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Cluster scattering diagrams and theta basis for reciprocal generalized cluster algebras

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Abstract. We give a construction of generalized cluster scattering diagrams, generalized cluster varieties, and theta bases for reciprocal generalized cluster algebras, the latter of which were defined by Chekhov and Shapiro. These constructions are analogous to the structures given for ordinary cluster algebras in the work of Gross, Hacking, Keel, and Kontsevich.

Keywords: cluster algebra, generalized cluster algebra, scattering diagram, theta function

1 Introduction

This work concerns constructing analogues of scattering diagrams, cluster varieties, and theta bases beyond the ordinary cluster algebra setting. Cluster algebras were originally introduced by Fomin and Zelevinsky as a tool for studying total positivity [4]. Two particularly celebrated structural features of cluster algebras are the *Laurent phenomenon* and *positivity*. Although the Laurent phenomenon was proved in the original work of Fomin and Zelevinsky, a proof of positivity for arbitrary cluster algebras did not appear until the work of Gross, Hacking, Keel, and Kontsevich [9].

From a geometric perspective, one can study cluster algebras by studying *cluster varieties*, which are studied in [3] as *cluster ensembles* $(\mathcal{A}, \mathcal{X})$. The $\mathcal{A}$-variety encodes information about the cluster variables and the $\mathcal{X}$-variety encodes information about the coefficients in the sense of [5]. For the special case of cluster algebras with principal coefficients, one can talk about the $\mathcal{A}$ *cluster variety with principal coefficients*, $\mathcal{A}_{\text{prin}}$; the $\mathcal{A}$ and $\mathcal{X}$ varieties appear, respectively, as a fiber and quotient of $\mathcal{A}_{\text{prin}}$.

An important structural question in the study of cluster algebras concerns the existence of bases. Because the original definition of cluster algebras arose from a desire to understand dual canonical bases, it is natural to wonder if "desirable" bases for cluster algebras exist. Many subclasses of cluster algebras have known bases, including: the cluster monomial basis for finite type, the generic basis for affine type [1], the generic basis for acyclic type [7, 6], and the bangle and band bases for cluster algebras of surface type [14]. Gross, Hacking, Keel, and Kontsevich proved the existence of the *theta basis* for arbitrary cluster algebras [9].

Their proofs of positivity and the existence of the theta basis used *scattering diagrams*, a tool from algebraic geometry. Scattering diagrams were first introduced in two dimensions by Kontsevich and Soibelman in [12] and then in arbitrary dimension by Gross and Siebert in [10] as a tool for constructing mirror spaces in mirror symmetry. Gross, Hacking, Keel, and Kontsevich constructed cluster scattering diagrams and then defined the theta basis [9] for cluster algebras.

We parallel the work of Gross, Hacking, Keel, and Kontsevich in the context of reciprocal generalized cluster algebras. In particular, we construct generalized cluster varieties whose rings of regular functions are generalized cluster algebras. Simultaneously, we develop scattering diagrams which lie in the tropicalization of the Fock–Goncharov dual of said varieties. This allows us to establish theta bases for such algebras.

We remark that in his recent Ph.D. thesis, Lang Mou outlines a related approach for constructing cluster scattering diagrams for reciprocal generalized cluster algebras [13]. His work utilizes a different lattice structure and does not provide theta functions. Our work occurred contemporaneously and independently.

Section 2 contains necessary background information for generalized cluster algebras. Our construction is then given in Section 3. In Section 4, we define *theta functions* in terms of *broken lines* and state our main result, Theorem 4.6, that given generalized fixed data Γ and any choice of seed $\mathbf{s}$ and corresponding cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$, a particular collection Θ of theta functions defined on $\mathfrak{D}_{\mathbf{s}}$ forms a basis for the associated generalized cluster algebra (see Theorem 4.6 for the precise statement).

2 Generalized Cluster Algebras

One natural generalization of a cluster algebra, introduced by [2], is to allow the characteristic binomial exchange relations to instead contain arbitrarily many terms. Let $(\mathbb{P}, \oplus)$ be a semifield and $\mathcal{F}$ be isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{P}$.

Definition 2.1. *A* labeled generalized cluster seed *is a quintuple* $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ *such that* $\mathbf{x} = (x_1, \ldots, x_n)$ *is a free generating set for* $\mathcal{F}$, $\mathbf{y}$ *is an* n*-tuple with elements in* $\mathbb{P}$, $B = [b_{ij}]$ *is an* $n \times n$ *skew-symmetrizable matrix with entries in* $\mathbb{Z}$, R *is an* $n \times n$ *diagonal matrix with positive integer entries whose* i*-th diagonal entry is denoted by* r_i, *and* $\mathbf{a} = (a_{i,s})_{i \in [n], s \in [r_i - 1]}$ *is a collection of elements in* $\mathbb{P}$. *We refer to* $\mathbf{x} = (x_1, \ldots, x_n)$ *as the* cluster *of* Σ, $\mathbf{y} = (y_1, \ldots, y_n)$ *as the* coefficient tuple, B *as the* generalized exchange matrix, R *as the exchange degree matrix, and* $\mathbf{a}$ *as the exchange coefficient collection. The elements* $x_1, \ldots, x_n$ *are the* cluster variables *of* Σ *and* $y_1, \ldots, y_n$ *are its* coefficient variables.

