Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #85, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Naruse hook formula for linear extensions of mobile posets

GaYee Park*[1]

[1]*Department of Mathematics and Statistics, University of Massachusetts, Amherst*

Abstract. Linear extensions of posets are important objects in enumerative and algebraic combinatorics that are difficult to count in general. Families of posets like straight shapes and d-complete posets have hook-length product formulas to count linear extensions whereas families like skew shapes have determinant or positive sum formulas like the Naruse hook length formula from 2014. In 2020, Garver et. al. gave determinant formulas to count linear extensions of a family of posets called mobile posets that refine d-complete posets and border strip skew shapes. We give a Naruse type hook length formula to count linear extensions of such posets as well as q-analogues of our formula in both major and inversion index.

Keywords: standard tableaux, skew shapes, hook formula

1 Introduction

The number of standard Young tableaux (SYT) of a shape λ is counted by the famous hook-length formula:

Theorem 1.1 (Frame-Robinson-Thrall [2]). *Let λ be a partition of n. We have*

$$|\mathrm{SYT}(\lambda)| = n! \prod_{u\in[\lambda]} \frac{1}{h(u)},$$

where $h(u) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length of the square $u = (i, j)$.

Naruse introduced a more general formula to count the number of SYT of skew-shape as a sum over excited diagrams of products of hook-lengths.

Theorem 1.2 (Naruse [7]). *For a skew shape λ/μ of size n, we have*

$$|\mathrm{SYT}(\lambda/\mu)| = n! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in[\lambda]\setminus D} \frac{1}{h(u)}, \qquad \text{(NHLF)}$$

where $\mathcal{E}(\lambda/\mu)$ is the set of excited diagrams of λ/μ.

*park@math.umass.edu received support form the NSF grant (DMS-1855536).

In [4], Morales, Pak and Panova introduces a combinatorial proof of NHLF based on the case of *border strips*, connected skew-shapes with no 2×2 box.

The number of SYT of shape λ/μ can also be interpreted as the number of linear extension of a poset induced by the Young diagram of λ/μ. In [10], Proctor defined the family of *d-complete* posets, that include Young diagrams and rooted trees, and proved a hook-length formula to count the number of linear extensions.

Theorem 1.3 (Peterson-Proctor [10]). *The number of linear extensions of a d-complete poset $\mathcal{P}$ with n element is*

$$e(\mathcal{P}) = \frac{n!}{\prod_{x\in\mathcal{P}} h_{\mathcal{P}}(x)},$$

where $h_{\mathcal{P}}(x)$ is the hook-length of $x \in \mathcal{P}$.

A *mobile poset* is a recent common refinement of border strips and d-complete posets introduced in [3] (see Figure 1, (a)). The authors found a determinantal formula for the number of linear extensions of these posets, similar to *Jacobi–Trudi* formula and asked whether there was a Naruse-type formula [3, Sec. 6.1] for this number. The first main result of this extended abstract is a Naruse hook-length formula for mobile posets.

Theorem 1.4 (NHLF for mobiles). *Let $P_{\lambda/\mu}(\mathbf{p})$ be a free-standing mobile poset of size n with underlying border strip λ/μ and $\mathbf{p} = (p_{(r_1,s_1)}, \ldots, p_{(r_k,s_k)})$ the d-complete posets hanging on (r,s). Then we have,*

$$e(P_{\lambda/\mu}(\mathbf{p})) = \frac{n!}{H(\mathbf{p})} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\gamma} \frac{1}{h'(i,j)}, \tag{1.1}$$

where $h'(i,j) = \lambda_i - i + \lambda'_j - j + 1 + \sum_{r\geq i, s\geq j} |p_{(r,s)}|$, and $H(\mathbf{p})$ is the product of hook lengths of all elements in the d-complete posets hanging from λ/μ.

As an application to this main theorem, we give bounds to generalizations of Euler number defined in [3]. See Corollary 4.1 and Corollary 4.2.

Moreover, there is the following q-analogue of the NHLF for semistandard Young tableaux proved in [6]. We state this result in terms of $e_q^{\mathrm{maj}}(P,\omega) := \sum_\sigma q^{\mathrm{maj}(\sigma)}$ where (P,ω) is a labeled poset, and σ is a linear extension of it.

Theorem 1.5 (Morales-Pak-Panova [6]). *For a skew shape λ/μ with associated poset $Q_{\lambda/\mu}$ with ω Schur labeling, we have:*

$$\frac{e_q^{\mathrm{maj}}(Q_{\lambda/\mu},\omega)}{\prod_{i=1}^n (1-q^i)} = \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w(Br(D))} \prod_{u\in[\lambda]\setminus D} \frac{1}{1-q^{h(u)}} \tag{1.2}$$

where $w(Br(D))$ is the sum of hook-lengths of the support of broken diagonals.

