_n^\bullet$, where A_n is of type A and $A_n^\bullet$ is obtained from A_n by freezing the vertices corresponding to the end points of the underlying Dynkin diagram.

Remark 4.3. *We emphasize that we obtain $Q \boxtimes A_n^\bullet$ from Q and $A_n^\bullet$ using Definition 3.1 of the unfrozen triangle product and not Definition 3.4. Thus, $Q \boxtimes A_n^\bullet$ is obtained by freezing the vertices (q,r) in $Q \boxtimes A_n$ for those $r \in A_n$ corresponding to endpoints of the underlying Dynkin diagram.*

The quivers $Q \boxtimes A_n^\bullet$ satisfy properties (1) - (3) and thus by the work of Gross-Hacking-Keel-Kontsevich the full Fock–Goncharov conjecture holds:

Theorem 4.4. *Let Q be an acyclic quiver and A_n a Dykin quiver of type A. Then properties (1) - (3) hold for $Q \boxtimes A_n^\bullet$.*

Proof. Property (1) holds for $Q \boxtimes A_n^\bullet$ due to Theorem 3.8 applied to the triangle product of Q and the mutable part of $A_n^\bullet$. Furthermore, a simple induction argument yields (3) for $Q \boxtimes A_n^\bullet$.

It remains to show (2). We first assume that A_n is linearly ordered with arrows from r_i to r_{i+1}. By induction on $2 \leqslant k \leqslant n-1$ we show: In $R = Q \boxtimes A_n^\bullet(q,r_2),\dots,(q,r_k)$ there is a triangle $(q,r_1) \to (q,r_{k+1}) \to (q,r_k) \to (q,r_1)$ of arrows with value $(1,1)$. There is no other arrow departing from (q,r_1). The only other arrow possibly departing from (q,r_{k+1}) has value $(1,1)$ and is targeting (q,r_{k+2}).

For $k=2$ the claim follows directly by the construction of $Q \boxtimes A_n^\bullet$. We deduce the claim for $k+1$ from k as follows. By the induction hypothesis there are arrows $(q,r_1) \to (q,r_{k+1}) \to (q,r_k) \to (q,r_1)$ in R as well as an arrow $(q,r_{k+1}) \to (q,r_{k+2})$ all with value $(1,1)$. Therefore, in $R(q,r_{k+1})$ there is a triangle $(q,r_1) \to (q,r_{k+2}) \to (q,r_{k+1}) \to (q,r_1)$ of arrows with value $(1,1)$. Since the only arrow in R departing from (q,r_1) targets (q,r_{k+1}) with value $(1,1)$ and the only arrows departing from (q,r_{k+1}) target (q,r_k) and (q,r_{k+2}) both with value $(1,1)$, we conclude that in $R(q,r_{k+1})$ the only arrow departing from (q,r_1) targets (q,r_{k+2}) with value $(1,1)$. Furthermore, since to each arrow in $Q \boxtimes A_n^\bullet$ with value (a,b) departing from (q,r_{k+2}) targeting $v \neq (q,r_{k+3})$ there is an arrow $v \to (q,r_{k+1})$ with value (b,a), and since both v and (q,r_{k+2}) are not connected by arrows to $\{(q,r_1),\dots,(q,r_k)\}$ in $Q \boxtimes A_n^\bullet$, the only arrows possibly departing from (q,r_{k+2}) in $R(q,r_{k+1})$ are targeting (q,r_{k+3}) and (q,r_{k+1}) both with value $(1,1)$.