Together, the data of the exchange degree matrix R and the exchange coefficient collection $\mathbf{a}$ determine a set of *exchange polynomials* $\rho_1, \ldots, \rho_n$, where for each i, we have

$\rho_i(u) = 1 + a_{i,1}u + \cdots + a_{i,r_i-1}u^{r_i-1} + u^{r_i} \in \mathbb{ZP}[u]$. The structure of the exchange relations for mutation in direction k are determined by the k-th exchange polynomial. We work in the specialized setting of [15] and impose the additional requirement that $a_{i,s} = a_{i,r_i-s}$ - i.e., that all exchange polynomials are *reciprocal polynomials*.

Definition 2.2. *For a generalized cluster seed $\Sigma = (\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$, generalized mutation in direction k, $\mu_k^{(r)}$, is defined by the following exchange relations:*

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + r_k\left([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+\right) & i, j \neq k \end{cases}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i \left(y_k^{[b_{ki}]_+}\right)^{r_k} \left(\bigoplus_{s=0}^{r_k} a_{k,s} y_k^s\right)^{-b_{ki}} & i \neq k \end{cases}$$

$$x'_i = \begin{cases} x_k^{-1}\left(\prod_{j=1}^n x_j^{[-b_{jk}]_+}\right)^{r_k} \frac{\sum_{s=0}^{r_k} a_{k,s}\widehat{y}_k^s}{\bigoplus_{s=0}^{r_k} a_{k,s} y_k^s} & i = k \\ x_i & i \neq k \end{cases}$$

$$a_{k,s} = a_{k,r_k-s}$$

where $[\cdot]_+ = max(\cdot, 0)$ and $\widehat{y}_i := y_i \prod_{j=1}^n x_j^{b_{ji}}$.

Remark 2.3. *The mutation relation for b'_{ij} given in Definition 2.2 is for the generalized exchange matrix, B. This is equivalent to writing that the matrix BR mutates according to the relation*

$$(br)'_{ij} = (br)_{ij} + ([-(br)_{ik}]_+ (br)_{kj} + (br)_{ik}[(br)_{kj}]_+).$$

This reflects the fact that $\mu_k(BR) = \mu_k^{(r)}(B)R$, where μ_k denotes ordinary matrix mutation.

A generalized cluster algebra is then defined as:

Definition 2.4. *The* generalized cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B, R, \mathbf{a})$ *associated to a generalized seed Σ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the cluster variables $\mathbf{x} = \{x_i\}_{i\in[n]}$ of Σ via the exchange relations given in Definition 2.2.*

Chekhov and Shapiro proved that generalized cluster algebras exhibit the Laurent phenomenon [2]. They further proved that positivity holds for the subclass of generalized cluster algebras from orbifolds and conjectured that it holds in all cases.

3 Generalized cluster scattering diagrams

We begin by establishing the notions of *generalized fixed data* and *generalized torus seed data* and defining an associated map.

Definition 3.1. *The following data is referred to as* generalized fixed data, *denoted by* Γ*:*

- *A lattice* N *called the* co-character lattice *with skew-symmetric bilinear form*

$$\{\cdot,\cdot\} : N \times N \to \mathbb{Q}.$$

- *A saturated sublattice* $N_{uf} \subseteq N$ *called the* unfrozen sublattice.
- *An index set* I *with* $|I| = rank(N)$ *and subset* $I_{uf} \subseteq I$ *such that* $|I_{unf}| = rank(N_{uf})$
- *A set of positive integers* $\{d_i\}_{i\in I}$ *such that* $gcd(d_i) = 1$.
- *A sublattice* $N^\circ \subseteq N$ *of finite index such that* $\{N_{uf}, N^\circ\} \subseteq \mathbb{Z}$ *and* $\{N, N_{uf} \cap N^\circ\} \subseteq \mathbb{Z}$.
- *A lattice* $M = Hom(N,\mathbb{Z})$ *called the* character lattice *and sublattice* $M^\circ = Hom(N^\circ,\mathbb{Z})$.
- *A set of positive integers* $\{r_i\}_{i\in I}$.

The adjective 'fixed' refers to the fact that this data is fixed under mutation.

To construct cluster scattering diagrams, we must assume that the map $p_1^* : N_{\mathrm{uf}} \to M^\circ$ given by $n \mapsto \{n,\cdot\}$ is injective. This is not true for all choices of fixed data, but is true in the principal coefficient case. Because arbitrary generalized cluster algebras can be obtained as specializations of the principal coefficient case, this is sufficient.

Definition 3.2. *Given a set of generalized fixed data, we can define* generalized torus seed data $\mathbf{s} = \{(e_i, \mathbf{a}_i)\}_{i\in I}$ *such that* $\{e_i\}_{i\in I}$ *is a basis for* N*,* $\{e_i\}_{i\in I_{uf}}$ *is a basis for* N_{uf}*,* $\{d_ie_i\}_{i\in I}$ *is a basis for* N°*, and each* $\mathbf{a}_i = (a_{i,s})$ *is a tuple of scalars of length* r_i *with* $a_{i,1} = a_{i,r_i} = 1$*. The torus seed data defines a new bilinear form* $[\cdot,\cdot]_{\mathbf{s}} : N \times N \to \mathbb{Q}$ *given by* $[e_i, e_j]_{\mathbf{s}} = \epsilon_{ij} = \{e_i, e_j\}d_j$*. This bilinear form is essentially the exchange matrix of [4] and is skew-symmetrizable.*

A choice of a generalized torus seed $\mathbf{s} = \{(e_i, \mathbf{a}_i)\}_{i\in I}$ defines a dual basis $\{e_i^*\}_{i\in I}$ for M and a basis $\{f_i = d_i^{-1}e_i^*\}_{i\in I}$ for M°. A generalized torus seed $\mathbf{s}$ is called *reciprocal* if its scalar tuples $\mathbf{a}_i = (a_{i,s})$ satisfy the reciprocity condition $a_{i,s} = a_{i,r_i-s}$. We refer to the associated algebra as a *reciprocal* generalized cluster algebra. Here, we confine our attention to this subclass of generalized cluster algebras.