Naruse-Okada [8] have a different q-analogue of $e_q^{\mathrm{maj}}(P,\omega)$ for a family called *skew d-complete* posets with *natural labelings*.

Our second result is the q-analgoue of Theorem 1.4 in terms of major index for all mobile posets.

Theorem 1.6. *For a labeled mobile poset $(P_{\lambda/\mu},\omega)$ with ω reverse Schur labeling on $[\lambda/\mu]$ and natural labeling on d-complete posets,*

$$\frac{e_q^{\mathrm{maj}}(P_{\lambda/\mu},\omega)}{\prod_{i=1}^n(1-q^i)} = \prod_{v\in\mathbf{p}} \frac{1}{1-q^{h(v)}} \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w'(\mathrm{Br}(D))} \prod_{u\in[\lambda]\setminus D} \frac{1}{1-q^{h'(u)}}, \tag{1.3}$$

where $w'(\mathrm{Br}(D)) = \sum_{u\in\mathrm{Br}(D)} h'(u)$.

We also have a q-analogue in terms of inversion statistic for the *mobile trees* (*d*-complete posets that are restricted to rooted trees).

Theorem 1.7. *For a labeled mobile poset $(P_{\lambda/\mu},\omega)$ with ω reverse Schur labeling on $[\lambda/\mu]$ and natural labeling on d-complete posets,*

$$\frac{e_q^{\mathrm{inv}}(P_{\lambda/\mu},\omega)}{\prod_{i=1}^n(1-q^i)} = \prod_{v\in\mathbf{p}} \frac{1}{1-q^{h(v)}} \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w(Br(D))+q^{p_D}} \prod_{u\in[\lambda]\setminus D} \frac{1}{1-q^{h'(u)}}, \tag{1.4}$$

where $w(Br(D)) = \sum_{u\in Br(D)} h(u)$ and $p_D = \sum_{(i,j)\in[\mu]\setminus D} \sum_{b=j} p_{a,b}$.

In Section 2, we give definitions and background results required for the proof. In Section 3, we give an example and the proof for Theorem 1.4. In Section 4, we show an application to the main theorem. Finally, In Section 5, we give examples and sketch the proofs of the q-analogues.

The full length of this extended abstract is available at [9].

2 Background

2.1 Posets and linear extensions

A *linear extension* of an n-element poset $\mathcal{P}$ is a bijection $f : \mathcal{P} \to [n]$ that is order-preserving. The number of SYT of a shape λ/μ is equal to the number of *linear extensions* of a poset of shape λ/μ. We denote the set of linear extensions of $\mathcal{P}$ as $\mathcal{L}(\mathcal{P})$, and $e(\mathcal{P}) = |\mathcal{L}(\mathcal{P})|$. We have the following formula for the number of linear extensions of a disjoint sum.

Proposition 2.1. *The number of linear extensions of a disjoint sum of posets $\mathcal{P}_i$, each of size n_i, is $e(\mathcal{P}_1+\cdots+\mathcal{P}_k) = \binom{n_1+\cdots n_k}{n_1,\cdots n_k} e(\mathcal{P}_1)\cdots e(\mathcal{P}_k)$.*

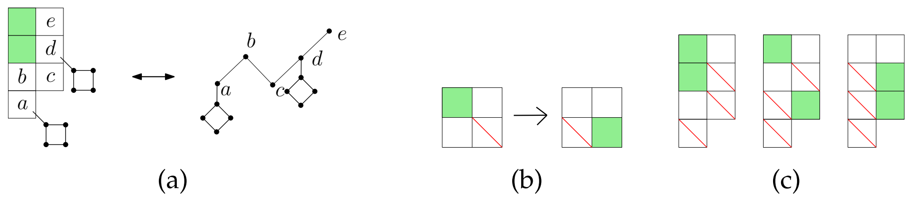

Figure 1: (a) The conversion of tableaux to mobile posets. (b) Excited move (c) Excited diagrams and the corresponding broken diagonals of $(2,2,2,1)/(1,1)$

2.2 Border strips and Mobile Posets

A *border strip* is a connected skew shape λ/μ containing no 2×2 box. A family of *d-complete posets* are a large class of posets containing rooted tree posets and posets arising from Young diagrams. Given a border strip, we can convert it into a poset by letting the inner corners of the diagram be the maximal points of the corresponding poset. Now we can construct a mobile poset (See Figure 1 (a) for an example).