From our claim for $k = n$ we conclude that

$$\mathcal{L}_q := (q,r_2),\dots,(q,r_{n-1}) \tag{4.1}$$

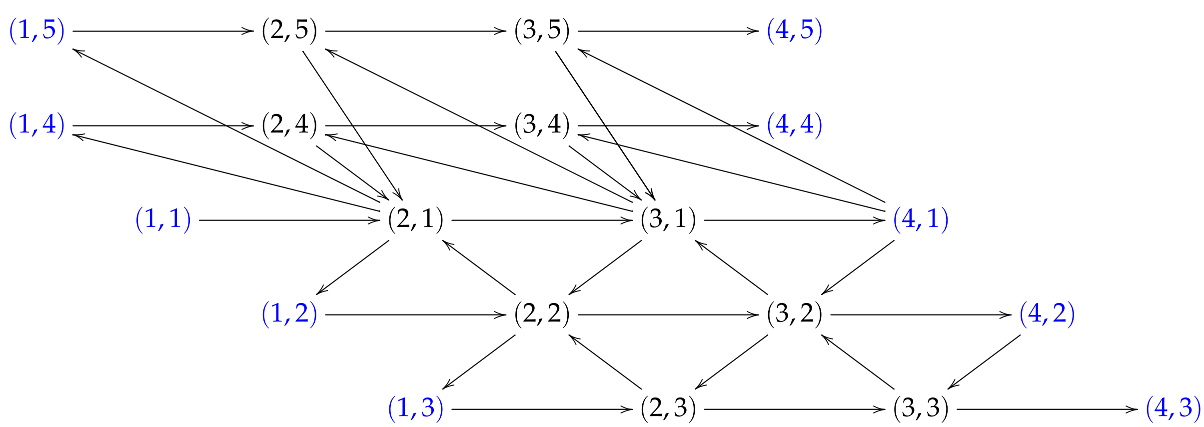
is an optimization sequence for (q,r_1).

For the general case we denote the frozen vertices of A_n with a_1 and a_n. Then there exist source sequences $\mathcal{T}^{(1)} \subset A_n \backslash \{a_n\}$ and $\mathcal{T}^{(n)} \subset A_n \backslash \{a_1\}$ such that $A_n \backslash \{a_n\}\mathcal{T}^{(1)}$ and $A_n \backslash \{a_1\}\mathcal{T}^{(n)}$ are linearly ordered with sources a_1 and a_n, respectively. Choosing a sink maximal red sequence $\mathcal{S}$ for Q we obtain by (4.1) and the dual version of Proposition 3.6 that

$$(q,r_1) \in (Q \boxtimes A_n^\bullet)(\mathcal{S} \boxtimes \mathcal{T}^{(1)})\mathcal{L}_q \qquad \text{and}$$
$$(q,r_n) \in (Q \boxtimes A_n^\bullet)(\mathcal{S} \boxtimes \mathcal{T}^{(n)})\mathcal{L}_q^{-1}$$

are optimized, where $\mathcal{L}_q^{-1} = (q,r_{n-1}),(q,r_{n-2})\dots,(q,r_2)$. □

Example 4.5. *Let D_4 correspond to a suitable orientation of the Dynkin diagram with the same name and A_4 be a linear oriented type A quiver with 4 vertices. Then $D_4 \boxtimes A_4^\bullet$ looks as follows:*



Here we depict the frozen vertices using blue. We then obtain the maximal green sequence

$$(3,5),(3,4),(3,3),(3,2),(3,1),(2,5),(2,4),(2,3),(2,2),(2,1),(3,5),(3,4),(3,3),(3,2),(3,1)$$

for the mutable part $D_4 \boxtimes A_2$ of $D_4 \boxtimes A_4^{\bullet}$. Furthermore, we obtain for $i \in [5]$ the following optimization sequences. For the vertex $(1,i)$

$$\mathcal{L}_i = (2,i),(3,i)$$

and for $(4,i)$:

$$\begin{aligned}\left(\mathcal{S} \boxtimes \mathcal{T}^{(4)}\right) \mathcal{L}_i^{-1} =&(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(3,i),(2,i).\end{aligned}$$

4.1 The Fock–Goncharov conjecture for large double Bruhat cells

Let G be a simply-connected, connected, semisimple complex algebraic group with Weyl group W. Let $B_{\pm} \subset G$ be a pair of opposite Borel subgroups of G. Berenstein, Fomin and Zelevinsky equipped in [2] the coordinate ring of the double Bruhat cells

$$G^{v,w} := B_+ v B_+ \cap B_- w B_-$$

associated to $v, w \in W$ with the structure of an upper cluster algebra by introducing appropriate ice quivers.