Example 3.3. *Consider the generalized cluster algebra*

$$\mathcal{A}\left(\mathbf{x},\mathbf{y},\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix},((1,a,a,1),(1,1))\right).$$

In the language of Definition 3.1, this corresponds to the generalized fixed data Γ *with* $d = (1,1)$*,* $r = (3,1)$*,* $I = I_{uf} = \{1,2\}$*,* $N = N^\circ = \langle(1,0),(0,1)\rangle$*,* $M = M^\circ = \langle(1,0),(0,1)\rangle$*, and skew-symmetric bilinear form specified by the exchange matrix. The generalized torus seed data is* $\mathbf{s} = \{((1,0),(1,a,a,1)),((0,1),(1,1))\}$.

Just as in Section 2, we can define generalized mutation in direction k.

Definition 3.4. *Given generalized torus seed data* $\mathbf{s}$ *and some* $k \in I_{uf}$*, a* mutation in direction k *of the generalized torus seed data is defined by the following transformations of basis vectors and exchange polynomial coefficients:*

$$e'_i := \begin{cases} e_i + r_k[\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$

$$f'_i := \begin{cases} -f_k + r_k \sum_{j \in I_{uf}} [-\epsilon_{kj}]_+ f_j & i = k \\ f_i & i \neq k \end{cases}$$

$$a'_{k,s} := a_{k,r_k - s}$$

The basis mutation induces the following mutation of the matrix $[\epsilon_{ij}]$*:*

$$\epsilon'_{ij} := \{e'_i, e'_j\} d_j = \begin{cases} -\epsilon_{ij} & k = i \text{ or } k = j \\ \epsilon_{ij} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \leq 0 \\ \epsilon_{ij} + r_k |\epsilon_{ik}| \epsilon_{kj} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \geq 0 \end{cases}$$

Example 3.5. *Mutating the generalized torus seed from Example 3.3 once in either direction yields*

$$\mu_1(\mathbf{s}) = \{((-1,0),(1,a,a,1)),((0,1),(1,1))\}$$
$$\mu_2(\mathbf{s}) = \{((1,1),(1,a,a,1)),((0,-1),(1,1))\}.$$

Given a generalized torus seed $\mathbf{s}$, we can then define *generalized cluster varieties*. Let $\mathfrak{T}$ be an infinite $|I_{\text{uf}}|$-regular oriented tree rooted at vertex v whose edges are labeled by elements of I_{uf}. If we attach a choice of $\mathbf{s}$ to v, then paths within $\mathfrak{T}$ correspond to sequences of mutations in the directions specified by the edge labels. We write $\mathfrak{T}_{\mathbf{s}}$ to record the choice of seed. For each vertex v, we can associate the tori $\mathcal{X}_{\mathbf{s}} = T_M = \text{Spec } \Bbbk[N]$ and $\mathcal{A}_{\mathbf{s}} = T_{N^\circ} = \text{Spec } \Bbbk[M^\circ]$.

The cluster variables of Definition 2.1 appear in this setting as $x_i = z^{f_i}$ and $y_i = z^{e_i}$. We can then define birational maps between these tori which encode the exchange relations of Definition 2.2.

Definition 3.6. *For* $n \in N$ *and* $m \in M^\circ$*, we define birational maps* $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$ *and* $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ *via the pull-back of functions*

$$\mu_k^* z^n = z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[n,e_k]}$$
$$\mu_k^* z^m = z^m \left(1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{-\langle d_k e_k, m\rangle}$$

Here the bilinear form $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$ *naturally defines a map* $p_1^* : N_{uf} \to M^\circ$*, and we use* $v_k = p_1^*(e_k)$ *in the above.*

By [8, Proposition 2.4], we can glue along the open piece of $\mathcal{A}_{\mathbf{s}}$, where μ is defined, to obtain the scheme $\mathcal{A}$. We can similarly obtain the scheme $\mathcal{X}$.

Definition 3.7. *The $\mathcal{A}$ (resp. $\mathcal{X}$) generalized cluster varieties are schemes which are isomorphic up to codimension 2 to the schemes $\mathcal{A} := \bigcup_{w\in\mathfrak{T}} T_{N^\circ,\mathbf{s}_w}$ and $\mathcal{X} := \bigcup_{w\in\mathfrak{T}} T_{M,\mathbf{s}_w}$ where the tori are glued according to the birational functions maps from Definition 3.6.*

3.1 Generalized Cluster Scattering Diagrams

We begin by establishing the setting for our construction. Let $\Bbbk$ be a field of characteristic zero and $\sigma \subseteq M_{\mathbb{R}}$ a convex top-dimensional cone which contains $\{p_1^*(e_i)\}_{i\in I_{\mathrm{uf}}}$. Then $\Bbbk[P]$ and $\widehat{\Bbbk[P]}$ denote, respectively, the rings of polynomials and formal power series in the monoid $P := \sigma \cap M^\circ$. Choose a linear function $d : N \to \mathbb{Z}$ such that $d(n) > 0$ for $n \in N^+$.