Definition 2.2 (Garver-Grosser-Matherne-Morales [3])**.** *A* mobile[1] (tree) poset *is a poset obtained from a border strip Q, by allowing every element $x \in Q$ to cover the maximal element of a nonnegative number of disjoint d-complete (rooted tree) posets.*

2.3 Excited diagrams and broken diagonals

Denote $[\lambda/\mu]$ as the skew shape Young diagram of a shape λ/μ. An *excited diagram* of λ/μ, denoted by D, is a subset of $[\lambda]$ obtained from μ by applying a sequence of *excited moves* that we define next. Let $D \in \mathcal{E}(\lambda/\mu)$, then $(i,j) \in D$ is an *active cell* if $(i+1,j),(i,j+1)$, and $(i+1,j+1)$ are not in D. We obtain a new excited diagram by replacing an active cell by $(i+1,i+j)$ (see Figure 1, (b)). Note that for border strips, the excited diagrams can also be interpreted as the complement of its lattice paths γ from $(\lambda'_1,1) \to (1,\lambda_1)$ that stay inside $[\lambda]$ (see[4, Sec. 3]).

For each excited diagram $D \in \mathcal{E}(\lambda/\mu)$ we associate a set of *broken diagonals* $Br(D) \subset [\lambda] \setminus D$ as follows. Start with $D = [\mu]$, then $\text{Br}(D) = \{(i,j) \in \lambda/\mu | i-j = \mu_t - t\}$, for $1 \le t \le \ell(\lambda) - 1$ and $\mu_t = 0$ if $\ell(\mu) < t \le \ell(\lambda)$. For each active cell $u = (i,j)$ and its excited move $\alpha_u : D \to D'$, we have a corresponding move for the broken diagonal where $\text{Br}(D') = \text{Br}(D) \setminus \{(i+1,j+1)\} \cup \{(i+1,j)\}\}$. See Figure 1, (b) and (c).

[1]What we call a mobile poset is called a free-standing mobile poset in [3].

2.4 Multivariate function

For border strip λ/μ, let

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = F_{\lambda/\mu}(x_1,\ldots,x_{\lambda'_1},y_1,\ldots,y_{\lambda_1}) := \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \frac{1}{x_i - y_j}. \tag{2.1}$$

Let λ/ν be the shape obtained by removing an inner corner of λ/μ. We denote this subtraction by $\mu \to \nu$. Note that for border strips, λ/ν is disconnected. We denote each disconnected component as λ/ν_1 and λ/ν_2.

We need the following identity of $F_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ from [4].

Lemma 2.3 (Pieri-Chevalley formula [4, Eq. (6.3)])**.**

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = \frac{1}{x_1 - y_1} \sum_{\mu\to\nu} F_{\lambda/\nu^1}(\mathbf{x},\mathbf{y}) F_{\lambda/\nu^2}(\mathbf{x},\mathbf{y}), \tag{2.2}$$

where λ/ν^1 and λ/ν^2 are the two connected border strips that form λ/ν.

2.5 q-analogues of linear extensions

A *labeled poset* $(\mathcal{P},\omega)$ is a poset $\mathcal{P}$ with a labeling $\omega : \mathcal{P} \to [n]$. We call ω a *natural labeling* if for any $x, y \in \mathcal{P}$ with $x <_{\mathcal{P}} y$, we have $\omega(x) < \omega(y)$ [11]. We call ω a *reverse Schur labeling* if the labeling increases as it follows the path from the right end of the poset [11]. For Theorem 1.7 and Conjecture 1.6, we use reverse Schur labeling on $[\lambda/\mu]$ and natural labeling on the d-complete posets. In the case of inversion statistic, for each $\mu \to \nu$, we need $\omega(x_1) > \omega(x_2)$ for all $x_i \in \lambda/\nu_i$ to satisfy the condition for Proposition 2.6

Given a linear extension $f : \mathcal{P} \to [n]$, the permutation $\omega \circ f^{-1} \in \mathfrak{S}_n$ is a *linear extension of the labeled poset*, $\mathcal{L}(\mathcal{P},\omega)$. Recall, $\mathrm{maj}(\sigma) = \sum_{i\in \mathrm{Des}(\sigma)} i$ and $\mathrm{inv}(\sigma) = \#\{i \in [n-1] | \sigma_i > \sigma_j \text{ where } i < j\}$ where $\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}$. The two common statistics for q-analogues of the number of linear extensions for a labeled poset $(\mathcal{P},\omega)$ are the major index and inversions.

Let stat $\in \{\mathrm{maj}, \mathrm{inv}\}$, the *major index (inversion) q-analogue* of the number of linear extensions of a labeled poset $(\mathcal{P},\omega)$ is

$$e_q^{\mathrm{stat}}(\mathcal{P},\omega) := \sum_{\sigma\in\mathcal{L}(\mathcal{P},\omega)} q^{\mathrm{stat}(\sigma)}.$$

Now we state the q-analogue of the hook-length formulas. We define the *q-integer* $[n]_q := 1 + q + \cdots + q^{n-1}$ and the *q-factorial* $[n]_q! := [n]_q \cdots [2]_q[1]_q$. We also define the *$q$-multinomial coefficients* $\begin{bmatrix} n \\ p \end{bmatrix}_q := \frac{[n]!}{[p]_q![n-p]_q!}$.

