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Generalized Quantum Yang–Baxter Moves and Their Application to Schubert Calculus

Takafumi Kouno*[1], Cristian Lenart†[2], and Satoshi Naito‡[3]

[1]*Department of Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan*

[2]*Department of Mathematics ans Statistics, State University of New York at Albany, NY 12222, U.S.A.*

[3]*Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152-8551, Japan*

Abstract. The quantum alcove model is a uniform combinatorial model in the representation theory of quantum affine algebras and Schubert calculus on flag manifolds. Given a weight λ, the model is based on a sequence of roots called a λ-chain. When λ is dominant, the independence of the model from the chosen λ-chain was shown using certain elementary transformations called quantum Yang–Baxter moves. The purpose of the present work is to generalize the quantum Yang–Baxter moves to an arbitrary weight λ. As an application, we give a combinatorial proof of the Chevalley formula in the equivariant K-group of semi-infinite flag manifolds, first proved by Lenart–Naito–Sagaki.

Résumé. Le modèle des alcôves quantique est un modèle combinatoire uniforme dans la théorie des représentations des algèbres affines quantiques et le calcul de Schubert sur les variétés de drapeaux. Étant donné un poids λ, le modèle est basé sur une séquence de racines appelée λ-chaîne. Lorsque λ est dominant, l'indépendance du modèle de la λ-chaîne choisie a été montrée en utilisant certaines transformations élémentaires appelées mouvements de Yang–Baxter quantiques. Le but de ce travail est de généraliser les mouvements de Yang–Baxter quantiques à un poids arbitraire. Comme application, nous donnons une preuve combinatoire de la formule de Chevalley dans la K-théorie équivariante des variétés de drapeaux semi-infinies, prouvé pour la première fois par Lenart–Naito–Sagaki.

Keywords: quantum Yang–Baxter move, quantum Bruhat graph, quantum alcove model, semi-infinite flag manifold, Chevalley formula

*t.kouno@kurenai.waseda.jp. Research Fellow of Japan Society for the Promotion of Science. Supported in part by JSPS Grant-in-Aid for Scientific Research 20J12058 and 22J00874.

†clenart@albany.edu. Supported in part by the NSF grant DMS-1855592.

‡naito@math.titech.ac.jp. Supported in part by JSPS Grant-in-Aid for Scientific Research (B) 16H03920.

1 Introduction

The quantum alcove model, introduced by Lenart–Lubovsky [7], is a uniform combinatorial model, which appears in many branches of mathematics related to root systems; for example, Schubert calculus of semi-infinite/ordinary flag manifolds, the representation theory of quantum affine algebras, and the theory of Macdonald polynomials.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Weyl group W, weight lattice P, and coroot lattice $Q^\vee$. In the theory of the quantum alcove model, one fixes $w \in W$, $\lambda \in P$, and a certain sequence Γ of roots corresponding to λ, called a *λ-chain*; then, one considers the collection $\mathcal{A}(w, \Gamma)$ of so-called *admissible subsets*. In general, for a fixed $\lambda \in P$, we have several λ-chains, and hence several collections of admissible subsets (depending on them and some $w \in W$). If λ is dominant and the λ-chains are "reduced" (defined later), then these collections are known to be mutually isomorphic (as combinatorial models for certain Kashiwara crystals). More precisely, Lenart–Lubovsky [8] proved that, given reduced λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by a certain deformation procedure, called a *Yang–Baxter transformation*, there exists a bijection $Y\colon \mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$, with e the identity of W; moreover, the bijection preserves some important statistics, including $\mathrm{wt}(\cdot)$ and $\mathrm{height}(\cdot)$. This bijection is called a *quantum Yang–Baxter move*, and the above statement works for any reduced λ-chains Γ_1 and Γ_2 if we use a sequence of such moves. In [8] it is explained that the quantum Yang–Baxter moves provide a root system generalization of the well-known jeu de taquin on semi-standard Young tableaux.