Definition 3.8 (Definition 1.2 of [9])**.** *For $n_0 \in N^+$, let $m_0 := p_1^*(n_0)$ and $f = 1 + \sum_{k=1}^{\infty} c_k z^{km_0} \in \widehat{\Bbbk[P]}$. Then $\mathfrak{p}_f \in \widehat{\Bbbk[P]}$ denotes the automorphism given by $\mathfrak{p}_f(z^m) = z^m f^{\langle n_0', m\rangle}$ where n_0' generates the monoid $\mathbb{R}_{\geq 0} n_0 \cap N^\circ$.*

Definition 3.9 (Definition 1.4 of [9])**.** *A* wall *in $M_{\mathbb{R}}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}}) \in (N^+, \widehat{\Bbbk[P]})$ such that for some primitive $n_0 \in N^+$,*

1. *$f_{\mathfrak{d}} \in \widehat{\Bbbk[P]}$ has the form $1 + \sum_{j=1}^{\infty} c_j z^{j p_1^*(n_0)}$ with $c_j \in \Bbbk$.*
2. *$\mathfrak{d} \subset n_0^{\perp} \subset M_{\mathbb{R}}$ is a convex rational polyhedral cone with dimension* rank $M - 1$.

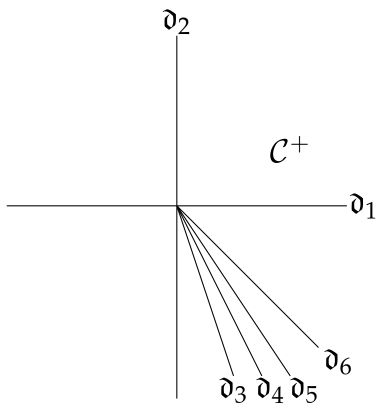
We refer to $\mathfrak{d} \subset M_{\mathbb{R}}$ as the support *of the wall $(\mathfrak{d}, f_{\mathfrak{d}})$.*

We can then define scattering diagrams, an example of which appears in Figure 1.

Definition 3.10 (Definition 1.6 of [9])**.** *A* scattering diagram *$\mathfrak{D}$ for N^+ is a set of walls $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$ such that for every degree $k > 0$, there are a finite number of walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ with $f_{\mathfrak{d}} \neq 1 \mod \mathfrak{m}^{k+1}$, where $\mathfrak{m}$ is the maximal monomial ideal generated by $\{z^m : m \in P\backslash\{0\}\}$.*

Let $\gamma : [0,1] \to M_{\mathbb{R}}\backslash \mathrm{Sing}(\mathfrak{D})$ be a smooth immersion which crosses walls transversely and whose endpoints aren't in the support of $\mathfrak{D}$. Let $0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$ be a sequence such that at time t_i the path γ crosses the wall $\mathfrak{d}_i$ such that $f_i \neq 1 \mod \mathfrak{m}^{k+1}$. Definition 3.10 ensures that this is a finite sequence. Set $\epsilon_i := -\mathrm{sgn}(\langle n_i, \gamma'(t_i)\rangle)$ where $n_i \in N^+$ is the primitive vector normal to $\mathfrak{d}_i$. For each degree $k > 0$, define $\mathfrak{p}^k_{\gamma,\mathfrak{D}} := \mathfrak{p}^{\epsilon_s}_{f_{\mathfrak{d}_{t_s}}} \circ \cdots \circ \mathfrak{p}^{\epsilon_1}_{f_{\mathfrak{d}_{t_1}}}$, where $\mathfrak{p}_{f_{\mathfrak{d}_{t_i}}}$ is defined as in Definition 3.8. Then let $\mathfrak{p}_{\gamma,\mathfrak{D}} := \lim_{k\to\infty} \mathfrak{p}^k_{\gamma,\mathfrak{D}}$. We refer to $\mathfrak{p}_{\gamma,\mathfrak{D}}$ as a *path-ordered product*.

Definition 3.11 (Definition 1.8 of [9])**.** *Let $\mathfrak{D}$, $\mathfrak{D}'$ be scattering diagrams. They are* equivalent *if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for all paths γ for which both path-ordered products are defined.*



$$
\begin{aligned}
f_{\mathfrak{d}_1} &= 1 + z^{(-1,0)} \\
f_{\mathfrak{d}_2} &= 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)} \\
f_{\mathfrak{d}_3} &= 1 + z^{(-1,3)} \\
f_{\mathfrak{d}_4} &= 1 + az^{(-1,2)} + az^{(-2,4)} + z^{(-3,6)} \\
f_{\mathfrak{d}_5} &= 1 + z^{(-2,3)} \\
f_{\mathfrak{d}_6} &= 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)}
\end{aligned}
$$

Figure 1: A cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$ for the algebra from Example 3.3.

A scattering diagram $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_{\gamma,\mathfrak{D}}$ depends only on the endpoints of γ. Let $v_i = p_1^*(e_i)$ for $i \in I_{\mathrm{uf}}$. Then the *initial scattering diagram* for generalized torus seed $\mathbf{s}$ is

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} := \{(e_i^{\perp}, 1 + a_{i,1}z^{v_i} + a_{i,2}z^{2v_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i} + z^{r_i v_i}) : i \in I_{\mathrm{uf}}\},$$

and we have the following:

Proposition 3.12 (cf. [10, 11])**.** *Given a generalized torus seed* $\mathbf{s}$*, there is a consistent scattering diagram* $\mathfrak{D}_{\mathbf{s}}$ *which contains* $\mathfrak{D}_{in,\mathbf{s}}$ *such that* $\mathfrak{D}_{\mathbf{s}} \backslash \mathfrak{D}_{in,\mathbf{s}}$ *consists only of walls* $\mathfrak{d} \subset n_0^{\perp}$ *with* $p_1^*(n_0) \notin \mathfrak{d}$ *for some primitive* $n_0 \in N^+$*. The scattering diagram* $\mathfrak{D}_{\mathbf{s}}$ *is unique up to equivalence.*