Peterson and Proctor [10] gave a q-analogue formula in terms of major index for d-complete posets.

Theorem 2.4 (Peterson and Proctor [10])**.** *Let $(\mathcal{P},\omega)$ be a labeled d-complete poset of size n with any labeling. Then,*

$$e_q^{\mathrm{maj}(\mathcal{P},\omega)} = q^{\mathrm{maj}(\mathcal{P},\omega)} \frac{[n]_q!}{\prod_{x\in\mathcal{P}}[h_{\mathcal{P}}(x)]_q}.$$

Björner and Wachs [1] gave a q-analogue formula in terms of inversion index for rooted tree posets.

Theorem 2.5 (Björner and Wachs [1])**.** *Let $(\mathcal{P},\omega)$ be a rooted tree poset with a natural labeling. Then,*

$$e_q^{\mathrm{inv}}(\mathcal{P},\omega) = q^{\mathrm{inv}(\mathcal{P},\omega)} \frac{[n]_q!}{\prod_{x\in\mathcal{P}}[h_{\mathcal{P}}(x)]_q}.$$

For both major [11, Exercise 3.162(a)] and inversion [1] statistics, we have the following proposition.

Proposition 2.6. *Let $(P+Q,\omega)$ be a labeled disjoint sum of posets with $|P+Q|=n$ and $|P|=p$. Suppose that ω has the property that the label of every element of P is smaller than the label of every element of Q. We have*

$$e_q^{\mathrm{inv}}(P+Q,\omega) = \begin{bmatrix} n \\ p \end{bmatrix}_q e_q^{\mathrm{inv}}(P,\omega_1)\cdot e_q^{\mathrm{inv}}(Q,\omega_2),$$

where ω_1 and ω_2 are the labeling obtained by restricting ω to P and Q respectively.

The formula for the disjoint sum in major index is the same, and it applies for all labeling.

3 Proof of the NHLF for Mobiles

In this section we sketch the proof of Theorem 1.4. The proof follows the proof of the NHLF for border strips in [4]. We need to first define the hook lengths of mobile posets. Given a mobile poset $P_{\lambda/\mu}(\mathbf{p})$, define the hook length of $(i,j)\in[\lambda]$ as follows:

$$h'(i,j) = \lambda_i - i + \lambda'_j - j + 1 + \sum_{a\geq i, b\geq j} p_{a,b} \tag{3.1}$$

In other words, it is the usual hook length of the cell in λ plus the sizes of the d-complete posets that are attached on the segment of the border strip inside of the hook of (i,j) (see Figure 2 (a)). We provide an example of the main theorem below.

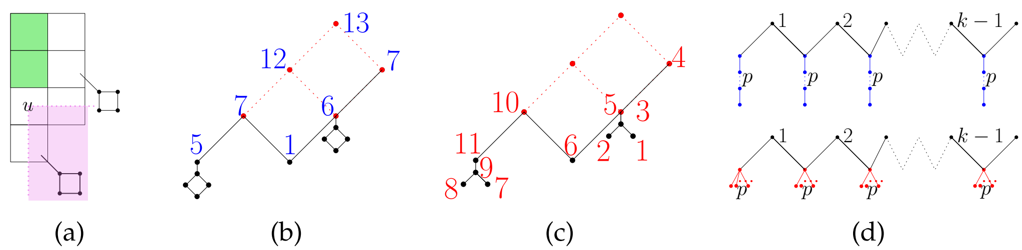

Figure 2: (a): $h'(u)$ is the usual hook-length plus the size of d-complete posets in the shaded area. (b): mobile poset with hook-lengths marked. (c): a labeled mobile tree. (d): illustration of poset $\mathcal{C}_p(k)$ (top) and $\mathcal{A}_p(k)$ (bottom).

Example 3.1. *Consider the mobile poset $P_{2221/11}$ from Figure 2 (b). By Theorem 1.4, one can check that*

$$e(P) = \frac{13!}{2^4 \cdot 4^2}\left(\frac{1}{5\cdot 6\cdot 7^2} + \frac{1}{5\cdot 6\cdot 7^2\cdot 12} + \frac{1}{5\cdot 7^2\cdot 12\cdot 13}\right) = 33000. \tag{3.2}$$

Now we give the proof of the theorem. Let $H_{\lambda/\mu}$ be the sum on the RHS of (1.1). Then we have the following lemma.