The purpose of the present work is to give a generalization of the quantum Yang–Baxter moves to the case of an arbitrary (not necessarily dominant) weight $\lambda \in P$, and an arbitrary $w \in W$. This proves the independence of the quantum alcove model associated with λ and w from the chosen λ-chain. Here we should mention that our generalized quantum Yang–Baxter moves are no longer bijections, but *sijections*, *i.e.*, "signed bijections"; recall from [2] that for two signed sets S, T, a sijection from S to T is a triple $(\iota_S, \iota_T, \varphi)$ consisting of the maps ι_S, ι_T, and φ, where φ is a sign-preserving bijection between subsets $S_0 \subset S$ and $T_0 \subset T$, and ι_S (resp., ι_T) is a sign-reversing involution on $S \setminus S_0$ (resp., $T \setminus T_0$). By regarding collections of admissible subsets as signed sets equipped with certain sign functions, our generalized quantum Yang–Baxter moves can be viewed as sijections.

We give an application of generalized quantum Yang–Baxter moves to Schubert calculus of semi-infinite flag manifolds. Let G be a connected, simply-connected simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, $H \subset G$ a maximal torus of G, and $(H \subset)\ B \subset G$ a Borel subgroup with unipotent radical N. The *semi-infinite flag manifold* $\mathbf{Q}_G^{\mathrm{rat}}$ is the ind-scheme of infinite type whose set of $\mathbb{C}$-valued points is $\mathbf{Q}_G^{\mathrm{rat}}(\mathbb{C}) = G(\mathbb{C}((z)))/(H(\mathbb{C}) \cdot N(\mathbb{C}((z))))$, where z is an indeterminate (for details, see [3]). For each element x of the affine Weyl group W_{af} associated to $\mathfrak{g}$, $\mathbf{Q}_G^{\mathrm{rat}}$ has the corresponding orbit under the action of the Iwahori subgroup of $G(\mathbb{C}[\![z]\!])$; the closure of this orbit is denoted by $\mathbf{Q}_G(x)$, and called a *semi-infinite Schubert variety*. We set $\mathbf{Q}_G := \mathbf{Q}_G(e)$, and let $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ denote the $(H\times\mathbb{C}^*)$-equivariant K-group of $\mathbf{Q}_G$, introduced in [4]. Let $[\mathcal{O}(\lambda)] \in K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ be the class of the line bundle associated to $\lambda \in P$, and $[\mathcal{O}_x]$ the class of the structure sheaf of the semi-infinite Schubert variety $\mathbf{Q}_G(x)$ corresponding to $x \in W_{\mathrm{af}}$. We set $W_{\mathrm{af}}^{\geq 0} := \{wt_\xi \mid w \in W,\ \xi \in Q^{\vee,+}\}$; here $Q^{\vee,+} := \{\xi \in Q^\vee \mid \xi \geq 0\}$, and t_ξ for $\xi \in Q^\vee$ denotes the translation element. The expansion of the product $[\mathcal{O}(\lambda)]\cdot[\mathcal{O}_x]$ in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ of the following form plays an important role in the study of the structure of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$: for $\lambda \in P$ and $x \in W_{\mathrm{af}}^{\geq 0}$,

$$[\mathcal{O}(\lambda)]\cdot[\mathcal{O}_x] = \sum_{y\in W_{\mathrm{af}}^{\geq 0}} c_{x,\lambda}^{y}[\mathcal{O}_y], \tag{1.1}$$

where $c_{x,\lambda}^{y} \in \mathbb{Z}[P](\!(q^{-1})\!)$. An explicit description (in terms of the quantum alcove model) of the coefficients $c_{x,\lambda}^{y}$ in equation (1.1) is given by the *Chevalley formula*, which is proved by Lenart–Naito–Sagaki [9] based on the Yang–Baxter equation for quantum Bruhat operators. By making use of generalized quantum Yang–Baxter moves, we can give a combinatorial proof of this Chevalley formula.

This paper is an extended abstract of our paper [6], which is a part of the first author's Ph. D. thesis [5].