3.2 Principal Coefficients

As with ordinary cluster algebras, we can obtain a generalized cluster algebra with *principal coefficients* by extending the matrix $[\{e_i, e_j\}]$ to a $2n \times 2n$ skew-symmetric matrix with the $n \times n$ identity matrix as its upper right block and an $n \times n$ zero matrix as its lower right block. This requires the following modifications to our construction:

Definition 3.13. *Given generalized fixed data* Γ*, the generalized fixed data for the cluster variety with principal coefficients,* Γ_{prin}*, is defined by:*

- *The* double *of the lattice* N*,* $\widetilde{N} := N \oplus M^{\circ}$*, with skew-symmetric bilinear form given by*

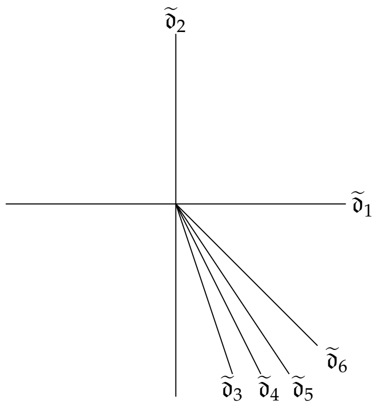
$$\{(n_1, m_1), (n_2, m_2)\} = \{n_1, n_2\} + \langle n_1, m_2 \rangle - \langle n_2, m_1 \rangle.$$

- *The unfrozen sublattice* $\widetilde{N}_{uf} := N_{uf} \oplus 0 \cong N_{uf}$*.*
- *The sublattice* $\widetilde{N}^{\circ} := N^{\circ} \oplus M$*.*
- *The index set* $\widetilde{I}$ *is given by the disjoint union of two copies of* I*.*
- *The unfrozen index set,* $\widetilde{I}_{uf}$*, is the original* I_{uf} *thought of as a subset of the first copy of* I*.*

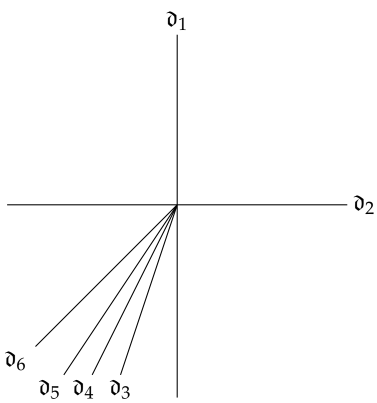
- *The integer collections* $\widetilde{d} = (d_i)_{i \in \widetilde{I}}$ *and* $\widetilde{r} = (r_i)_{i \in \widetilde{I}}$ *taken such that within each disjoint copy of* I*, the* d_i *and* r_i *agree with the original generalized torus seed* $\mathbf{s}$*.*
- *The character lattice* $\widetilde{M} = Hom(\widetilde{N}, \mathbb{Z}) = M \oplus N^\circ$ *with sublattice* $\widetilde{M}^\circ = M^\circ \oplus N$*.*

Given a torus seed $\mathbf{s}$, the torus seed with principal coefficients $\mathbf{s}_{\text{prin}}$ is defined as $\mathbf{s}_{\text{prin}} := \{((e_i, 0), \mathbf{a}_i), ((0, f_i), \mathbf{a}_i)\}_{i \in I}$. The choice of $\mathbf{s}_{\text{prin}}$ defines dual bases for $\widetilde{M}$ and $\widetilde{M}^\circ$. Let $\widetilde{v}_i := (v_i, e_i) = (p_1^*(e_i), e_i)$. Then we can define

$$\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\text{in},\mathbf{s}} = \left\{ \left((e_i, 0)^\perp, 1 + a_{i,1} z^{\widetilde{v}_1} + \cdots + a_{i,r_i-1} z^{(r_i-1)\widetilde{v}_i} + z^{r_i \widetilde{v}_i} \right) \right\}.$$



$$f_{\widetilde{\mathfrak{d}}_1} = 1 + z^{(-1,0,0,1)}$$
$$f_{\widetilde{\mathfrak{d}}_2} = 1 + az^{(0,1,1,0)} + az^{(0,2,2,0)} + z^{(0,3,3,0)}$$
$$f_{\widetilde{\mathfrak{d}}_3} = 1 + z^{(-1,3,3,1)}$$
$$f_{\widetilde{\mathfrak{d}}_4} = 1 + az^{(-1,2,2,1)} + az^{(-2,4,4,2)} + z^{(-3,6,6,3)}$$
$$f_{\widetilde{\mathfrak{d}}_5} = 1 + z^{(-2,3,3,2)}$$
$$f_{\widetilde{\mathfrak{d}}_6} = 1 + az^{(-1,1,1,1)} + az^{(-2,2,2,2)} + z^{(-3,3,3,3)}$$



$$f_{\mathfrak{d}_1} = 1 + z^{(0,1)}$$
$$f_{\mathfrak{d}_2} = 1 + az^{(1,0)} + az^{(2,0)} + z^{(3,0)}$$
$$f_{\mathfrak{d}_3} = 1 + z^{(3,1)}$$
$$f_{\mathfrak{d}_4} = 1 + az^{(2,1)} + az^{(4,2)} + z^{(6,3)}$$
$$f_{\mathfrak{d}_5} = 1 + z^{(3,2)}$$
$$f_{\mathfrak{d}_6} = 1 + az^{(1,1)} + az^{(2,2)} + z^{(3,3)}$$

Figure 2: The $\mathcal{A}_{\text{prin}}$ (top) and $\mathcal{X}$ (bottom) scattering diagrams for the generalized cluster algebra $\mathcal{A}\left(\mathbf{x}, \mathbf{y}, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right], ((1, a, a, 1), (1, 1))\right)$. The $\mathcal{X}$-diagram is obtained from the $\mathcal{A}_{\text{prin}}$-diagram via the slice $\{m \in M^\circ : m = p^*(n)\}$, with $\widetilde{\mathfrak{d}}_i$ in the $\mathcal{A}_{\text{prin}}$-diagram yielding the wall $\mathfrak{d}_i$ in the $\mathcal{X}$-diagram. Note that in this example, the diagram for $\mathcal{A}_{\text{prin}}$ is actually four dimensional and it is drawn here via projection onto M°.