Lemma 3.2. $H_{\lambda/\mu} = \frac{1}{n}\sum_{\mu\to\nu} H_{\lambda/\nu^1} H_{\lambda/\nu^2}$

Proof. We evaluate $F(\mathbf{x},\mathbf{y})$ from (2.1) at the following values of x_i and y_j:

$$x_i = \lambda_i - i + 1 - \sum_{a<i} p_{a,b} \quad \text{and} \quad y_j = j - \lambda'_j - \sum_{b\geq j} p_{a,b}.$$

Note that $x_1 - y_1 = n$ and $x_i - y_j = h'(i,j)$. Evaluating $F_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ at such x_i and y_j, we obtain the sum on the RHS of (1.1).

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) \;\Big|_{\substack{x_i=\lambda_i-i+1+\sum_{a\geq i} p_{a,b}\\ y_j=j-\lambda'_j+\sum_{b<j} p_{a,b}}} = \sum_{\gamma:(\lambda'_1,1)\to(1,\lambda_1),\gamma\subset\lambda}\ \prod_{(i,j)\in\gamma} \frac{1}{h'(i,j)} = H_{\lambda/\mu} \tag{3.3}$$

Then, evaluating (2.2) at such x_i and y_j, we obtain the desired equation. $\square$

Similar to skew shaped SYT, we have the following recurrence for mobile posets:

Lemma 3.3. *For a mobile poset $P_{\lambda/\mu}$, we have*

$$e(P_{\lambda/\mu}) = \sum_{\mu\to\nu} e(P_{\lambda/\nu}), \tag{3.4}$$

where ν is obtained by adding an inner corner of λ/μ to μ.

Proof. For a fixed mobile $P_{\lambda/\mu}$, a linear extension of this poset consists of an inner corner of λ/μ followed by a linear extension of the remaining poset of shape λ/ν, where $\mu \to \nu$. Conversely, given a linear extension of $P_{\lambda/\nu}$, by inserting the new element in the beginning we obtain a linear extension of $P_{\lambda/\mu}$. $\square$

We are now ready to give the proof for Theorem 1.4.

Proof of Theorem 1.4. We induct on $|\lambda/\mu|$ to show that $e(P_{\lambda/\mu}(\mathbf{p}))/n! = H_{\lambda/\mu}/H(\mathbf{p})$ using Lemma 3.3. Note that λ/ν is disconnected and

$$P_{\lambda/\nu} = P_{\lambda/\nu^1} + P_{\lambda/\nu^2} + T_1 + \cdots + T_k, \tag{3.5}$$

where $P_{\lambda/\nu^1}, P_{\lambda/\nu^2}$ are the two connected free standing mobiles of size p_1 and p_2, and $T_1, \ldots, T_k$, for $k \geq 0$, are the d-complete posets of sizes $t_1, \ldots, t_k$ hanging from the removed inner corner in $P_{\lambda/\mu}$. By Proposition 2.1 we have

$$e(P_{\lambda/\nu}) = \binom{n-1}{p_1, p_2, t_1, \ldots, t_k} e(P_{\lambda/\nu^1}) e(P_{\lambda/\nu^2}) e(T_1) \cdots e(T_k),$$

then using this, (3.4) becomes

$$\frac{e(P_{\lambda/\mu})}{(n-1)!} = \sum_{\mu \to \nu} \left(\prod_{i=1}^{k} \frac{e(T_i)}{t_i!} \right) \frac{e(P_{\lambda/\nu^1})}{p_1!} \frac{e(P_{\lambda/\nu^2})}{p_2!}. \tag{3.6}$$

Since T_i is d-complete then $e(T_i)/t_i! = 1/H(T_i)$, by Thoerem 1.3. Also, both

$$|\lambda/\nu^1|, |\lambda/\nu^2| < |\lambda/\mu|,$$

by induction $e(P_{\lambda/\nu^j})/p_j! = H_{\lambda/\nu^j}/H(\mathbf{p_j})$ for $j = 1, 2$, where $H(\mathbf{p_j})$ is the product of the hook lengths of the elements in the d-complete posets hanging from $[\lambda/\nu^j]$. Note that $H(\mathbf{p}) = H(\mathbf{p_1})H(\mathbf{p_2})H(T_1) \cdots H(T_k)$. Then (3.6) becomes

$$e(P_{\lambda/\mu}) = \frac{(n-1)!}{H(\mathbf{p})} \sum_{\mu \to \nu} H_{\lambda/\nu^1} H_{\lambda/\nu^2}. \tag{3.7}$$

By Lemma 3.2, the sum on the RHS of (3.7) equals $n \cdot H_{\lambda/\mu}$, completing the proof. $\square$

4 Application

As an application to Theorem 1.4, it gives bounds to $e(P_{\lambda/\mu}(\mathbf{p}))$ just as in [5].