2 Basic definitions

First, we recall the definition of the quantum Bruhat graph, which was introduced by Brenti–Fomin–Postnikov [1]. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and set $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$. We denote by $\langle\cdot,\cdot\rangle$ the canonical pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$. Let Δ be the root system of $\mathfrak{g}$, Δ^+ the set of all positive roots, and set $\rho := (1/2)\sum_{\alpha\in\Delta^+}\alpha$. For each $\alpha \in \Delta$, we denote by $\alpha^\vee$ the coroot of α, and by $s_\alpha \in W$ the corresponding reflection. Also, we denote by $\ell(\cdot)$ the length function on W.

Definition 1 ([1, Definition 6.1])**.** The *quantum Bruhat graph* $\mathrm{QBG}(W)$ is the Δ^+-labeled directed graph whose vertex set is W, and whose (directed) edges are defined as follows: $x \xrightarrow{\alpha} y$ for $x, y \in W$ and $\alpha \in \Delta^+$ if and only if $y = xs_\alpha$ and one of the following conditions holds:

(B) $\ell(y) = \ell(x) + 1$;

(Q) $\ell(y) = \ell(x) - 2\langle\rho,\alpha^\vee\rangle + 1$.

If condition (B) (resp., (Q)) holds, then the corresponding edge is called a *Bruhat* (resp., *quantum*) *edge*.

Next, following [10] and [9], we briefly review the quantum alcove model. Set $\mathfrak{h}^*_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$. For each $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we define a hyperplane $H_{\alpha,k}$ by

$$H_{\alpha,k} := \{\nu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \nu, \alpha^\vee \rangle = k\}.$$

In this setting, connected components of the space

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta,\ k \in \mathbb{Z}} H_{\alpha,k}$$

are called *alcoves*. In particular, the specific alcove $A_\circ$ defined by

$$A_\circ := \{\nu \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \nu, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}$$

is called the *fundamental alcove*. Also, for each $\lambda \in P$, we define A_λ by

$$A_\lambda := A_\circ + \lambda = \{\nu + \lambda \mid \nu \in A_\circ\}.$$

For two adjacent alcoves A and B and for $\beta \in \Delta$, we write $A \xrightarrow{\beta} B$ if the common wall of A and B is contained in the hyperplane $H_{\beta,k}$ for some $k \in \mathbb{Z}$, and β (viewed as a direction vector) points in the direction from A to B.

Definition 2 ([10, Definitions 5.2, 5.4])**.**

(1) A sequence $(A_0, \ldots, A_r)$ of alcoves is called an *alcove path* if A_{i-1} and A_i are adjacent for all $i = 1, \ldots, r$.

(2) An alcove path $(A_0, \ldots, A_r)$ is called *reduced* if r is minimal among all alcove paths from A_0 to A_r.

(3) Let $\lambda \in P$. A sequence $\Gamma = (\beta_1, \ldots, \beta_r)$ of roots is called a *λ-chain* if there exists an alcove path $(A_0, \ldots, A_r)$ of the form

$$A_\circ = A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda};$$

if this alcove path is reduced, then Γ is said to be *reduced*.

We recall the definition of an admissible subset, which is the main object of study in the theory of the quantum alcove model. For $\beta \in \Delta$, we set

$$\mathrm{sgn}(\beta) := \begin{cases} 1 & \text{if } \beta \in \Delta^+, \\ -1 & \text{if } \beta \in -\Delta^+, \end{cases}$$

and set $|\beta| := \mathrm{sgn}(\beta)\beta$.

Definition 3 ([9, Definition 17], [7, Definition 3.4])**.** Let $\lambda \in P$, and take a λ-chain $(\beta_1, \ldots, \beta_r)$. Let $w \in W$. The subset $A = \{j_1 < \cdots < j_t\} \subset \{1, \ldots, r\}$ is said to be *w-admissible* if

$$\mathbf{p}(A)\colon w = w_0 \xrightarrow{|\beta_{j_1}|} \cdots \xrightarrow{|\beta_{j_t}|} w_t =: \mathrm{end}(A)$$

is a directed path in $\mathrm{QBG}(W)$. We denote by $\mathcal{A}(w, \Gamma)$ the collection of all w-admissible subsets.