A choice of generalized torus seed with principal coefficients defines tori $\mathcal{X}_{\mathbf{s}_{\text{prin}}} := T_{\widetilde{M}} = \text{Spec } \Bbbk[\widetilde{N}]$ and $\mathcal{A}_{\mathbf{s}_{\text{prin}}} := T_{\widetilde{N}^\circ} = \text{Spec } \Bbbk[\widetilde{M}^\circ]$. The varieties $\mathcal{A}_{\text{prin}}$ and $\mathcal{X}_{\text{prin}}$ are then obtained by gluing according to the birational mutation maps, as before. By Theorem 3.12, there exists a unique consistent scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\text{prin}}}$ corresponding to $\mathcal{A}_{\mathbf{s}_{\text{prin}}}$. The varieties $\mathcal{A}_{\mathbf{s}}$ and $\mathcal{X}_{\mathbf{s}}$ are given, respectively, by the fiber $\mathcal{A}_e$, where $e \in T_M$ is the identity element, and the quotient $\mathcal{A}_{\text{prin}}/T_{N^\circ}$. Figure 2 illustrates an example of a consistent scattering diagram for $\mathcal{A}_{\text{prin}}$ and the corresponding $\mathcal{X}$-diagram obtained by slicing.

3.3 Mutation invariance

In Section 3, we gave both a construction for $\mathfrak{D}_{\mathbf{s}}$ and relations for obtaining a mutated seed $\mu_k(\mathbf{s})$. If applied to $\mu_k(\mathbf{s})$, our construction would yield another diagram $\mathfrak{D}_{\mu_k(\mathbf{s})}$. Because $\mathbf{s}$ and $\mu_k(\mathbf{s})$ correspond to the same generalized cluster algebra, we should expect $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mu_k(\mathbf{s})}$ to be equivalent. To see this, let $\mathcal{H}_{k,+} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \geq 0\}$ and $\mathcal{H}_{k,-} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \leq 0\}$. Then define the piecewise linear transformation $T_k : M^\circ \to M^\circ$ as

$$T_k(m) := \begin{cases} m + r_k v_k \langle d_k e_k, m\rangle & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases}$$

The shorthand $T_{k,-}$ and $T_{k,+}$ is sometimes used for T_k in the regions $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$.

Definition 3.14. *The scattering diagram $T_k(\mathfrak{D}_{\mathbf{s}})$ is obtained from $\mathfrak{D}_{\mathbf{s}}$ via the following procedure:*

1. *For each wall $(\mathfrak{d}, f_{\mathfrak{d}})$ in $\mathfrak{D}_{\mathbf{s}}$ other than $\mathfrak{d}_k := (e_k^\perp, 1 + a_1 z^{v_k} + \cdots + a_{j-1} z^{(j-1)v_k} + z^{jv_k})$, there are either one or two corresponding walls in $T_k(\mathfrak{D}_{\mathbf{s}})$. If $dim(\mathfrak{d} \cap \mathcal{H}_{k,-}) \geq rank(M) - 1$, then add to $T_k(\mathfrak{D}_s)$ the wall $(T_k(\mathfrak{d} \cap \mathcal{H}_{k,-}), T_{k,-}(f_{\mathfrak{d}}))$ where the notation $T_{k,\pm}(f_{\mathfrak{d}})$ indicates the formal power series obtained by applying $T_{k,\pm}$ to the exponent of each term of $f_{\mathfrak{d}}$. If $dim(\mathfrak{d} \cap \mathcal{H}_{k,+}) \geq rank(M) - 1$, add the wall $(T_k(\mathfrak{d} \cap \mathcal{H}_{k,+}), T_{k,+}(f_{\mathfrak{d}}))$.*

2. *The wall $\mathfrak{d}_k$ in $\mathfrak{D}_{\mathbf{s}}$ becomes the wall $\mathfrak{d}'_k = (e_k^\perp, 1 + a_1 z^{-v_k} + \cdots + a_{r_k - 1} z^{-(r_k-1)v_k} + z^{-r_k v_k})$*

It is important to note that T_k is an involution up to equivalence of diagrams.

Theorem 3.15 (Analogue of Theorem 1.24 of [9])**.** *If the injectivity assumption holds, then $T_k(\mathfrak{D}_{\mathbf{s}})$ is a consistent scattering diagram for $N^+_{\mu_k(\mathbf{s})}$. Moreover, $\mathfrak{D}_{\mu_k(\mathbf{s})}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent.*

Hence, we conclude that the definition of a scattering diagram is mutation invariant. That is, even though two distinct seeds $\mathbf{s}$ and $\mathbf{s}'$ for the same generalized cluster algebra will yield distinct scattering diagrams $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$ (see Figure 3), the diagrams are "mutable" to each other via some sequence of applications of T_k. Note that that the exchange polynomial coefficients are reciprocal, i.e. $a_{i,s} = a_{i,r_i - s}$, are necessary for the theorem.