To be more precise, Berenstein, Fomin and Zelevinsky attach to each pair of reduced words for (v,w) an ice quiver $\widetilde{Q}$ defining an upper cluster algebra structure on the coordinate ring of $G^{v,w}$. The resulting structure does not depend on the chosen pair of reduced words. In [14] a canonical basis, called theta-basis, has been constructed for cluster algebras, which satisfy properties (1) - (3). As a byproduct of [14] and (1) - (3) one obtains that the associated upper cluster algebra turns out to be an honest cluster algebra. For $G^{v,w}$ this previously is established in [11].

We are intersted in the properties (1) - (3), which by [14] imply the full Fock–Goncharov conjecture for $G^{v,w}$.

In [2, Proposition 2.6] property (3) is proved for the quivers under consideration. Furthermore, the mutable part Q of $\widetilde{Q}$ may be chosen as a full subquiver of $D \boxtimes A_N$, where D is an orientation of the Dynkin diagram associated to G. Using [22, Theorem 9] we therefore obtain property (1) for $G^{v,w}$ from Theorem 4.4. For the big cells $G^{\mathrm{e},\mathrm{w}_0}$, $G^{\mathrm{w}_0,\mathrm{e}}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$ we furthermore obtain explicit optimization sequences from Theorem 4.4. Here w_0 denotes the longest element and e the identity in W.

Theorem 4.6. *Properties (1) and (3) hold for the cluster structure on $G^{u,v}$. Furthermore, the big double Bruhat cells $G^{\mathrm{e},\mathrm{w}_0}$, $G^{\mathrm{w}_0,\mathrm{e}}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$ additionally satisfy (2).*

Proof. It remains to provide optimization sequences for the big double Bruhat cells $G^{\mathrm{e},\mathrm{w}_0}$, $G^{\mathrm{w}_0,\mathrm{e}}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$. If G is not of type A, we may choose a reduced word for w_0 of the form $\mathbf{i}^{h/2}$, where $\mathbf{i}$ is a reduced word for the Coxeter element and h the Coxeter number of G. The construction [2, chapter 2.1] then yields the ice quivers $D \boxtimes \overleftarrow{A}^{\bullet}_N$, $D \boxtimes \overrightarrow{A}^{\bullet}_N$ and $D \boxtimes \overleftrightarrow{A}^{\bullet}_M$ defining the cluster structures on $G^{\mathrm{w}_0,\mathrm{e}}$, $G^{\mathrm{e},\mathrm{w}_0}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$, respectively. Here D is any orientation of the Dynkin diagram of G, $\overleftarrow{A}_N$ and $\overrightarrow{A}_N$ denote the linearly oriented quiver with $N = 1 + h/2$ vertices and $\overleftrightarrow{A}_M$ denotes the type A quiver with $M = 1 + h$ vertices possessing a unique source in the middle vertex. The claim therefore follows from Theorem 4.4.

For G of type A_n we choose lexicographic minimal reduced words. Optimization sequences for the resulting quivers for $G^{\mathrm{e},\mathrm{w}_0}$, $G^{\mathrm{w}_0,\mathrm{e}}$ and $G^{\mathrm{w}_0,\mathrm{w}_0}$ produced by [2, chapter 2.1] can easily be constructed similar to the linear oriented case of Theorem 4.4. □

Remark 4.7. *For $SL^{\mathrm{e},\mathrm{w}_0}$ and $SL^{\mathrm{w}_0,\mathrm{e}}$ Theorem 4.6 is due to [20]. Compare also with [21]. In the case of double Bruhat cells, Goodearl and Yakimov announced in [13] (see also [14, Example 0.15]) the existence of maximal green sequences. The existence of maximal green sequences for $G^{u,v}$ has been shown previously in [23, Theorem 4.1] (see also [24]).*

Acknowledgements

We would like to thank the anonymous referee of [12] for pointing out that there exists no general account on the Fock–Goncharov conjecture for large double Bruhat cells in the literature. We furthermore thank Bea Schumann and Linhui Shen for insightful discussions.

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