Following [9, Equations (12), (13)], we define some important statistics for admissible subsets. Let $\lambda \in P$ and $w \in W$. Let $\Gamma = (\beta_1, \ldots, \beta_r)$ be a λ-chain. For $A = \{j_1 < \cdots < j_r\} \in \mathcal{A}(w, \Gamma)$, we set

$$A^- := \left\{ j \in A \;\middle|\; \text{the edge } w_{j-1} \xrightarrow{|\beta_j|} w_j \text{ in } \mathbf{p}(A) \text{ is a quantum one} \right\},$$

and define $\mathrm{down}(A)$ by

$$\mathrm{down}(A) := \sum_{j \in A^-} |\beta_j|^\vee.$$

Let

$$A_\circ = A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}$$

be the alcove path corresponding to Γ. For $i = 1, \ldots, r$, let $l_i \in \mathbb{Z}$ be such that the common wall of A_{i-1} and A_i is contained in $H_{\beta_i, -l_i}$. For each $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection with respect to $H_{\alpha,k}$. Then $\mathrm{wt}(\cdot)$ and $\mathrm{height}(\cdot)$ are defined as follows:

$$\mathrm{wt}(A) := -w s_{\beta_{j_1}, -l_{j_1}} \cdots s_{\beta_{j_t}, -l_{j_t}}(-\lambda),$$
$$\mathrm{height}(A) := \sum_{j \in A^-} \mathrm{sgn}(\beta_j)(\langle \lambda, \beta_j^\vee \rangle - l_j).$$

Also, for each $A \in \mathcal{A}(w, \Gamma)$, we define $n(A)$ by

$$n(A) := |\{ j \in A \mid \beta_j \in -\Delta^+ \}|.$$

3 Quantum Yang–Baxter moves

In this section, we explain our generalization of quantum Yang–Baxter moves. First, we review quantum Yang–Baxter moves introduced by Lenart–Lubovsky [8]. For this, we need to recall certain deformation procedures for λ-chains. Let $\lambda \in P$, and take a λ-chain $\Gamma = (\beta_1, \ldots, \beta_r)$. The procedures (YB) and (D) are given as follows (see [10, Lemma 9.3], and also [9, Remark 38]):

(YB) Take a segment $(\beta_{t+1}, \dots, \beta_{t+q})$ of Γ such that

- $\langle \beta_{t+1}, \beta_{t+q}^{\vee} \rangle \leq 0$, and
- if we set $\alpha := \beta_{t+1}$ and $\beta := \beta_{t+q}$, then

$$(\beta_{t+1}, \dots, \beta_{t+q}) = (\alpha, s_{\alpha}(\beta), s_{\alpha}s_{\beta}(\alpha), \dots, s_{\beta}(\alpha), \beta).$$

Then, we define a new λ-chain Γ' by

$$\Gamma' := (\beta_1, \dots, \beta_t, \beta_{t+q}, \beta_{t+q-1}, \dots, \beta_{t+1}, \beta_{t+q+1}, \dots, \beta_r);$$

namely, we reverse the segment $(\beta_{t+1}, \dots, \beta_{t+q})$.

(D) Take a segment $(\beta_{t+1}, \beta_{t+2})$ of Γ such that $\beta_{t+2} = -\beta_{t+1}$, and we define a new λ-chain Γ' by

$$\Gamma' := (\beta_1, \dots, \beta_t, \beta_{t+3}, \dots, \beta_r);$$

namely, we delete the segment $(\beta_{t+1}, \beta_{t+2})$.

The procedure (YB) (resp., (D)) is called a *Yang–Baxter transformation* (resp., *deletion*). It is known from [9, Remark 38] (see also [10, Lemma 9.3]) that one obtains an arbitrary reduced λ-chain from any (not necessarily reduced) λ-chain by repeated application of the procedures (YB) and (D).

Lenart–Lubovsky proved the following theorem in the case that λ is dominant.