The *positive chamber* of $\mathfrak{D}_{\mathbf{s}}$ is $\mathcal{C}^+_{\mathbf{s}} := \{m \in M_{\mathbb{R}} | \langle e_i, m\rangle \geq 0 \text{ for } i \in I_{\text{uf}}\}$. For seeds $\mathbf{s}'$ reachable from $\mathbf{s}$ via a finite sequence of mutations, let $\mathcal{C}^+_v$ denote the chamber of $\mathfrak{D}_{\mathbf{s}}$ corresponding to the positive chamber of $\mathfrak{D}_{\mathbf{s}'}$ and $\Delta^+_{\mathbf{s}}$ denote the set of chambers $\mathcal{C}^+_v$ as v runs across all vertices of $\mathfrak{T}_{\mathbf{s}}$. We refer to $\Delta^+_{\mathbf{s}}$ as the *cluster complex* and to its elements as *cluster chambers*. This identification of the chamber structure of $\mathfrak{D}_{\mathbf{s}}$ and the cluster complex follows from the fact that T_k is the Fock–Goncharov tropicalization of the mutation map μ_k on $\mathcal{A}^\vee$.

4 Theta basis for reciprocal generalized cluster algebras

To define theta functions, we will make use of *broken line* as in the ordinary setting:

Definition 4.1 (Definition 3.1 of [9]). *Let $\mathfrak{D}$ be a scattering diagram, m_0 be a point in $M^\circ \backslash \{0\}$, and Q be a point in $M_{\mathbb{R}} \backslash Supp(\mathfrak{D})$. A* broken line *with endpoint Q and initial slope m_0 is a piecewise linear path $\gamma : (-\infty, 0] \to M_{\mathbb{R}} \backslash Sing(\mathfrak{D})$ with finitely many domains of linearity. Each domain of linearity, L, has an associated monomial $c_L z^{m_L} \in k[M^\circ]$ such that:*

1. *$\gamma(0) = Q$.*
2. *If L is the first domain of linearity of γ, then $c_L z^{m_L} = z^{m_0}$.*
3. *Within the domain of linearity L, the broken line has slope $-m_L$, i.e. $\gamma'(t) = -m_L$.*
4. *If t is a point at which γ is non-linear and is passing from one domain of linearity L to another domain L' and $\mathfrak{D}_t := \{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} : \gamma(t) \in \mathfrak{d}.\}$, then the power series $\mathfrak{p}_{\gamma|_{(t-\epsilon, t+\epsilon)}, \mathfrak{D}_t}$ contains the term $c_{L'} z^{m_{L'}}$.*

One example of a broken line is shown in Figure 3.

Definition 4.2 (Definition 3.3 of [9]). *Suppose $\mathfrak{D}$ is a scattering diagram and consider points $m_0 \in M^\circ \backslash \{0\}$ and $Q \in M_{\mathbb{R}} \backslash Supp(\mathfrak{D})$. For a broken line γ with initial exponent m_0 and endpoint Q, we define $I(\gamma) = m_0$, $b(\gamma) = Q$, and $Mono(\gamma) = c(\gamma) z^{F(\gamma)}$ where $Mono(\gamma)$ is the monomial attached to the final domain of linearity of γ. We then define $\vartheta_{Q,m_0} := \sum_\gamma Mono(\gamma)$ where the summation ranges over all broken lines γ with initial exponent m_0 and endpoint Q. When $m_0 = 0$, we define $\vartheta_{Q,0} = 1$ for any endpoint Q.*

We can then observe how mutation acts on broken lines:

Proposition 4.3 (Analog of Proposition 3.6 of [9]). *The transformation T_k gives a bijection between broken lines with endpoint Q and initial slope m_0 in $\mathfrak{D}_{\mathbf{s}}$ and broken lines with endpoint $T_k(Q)$ and initial slope $T_k(m_0)$ in $\mathfrak{D}_{\mu_k(\mathbf{s})}$. In particular, $\vartheta^{\mu_k(\mathbf{s})}_{T_k(Q), T_k(m_0)} = T_{k,\pm}\left(\vartheta^{\mathbf{s}}_{Q,m_0}\right)$ for $Q \in \mathcal{H}_{k,\pm}$ where $T_{k,\pm}$ acts linearly on the exponents in $\vartheta^{\mathbf{s}}_{Q,m_0}$.*

This observation tells us theta functions is defined independent of the choice of the seed $\mathbf{s}$. The last important step is to prove that cluster monomials can be expressed as theta functions. To do so, we construct the scheme $\mathcal{A}_{\text{scat}}$ by associating tori to each chamber of the scattering diagram and gluing along the birational mutation maps induced by path-ordered products. Then we show that $\mathcal{A}_{\text{scat}}$ is isomorphic to the generalized cluster variety $\mathcal{A}$. The structure of our argument is the same as in the ordinary case (see Section

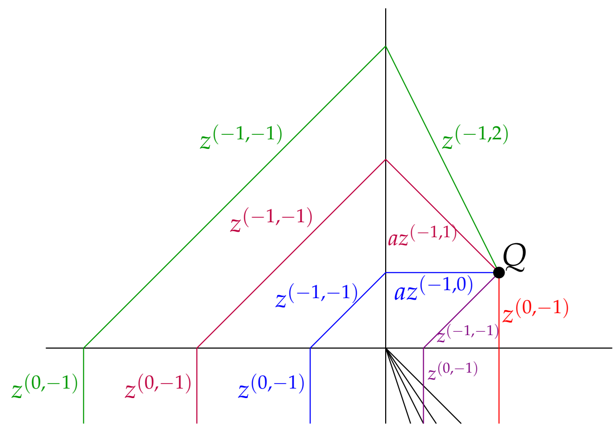


Figure 3: The broken line for $\vartheta_{(0,-1)}$ in $\mathfrak{D}_{\mathbf{s}}$ for the generalized cluster algebra from Example 3.3 with generalized torus seed $\mathbf{s} = ((1,0),(1,a,a,1),((0,1),(1,1))$.