Corollary 4.1. *For any mobile poset $e(P_{\lambda/\mu}(\mathbf{p}))$ of size n,*

$$\frac{n!}{H(\mathbf{p})\prod_{u\in[\lambda/\mu]} h'(u)} \leq e(P_{\lambda/\mu}(\mathbf{p})) \leq |\mathcal{E}(\lambda/\mu)| \cdot \frac{n!}{H(\mathbf{p})\prod_{u\in[\lambda/\mu]} h'(u)}$$

where $[\lambda/\mu]$ is the border strip of the mobile poset.

One application of the formula is that it provides bounds to generalizations of *Euler numbers* defined in [3]. The authors give two generalizations of Euler number using two different families of posets, up-down posets with $k-1$ downs and chains (or anti-chains) of size p hanging on every minimal element,denoted as $\mathcal{C}_p(k)$ and $\mathcal{A}_p(k)$ (see Figure 2 (d)). See A332471 and A332568 in the OEIS for examples of these sequences.

Corollary 4.2.

$$\frac{(2k+kp)!}{(p+1)!^k(2p+3)^{k-1}(p+2)} \leq e(\mathcal{C}_p(k)) \leq \text{Cat}(k) \cdot \frac{(2k+kp)!}{(p+1)!^k(2p+3)^{k-1}(p+2)}$$

$$\frac{(2k+kp)!}{(p+1)^k(2p+3)^{k-1}(p+2)} \leq e(\mathcal{A}_p(k)) \leq \text{Cat}(k) \cdot \frac{(2k+kp)!}{(p+1)^k(2p+3)^{k-1}(p+2)},$$

where $\mathcal{Z}$ is the up-down border strip with $k-1$ many down steps.

Proof. The result follows from Corollary 4.1, a routine calculations of hooks, and the fact that the excited diagrams of up-down posets are given by the Catalan numbers [4] □

5 q-analogues of the Naruse-type formula for mobiles

5.1 An inversion index q-analogue

In this section we give an example and sketch the proof of Theorem 1.7. Unless specified otherwise, $(P_{\lambda/\mu}(\mathbf{p}), \omega)$ is a labeled mobile tree poset.

Example 5.1. *Consider labeled the mobile tree poset $(P_{2221/11}, \omega)$ from Figure 2 (c). By Theorem 1.7, we have*

$$\begin{aligned} e_q^{\text{inv}}(P) &= \frac{[11]!}{[1]^4[3]^2}\left(\frac{q^4}{[4][6][1][5][6]} + \frac{q^9}{[4][6][10][5][6]} + \frac{q^{14}}{[4][6][10][11][6]}\right) \\ &= q^{38} + 4q^{37} + 9q^{36} + 17q^{35} + \cdots + 9q^6 + 4q^5 + q^4. \end{aligned}$$

Denote the sum on the RHS of (1.4) as $\widetilde{H}_{\lambda/\mu}(q)$. We have the following lemma.

Lemma 5.2. *We have* $(1-q^n)\cdot \widetilde{H}_{\lambda/\mu} = \sum_{\mu\to\nu} q^{\lambda'_1-1+c(u)}\cdot \widetilde{H}_{\lambda/\nu^1}(q)\cdot \widetilde{H}_{\lambda/\nu^2}(q)$, *where* $c(u) = j-i$ *for* $u=(i,j)$, *the inner corener from* $\mu\to\nu$

Proof. We first evaluate $F_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ at $x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}$ and $y_j = q^{j-\lambda'_j - \sum_{b\geq j} p_{a,b}}$.

$$F_{\lambda/\mu}(\mathbf{x},\mathbf{y}) \mid_{x_i = q^{\lambda_i - i+1-\sum_{a<i} p_{a,b}}, y_j = q^{j-\lambda'_j-\sum_{b\geq j} p_{a,b}}} = (-1)^n \cdot \sum_{\substack{\gamma: A\to B,\\ \gamma\subset\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - j+\sum_{b\geq j} p_{a,b}}}{1-q^{h'(i,j)}} \tag{5.1}$$

By [6, Prop 4.7] and [6, Lemma 7.17], we have

$$\begin{aligned}\sum_{(i,j)\in\lambda\setminus D}\Big((\lambda'_j - j) + \sum_{b\geq j} p_{a,b}\Big) &= \sum_{(i,j)\in[\lambda]\setminus D}\Big((\lambda'_j - i) + \sum_{b\geq j} p_{a,b}\Big) - \sum_{(i,j)\in[\lambda]\setminus[\mu]} c(i,j)\\ &= w(\mathrm{Br}(D)) + \sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geq j} p_{a,b} - \sum_{(i,j)\in[\lambda]\setminus[\mu]} c(i,j)\end{aligned} \tag{5.2}$$

where $c(i,j) = j-i$ and $w(\mathrm{Br}(D)) = \sum_{(i,j)\in Br(D)} h(i,j)$. Denote $\sum_{(i,j)\in[\lambda/\mu]}\sum_{b\geq j} p_{a,b}$ as p^*. Taking p^* and $c(i,j)$ outside of the sum, we can write (5.1) as