Theorem 4 ([8, Sections 3.1, 3.2])**.** *Let $\lambda \in P^{+}$, and take reduced λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by (YB). Then, there exists a bijection $Y \colon \mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$ such that for $A \in \mathcal{A}(e, \Gamma_1)$,*

- $\mathrm{end}(Y(A)) = \mathrm{end}(A)$,
- $\mathrm{down}(Y(A)) = \mathrm{down}(A)$,
- $\mathrm{wt}(Y(A)) = \mathrm{wt}(A)$, *and*
- $\mathrm{height}(Y(A)) = \mathrm{height}(A)$.

The bijection Y above is called a *quantum Yang–Baxter move*. In fact, Lenart–Lubovsky proved that quantum Yang–Baxter moves are isomorphisms of crystals (see [8, Theorem 3.8]). By taking composites of quantum Yang–Baxter moves, we see that the collections $\mathcal{A}(e, \Gamma)$ for all reduced λ-chains Γ are mutually isomorphic as crystals (see [8, Corollary 3.9]).

Remark 5. As mentioned in [8, Corollary 3.14], quantum Yang–Baxter moves for dominant weights realize combinatorial R-matrices for tensor products of column-shape Kirillov–Reshetikhin crystals; also, in type A, combinatorial R-matrices can be realized by Schützenberger's jeu de taquin (sliding algorithm) for type A root systems (see [8, Example 3.1]). More precisely, assume that $\mathfrak{g}$ is of type A_{n-1}. For $r \in \{1, \dots, n-1\}$, we denote by $B^{r,1}$ the column-shape Kirillov–Reshetikhin crystal (realized by column-shape semi-standard Young tableaux of height r with entries in $\{1, \dots, n\}$). Let $\mathbf{p} = (p_1, \dots, p_k)$, with $p_1, \dots, p_k \in \{1, \dots, n-1\}$, be a composition, and $\mathbf{p}' = (p'_1, \dots, p'_k)$ a composition obtained from $\mathbf{p}$ by permuting its parts. It is known (see, for example, [8, Remarks 2.2 (2)]) that there exists a unique $A^{(1)}_{n-1}$-crystal isomorphism

$$B^{\otimes \mathbf{p}} := B^{p_1,1} \otimes \cdots \otimes B^{p_k,1} \xrightarrow{\sim} B^{\otimes \mathbf{p}'} := B^{p'_1,1} \otimes \cdots \otimes B^{p'_k,1},$$

called a *combinatorial R-matrix*. Let I be the index set for the simple roots of $\mathfrak{g}$; in our case, $I = \{1, \dots, n-1\}$. Let α_i, $i \in I$, be the simple roots, and ϖ_i, $i \in I$, the fundamental weights. For each $k \in I$, the sequence $\Gamma(k)$ of roots given as

$$\begin{array}{rllll}
\Gamma(k) := (\alpha_k, & \alpha_k + \alpha_{k+1}, & \cdots, & \alpha_k + \cdots + \alpha_{n-1}, \\
\alpha_{k-1} + \alpha_k, & \alpha_{k-1} + \alpha_k + \alpha_{k+1}, & \cdots, & \alpha_{k-1} + \cdots + \alpha_{n-1}, \\
 & & \cdots & \\
\alpha_1 + \cdots + \alpha_k, & \alpha_1 + \cdots + \alpha_{k+1}, & \cdots, & \alpha_1 + \cdots + \alpha_{n-1})
\end{array}$$