4 of [9]) with some modifications to accommodate the differences in our wall-crossing automorphisms.

For any $m_0 \in M\backslash\{0\}$ such that there exists a path γ from m_0 to some point Q in the positive chamber $\mathcal{C}^+$ which passes through finitely many chambers, the theta function ϑ_{Q,m_0} can also be obtained via the path-ordered product $\mathfrak{p}_\gamma(z^{m_0})$. Within the cluster complex, the definitions of ϑ_{Q,m_0} by broken lines and by path-ordered products agree. Hence, the theta functions are regular functions on $\mathcal{A}_{\text{scat}}$. Because $\mathcal{A}_{\text{scat}} \simeq \mathcal{A}$ and the ring of regular functions on $\mathcal{A}$ is the generalized cluster algebra, we have

Theorem 4.4 (Analogue of Theorem 4.9 of [9])**.** *The generalized cluster monomials can be expressed in terms of theta functions.*

To demonstrate that the theta functions form a basis, we then define structure constants for multiplication and give a multiplication rule for theta functions:

Lemma 4.5 (Analogue of Definition-Lemma 6.2 and Proposition 6.3 in [9])**.** *Let p_1, p_2, and q be points in $\widetilde{M}^\circ_{\mathbf{s}}$ and z be a generic point in $\widetilde{M}^\circ_{\mathbb{R},\mathbf{s}}$. There are at most finitely many pairs of broken lines γ_1, γ_2 such that γ_i has initial slope p_i, both broken lines have endpoint z, and $F(\gamma_1) + F(\gamma_2) = q$. Let $a_z(p_1, p_2, q) := \sum_{(\gamma_1,\gamma_2)} c(\gamma_1)c(\gamma_2)$ for pairs γ_1, γ_2 such that $I(\gamma_i) = p_i$, $b(\gamma_i) = z$, and $F(\gamma_1) + F(\gamma_2) = q$. Then*

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{q \in \widetilde{M}^\circ_{\mathbf{s}}} \alpha_{z(q)}(p_1, p_2, q)\vartheta_q$$

for $z(q)$ sufficiently close to q. When z is sufficiently close to q, $a_z(p_1, p_2, q)$ is independent of the choice of z and we can simply write $\alpha(p_1, p_2, q) := a_z(p_1, p_2, q)$.

For a generic point Q in some chamber, let Θ be the collection of $m_0 \in M^\circ$ such that there are finitely many broken lines with initial slope $-m_0$ and endpoint Q. Then,

Theorem 4.6. *For fixed data Γ and torus seed $\mathbf{s} = \{(e_i, (a_{i,s})\}_{i \in I}$ with all $a_{i,s} \geq 0$, the collection $\{\vartheta_m\}_{m \in \Theta}$ forms a basis for the associated middle reciprocal generalized cluster algebra.*

When the upper generalized cluster algebra and generalized cluster algebra agree, Theorem 4.6 gives a basis for the generalized cluster algebra.

References

[1] P. Caldero and B. Keller. "From triangulated categories to cluster algebras". *Inventiones mathematicae* **172** (2008).

[2] L. Chekhov and M. Shapiro. "Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables". *International Mathematics Research Notices* **2014** (2013).

[3] V. Fock and A. Goncharov. "Cluster ensembles, quantization, and the dilogarithm". *Annales scientifiques de l'École Normale Supérieure* **42** (2009).

[4] S. Fomin and A. Zelevinsky. "Cluster Algebras I: Foundations". *Journal of the American Mathematical Society* **15**.2 (2002).

[5] S. Fomin and A. Zelevinsky. "Cluster Algebras IV: Coefficients". *Compositio Mathematica* **143**.1 (2007), pp. 112–164.

[6] C. Geiß, B. Leclerc, and J. Schröer. "Kac-Moody groups and cluster algebras". *Advances in Mathematics* **228** (2011).

[7] C. Geiß, B. Leclerc, and J. Schröer. "Generic bases for cluster algebras and the chamber ansatz". *Journal of the American Mathematical Society* 1 (2012).

[8] M. Gross, P. Hacking, and S. Keel. "Birational geometry of cluster algebras". *Algebraic Geometry* **2** (2015).

[9] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. "Canonical bases for cluster algebras". *Journal of the American Mathematical Society* **31** (2018).

[10] M. Gross and B. Siebert. "From affine geometry to complex geometry". *Annals of Mathematics* **174** (2011).

[11] M. Kontsevich and Y. Soibelman. "Affine Structures and Non-Archimedean Analytic Spaces". *Progress in Mathematics* (2006).

[12] M. Kontsevich and Y. Soibelman. Ed. by P. Etingof, V. Retakh, and I. Singer. Progress in Mathematics.

[13] L. Mou. "Wall-Crossing Structures in Cluster Algebras". PhD thesis. University of California - Davis, 2020.

[14] G. Musiker, L. Williams, and R. Schiffler. "Bases for cluster algebras from surfaces". *Compositio Mathematica* (2013).

[15] T. Nakanishi. "Structure of seeds in generalized cluster algebras". *Pacific Journal of Mathematics* **1** (2014).