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y}) \mid_{x_i = q^{\lambda_i - i+1-\sum_{a<i} p_{a,b}}, y_j = q^{j-\lambda'_j-\sum_{b\geq j} p_{a,b}}} = (-1)^n \cdot q^{p^* - \sum_{(i,j)\in[\lambda/\mu]} c(i,j)} \widetilde{H}_{\lambda/\mu}(q) \tag{5.3}$$

Then, evaluating (2.2) at such x_i and y_j, we obtain the desired formula. □

We generalize Lemma 3.3 to the inversion index q-analogue. We have the following.

Lemma 5.3.

$$e_q^{\mathrm{inv}}(P_{\lambda/\mu},\omega) = \sum_{\mu\to\nu} q^{n-\omega(u)} e_q^{\mathrm{inv}}(P_{\lambda/\nu},\omega_\nu) \tag{5.4}$$

where ω *is a reverse Schur labeling and* $\omega(u)$ *is the label of the inner corner* u *from* $\mu\to\nu$.

Proof. Recall that the linear extension σ of $P_{\lambda/\mu}$ consist of a linear extension σ' of λ/ν followed by an inner corner u. Note that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma') + n - \omega$, where $n-\omega$ is the number of inversion caused by the inner corner. The rest of the proof follows the same argument as Lemma 3.3. □

We are now ready to give the proof for Theorem 1.7.

Proof of Theorem 1.4. We show that $e_q^{\mathrm{inv}}(P_{\lambda/\mu}(\mathbf{p})) = \frac{\prod_{i=1}^n (1-q^i)}{\prod_{v\in T} 1-q^{h(v)}}\cdot \widetilde{H}_{\lambda/\mu}(q)$ by induction on $|\lambda/\mu|$ using Lemma 5.3. Recall λ/ν is disconnected and is expressed as (3.5). By induction and Theorem 2.5, for each P_{λ/ν^j} and T_i, we have
$\frac{e_q(P_{\lambda/\nu^j})}{[p_j]_q!} = \frac{(1-q)^{p_j}\cdot}{\prod_{v\in\mathbf{p_j}} 1-q^{h(v)}}\cdot \widetilde{H}_{\lambda/\nu^j}(q)$ and $\frac{e_q^{\mathrm{inv}}(T_i)}{[t_i]_q!} = \frac{q^{\mathrm{inv}(T_i)}}{\prod_{v\in T_i}[h(v)]_q} = \frac{(1-q)^{t_i}}{\prod_{v\in T_i}(1-q^{h(v)})}$.

Note that T_i are natural labeling, so $\mathrm{inv}(T_i) = 0$ vanishes. Using Proposition 2.6 and the equations above, we have

$$e_q(P_{\lambda/\nu}) = \frac{\prod_{i=1}^{n-1}(1-q^i)}{\prod_{v\in T}(1-q^{h(v)})}\widetilde{H}_{\lambda/\nu^1}(q)\cdot\widetilde{H}_{\lambda/\nu^2}(q)$$

We now apply the equation to (5.4).

$$e_q(P_{\lambda/\mu}) = \frac{\prod_{i=1}^{n-1}(1-q^i)}{\prod_{v\in T}(1-q^{h(v)})}\sum_{\mu\to\nu} q^{n-\omega(u)}\cdot\widetilde{H}_{\lambda/\nu^1}(q)\cdot\widetilde{H}_{\lambda/\nu^2}(q) \tag{5.5}$$

Note that $\lambda_1' - 1 + c(u) = n - \omega(u)$ for ω, reversed Schur labeling. By Lemma 5.2, the sum on the RHS of (5.5) equals $(1-q^n)\cdot\widetilde{H}_{\lambda/\mu}(q)$, completing the proof. □

5.2 A major index q-analogue

In this section, we give an example and sketch the proof of Theorem 1.6. The poset $(P_{\lambda/\mu}(\mathbf{p}),\omega)$ is labeled with a reversed Schur labeling on the border strip and a natural labeling on the d-complete posets.

Example 5.4. *Consider the poset $(P_{2221/11},\omega)$ from Figure 2 (c). By Theorem 1.6, we have*

$$\begin{aligned} e_q^{\mathrm{maj}}(P) &= q^{44} + 4q^{43} + 9q^{42} + 17q^{41} + \cdots + 9q^{12} + 4q^{11} + q^{10} \\ &= \frac{[11]!}{[1]^4[3]^2}\left(\frac{q^{10}}{[4][6][1][5][6]} + \frac{q^{15}}{[4][6][10][5][6]} + \frac{q^{20}}{[4][6][10][11][6]}\right). \end{aligned}$$

Similiarly as the case of the inversion index, we have the following lemmas in terms of the major index. Again, denote the sum of the RHS of (1.3) as $H_{\lambda/\mu}(q)$.