is a ϖ_k-chain (see [10, Corollary 15.4]). Let Γ (resp., Γ') be the concatenation $\Gamma(p_1) * \cdots * \Gamma(p_k)$ of $\Gamma(p_1), \dots, \Gamma(p_k)$ (resp., $\Gamma(p'_1) * \cdots * \Gamma(p'_k)$ of $\Gamma(p'_1), \dots, \Gamma(p'_k)$) in this order. Then, there exists a unique $A^{(1)}_{n-1}$-crystal isomorphism between $\mathcal{A}(e, \Gamma)$ (resp., $\mathcal{A}(e, \Gamma')$) and the subgraph of $B^{\otimes \mathbf{p}}$ (resp., $B^{\otimes \mathbf{p}'}$) having only the dual Demazure arrows ([8, Corollary 3.10 and Remark 3.13]). Thus the bijection $\mathcal{A}(e, \Gamma) \xrightarrow{\sim} \mathcal{A}(e, \Gamma')$, given by quantum Yang–Baxter moves, can be viewed as a realization of the combinatorial R-matrix $B^{\otimes \mathbf{p}} \xrightarrow{\sim} B^{\otimes \mathbf{p}'}$. To summarize, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}(e, \Gamma) & \xrightarrow{\text{quantum Yang–Baxter moves}} & \mathcal{A}(e, \Gamma') \\
\simeq \downarrow & & \downarrow \simeq \\
B^{\otimes \mathbf{p}} & \xrightarrow[\text{(jeu de taquin)}]{\text{combinatorial } R\text{-matrix}} & B^{\otimes \mathbf{p}'}
\end{array}$$

The purpose of this work is to generalize quantum Yang–Baxter moves to the case of an arbitrary (not necessarily dominant) $\lambda \in P$. However, for $\lambda \in P$ and $w \in W$, two collections $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$ of admissible subsets for λ-chains Γ_1 and Γ_2 can have different cardinalities; hence we cannot construct any bijection $\mathcal{A}(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$ in general. For this reason, we need the notion of "sijection", *i.e.*, signed bijection, to state a generalization of quantum Yang–Baxter moves.

Theorem 6 ([6, Theorems 3.2, 3.4; Proposition 5.3]). *Let $\lambda \in P$ and $w \in W$. Take λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by applying (YB), or by applying (D) in which a segment $(\beta, -\beta)$ of Γ_1, with β not a simple root, is deleted. Then, there exist explicit subsets $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}_0(w, \Gamma_2) \subset \mathcal{A}(w, \Gamma_2)$ satisfying the following.*

(1) *There exists a bijection $Y \colon \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$ such that for $A \in \mathcal{A}_0(w, \Gamma_1)$,*

- $\mathrm{end}(Y(A)) = \mathrm{end}(A)$,
- $\mathrm{down}(Y(A)) = \mathrm{down}(A)$,
- $\mathrm{wt}(Y(A)) = \mathrm{wt}(A)$, *and*
- $\mathrm{height}(Y(A)) = \mathrm{height}(A)$.

Also, this bijection Y is "sign-preserving"; that is, $(-1)^{n(Y(A))} = (-1)^{n(A)}$ for all $A \in \mathcal{A}(w, \Gamma_k)$.

(2) *For $k = 1, 2$, set $\mathcal{A}_0^C(w, \Gamma_k) := \mathcal{A}(w, \Gamma_k) \setminus \mathcal{A}_0(w, \Gamma_k)$. Then, there exists an involution I_k on $\mathcal{A}_0^C(w, \Gamma_k)$ such that*

- $\mathrm{end}(I_k(A)) = \mathrm{end}(A)$,
- $\mathrm{down}(I_k(A)) = \mathrm{down}(A)$,
- $\mathrm{wt}(I_k(A)) = \mathrm{wt}(A)$, *and*
- $\mathrm{height}(I_k(A)) = \mathrm{height}(A)$.

Also, this involution I_k is "sign-reversing"; that is, $(-1)^{n(I_k(A))} = -(-1)^{n(A)}$ for all $A \in \mathcal{A}_0^C(w, \Gamma_k)$.

Remark 7 ([6, Remark 3.3]). We now recall from Fischer–Konvalinka [2] the notion of sijection. Let S (resp., T) be a signed set equipped with a sign function $S \to \{\pm 1\}$ (resp., $T \to \{\pm 1\}$), and $S_0 \subset S$ (resp., $T_0 \subset T$) a subset of S (resp., T). Then, a *sijection* $S \Rightarrow T$ is a triple $(\iota_S, \iota_T, \varphi)$ consisting of a sign-reversing involution ι_S (resp., ι_T) on $S \setminus S_0$ (resp., $T \setminus T_0$) and a sign-preserving bijection $\varphi \colon S_0 \to T_0$. Roughly speaking, a sijection is a "signed bijection" between signed sets. In this terminology, the maps I_1, I_2, Y in Theorem 6 provide a sijection $(I_1, I_2, Y) \colon \mathcal{A}(w, \Gamma_1) \Rightarrow \mathcal{A}(w, \Gamma_2)$. This sijection can be thought of as a generalization of the quantum Yang–Baxter move $\mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$ in the case of a dominant weight.

Example 8. Assume that $\mathfrak{g}$ is of type A_2; in this example, $I = \{1, 2\}$. Let $\Gamma_1 := (-\alpha_1 - \alpha_2, -\alpha_1, \alpha_2, \alpha_1 + \alpha_2)$, $\Gamma_2 := (\alpha_2, -\alpha_1, -\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$, and $\Gamma_3 := (\alpha_2, -\alpha_1)$. We see that Γ_1, Γ_2, and Γ_3 are $(-\varpi_1 + \varpi_2)$-chains. Also, Γ_2 is obtained from Γ_1 by applying (YB) in which the segment $(-\alpha_1 - \alpha_2, -\alpha_1, \alpha_2)$ in Γ_1 is reversed, while Γ_3 is obtained from Γ_2 by applying (D) in which the segment $(-\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$ in Γ_2 is deleted; note that $\alpha_1 + \alpha_2$

is not a simple root. Below we will describe explicitly generalized quantum Yang–Baxter moves $\mathcal{A}(s_1, \Gamma_1) \Rightarrow \mathcal{A}(s_1, \Gamma_2)$ and $\mathcal{A}(s_1, \Gamma_2) \Rightarrow \mathcal{A}(s_1, \Gamma_3)$.

First of all, we give $A \in \mathcal{A}(s_1, \Gamma_k)$, $k = 1, 2, 3$, together with $\mathrm{end}(A)$ and $\mathrm{down}(A)$, in Tables 1, 2, and 3.

Table 1: List of $A \in \mathcal{A}(s_1, \Gamma_1)$

A	$n(A)$	$\mathrm{end}(A)$	$\mathrm{down}(A)$
$\emptyset$	0	s_1	0
$\{1\}$	1	$s_2 s_1$	0
$\{2\}$	1	e	$\alpha_1^\vee$
$\{3\}$	0	$s_1 s_2$	0
$\{4\}$	0	$s_2 s_1$	0
$\{1,2\}$	2	s_2	$\alpha_1^\vee$
$\{1,3\}$	1	$s_1 s_2 s_1$	0
$\{2,3\}$	1	s_2	$\alpha_1^\vee$
$\{1,2,3\}$	2	e	$\alpha_1^\vee + \alpha_2^\vee$
$\{1,2,4\}$	2	$s_1 s_2$	$\alpha_1^\vee$
$\{1,3,4\}$	1	e	$\alpha_1^\vee + \alpha_2^\vee$
$\{2,3,4\}$	1	$s_1 s_2$	$\alpha_1^\vee$

Table 2: List of $A \in \mathcal{A}(s_1, \Gamma_2)$

A	$n(A)$	$\mathrm{end}(A)$	$\mathrm{down}(A)$
$\emptyset$	0	s_1	0
$\{1\}$	0	$s_1 s_2$	0
$\{2\}$	1	e	$\alpha_1^\vee$
$\{3\}$	1	$s_2 s_1$	0
$\{4\}$	0	$s_2 s_1$	0
$\{1,2\}$	1	$s_1 s_2 s_1$	0
$\{1,2,3\}$	2	e	$\alpha_1^\vee + \alpha_2^\vee$
$\{1,2,4\}$	1	e	$\alpha_1^\vee + \alpha_2^\vee$