Lemma 5.5. *We have $(1-q^n)\cdot H_{\lambda/\mu}(q) = \sum_{\mu\to\nu} q^{|P_{\lambda/\nu^1}|}\cdot H_{\lambda/\nu^1}(q)\cdot H_{\lambda/\nu^2}(q)$, where T_ν is the union of the d-complete posets that were hanging on the removed inner corner u.*

Proof sketch. As done in the inversion index case, we first evaluate the multivariate formula $F_{\lambda/\mu}$ at $x_i = q^{\lambda_i - i + 1 - \sum_{a<i} p_{a,b}}$ and $y_j = q^{j-\lambda_j' - \sum_{b\geq j} p_{a,b}}$. We modify (5.2) as follows:

$$\mathrm{LHS} = w'(\mathrm{Br}(D)) + \left(\sum_{(i,j)\in[\lambda]\setminus D}\sum_{b\geq j} p_{a,b} - \sum_{(i,j)\in\mathrm{Br}(D)} p_{i,j}\right) - \sum_{(i,j)\in[\lambda]\setminus[\mu]} c(i,j)$$

where $w'(\mathrm{Br}(D)) = \sum_{(i,j)\in\mathrm{Br}(D)} h'(i,j)$. Let p_1 be the size of the d-complete posets on P_{λ/ν_1}. Applying this modification to (2.2), we get the following equation.

$$(1-q^n)\cdot H_{\lambda/\mu}(q) = \sum_{\mu\to\nu} q^{\lambda_1'-1+c(u)+p_1}\cdot H_{\lambda/\nu^1}(q)\cdot H_{\lambda/\nu^2}(q). \tag{5.6}$$

Note that $\lambda_1' - 1 + c(u) + p_1 = |P_{\lambda/\nu_1}|$. □

Lemma 5.6. *For a labeled mobile poset $(P_{\lambda/\mu}(\mathbf{p}), \omega)$, where ω is a reverse Schur labeling, $e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega) = \sum_{\mu \to \nu} q^{|P_{\lambda/\nu_1}|} e_q^{\mathrm{maj}}(P_{\lambda/\nu}, \omega_\nu)$ where λ/ν_1 is the left disconnected poset of λ/ν, and ω_ν is the restricted labeling of ω onto λ/ν.*

Acknowledgements

I would like to thank Alejandro Morales for introducing this problem and for all of his guidance throughout this project. I would also like to thank Stefan Grosser, Jacob Matherne, Soichi Okada, and the anonymous referees for helpful comments.

References

[1] A. Björner and M. L. Wachs. "q-hook length formulas for forests". *J. Combin. Theory Ser. A* **52**.2 (1989), pp. 165–187.

[2] J. S. Frame, G. d. B. Robinson, and R. M. Thrall. "The hook graphs of the symmetric groups". *Canad. J. Math.* **6** (1954), pp. 316–324.

[3] A. Garver, S. Grosser, J. P. Matherne, and A. H. Morales. "Counting linear extensions of posets with determinants of hook lengths". *SIAM J. Discrete Math.* **35**.1 (2021), pp. 205–233.

[4] A. H. Morales, I. Pak, and G. Panova. "Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications". *SIAM Journal on Discrete Mathematics* **31**.3 (2017), pp. 1953–1989.

[5] A. H. Morales, I. Pak, and G. Panova. "Asymptotics of the number of standard Young tableaux of skew shape". *European J. Combin.* **70** (2018), pp. 26–49.

[6] A. H. Morales, I. Pak, and G. Panova. "Hook formulas for skew shapes I. q-analogues and bijections". *J. Combin. Theory Ser. A* **154** (2018), pp. 350–405.

[7] H. Naruse. "Schubert calculus and hook formula". Slides at 73rd Séminaire Lotharingien de Combinatoire, Strobl. 2014. Link.

[8] H. Naruse and S. Okada. "Skew hook formula for d-complete posets via equivariant K-theory". *Algebr. Comb.* **2**.4 (2019), pp. 541–571. DOI.

[9] G. Park. "Naruse hook formula for linear extensions of mobile posets". arXiv:2104.11166.

[10] R. A. Proctor. "d-complete posets generalize Young diagrams for the hook product formula: Partial presentation of proof". *RIMS* **1913** (2014), pp. 120–140.

[11] R. P. Stanley. *Enumerative combinatorics. Vols. 1, 2.* Vol. 49, 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, 1999.