Table 3: List of $A \in \mathcal{A}(s_1, \Gamma_3)$

A	$n(A)$	$\mathrm{end}(A)$	$\mathrm{down}(A)$
$\emptyset$	0	s_1	0
$\{1\}$	0	$s_1 s_2$	0
$\{2\}$	1	e	$\alpha_1^\vee$
$\{1,2\}$	1	$s_1 s_2 s_1$	0

Let us describe the sijection $\mathcal{A}(s_1, \Gamma_1) \Rightarrow \mathcal{A}(s_1, \Gamma_2)$ as follows. We set

$$\begin{aligned}
\mathcal{A}_0(s_1, \Gamma_1) &:= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,2,3\}, \{1,3,4\}\}, \\
\mathcal{A}_0^C(s_1, \Gamma_1) &:= \{\{1,2\}, \{2,3\}, \{1,2,4\}, \{2,3,4\}\}, \\
\mathcal{A}_0(s_1, \Gamma_2) &:= \mathcal{A}(s_1, \Gamma_2), \\
\mathcal{A}_0^C(s_1, \Gamma_2) &:= \emptyset.
\end{aligned}$$

Then, the bijection $Y\colon \mathcal{A}_0(s_1, \Gamma_1) \to \mathcal{A}_0(s_1, \Gamma_2)$ and the involution I_1 on $\mathcal{A}_0^C(s_1, \Gamma_1)$ are given in Tables 4 and 5; here, the involution I_2 on $\mathcal{A}_0^C(s_1, \Gamma_2) = \emptyset$ is trivial.

Let us describe the sijection $\mathcal{A}(s_1, \Gamma_2) \Rightarrow \mathcal{A}(s_1, \Gamma_3)$ as follows. We set

$$\begin{aligned}
\mathcal{A}_0(s_1, \Gamma_2) &:= \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, \\
\mathcal{A}_0^C(s_1, \Gamma_2) &:= \{\{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}\}, \\
\mathcal{A}_0(s_1, \Gamma_3) &:= \mathcal{A}(s_1, \Gamma_3), \\
\mathcal{A}_0^C(s_1, \Gamma_3) &:= \emptyset.
\end{aligned}$$

Then, the bijection $Y\colon \mathcal{A}_0(s_1, \Gamma_2) \to \mathcal{A}_0(s_1, \Gamma_3)$ and the involution I_1 on $\mathcal{A}_0^C(s_1, \Gamma_2)$ are given in Tables 6 and 7; here, the involution I_2 on $\mathcal{A}_0^C(s_1, \Gamma_3) = \emptyset$ is trivial.

Table 4: Description of Y

$A \in \mathcal{A}_0(s_1, \Gamma_1)$	$Y(A) \in \mathcal{A}_0(s_1, \Gamma_2)$
$\emptyset$	$\emptyset$
$\{1\}$	$\{3\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{1\}$
$\{4\}$	$\{4\}$
$\{1,3\}$	$\{1,2\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\{1,3,4\}$	$\{1,2,4\}$

Table 5: Description of I_1

$A \in \mathcal{A}_0^C(s_1, \Gamma_1)$	$I_1(A) \in \mathcal{A}_0^C(s_1, \Gamma_1)$
$\{1,2\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2\}$
$\{1,2,4\}$	$\{2,3,4\}$
$\{2,3,4\}$	$\{1,2,4\}$

Table 6: Description of Y

$A \in \mathcal{A}_0(s_1, \Gamma_2)$	$Y(A) \in \mathcal{A}_0(s_1, \Gamma_3)$
$\emptyset$	$\emptyset$
$\{1\}$	$\{1\}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{1,2\}$

Table 7: Description of I_1

$A \in \mathcal{A}_0^C(s_1, \Gamma_2)$	$I_1(A) \in \mathcal{A}_0^C(s_1, \Gamma_2)$
$\{3\}$	$\{4\}$
$\{4\}$	$\{3\}$
$\{1,2,3\}$	$\{1, 2, 4\}$
$\{1,2,4\}$	$\{1, 2, 3\}$

For details of the construction of Y, I_1, and I_2, see [6, Section 4.6 and Proposition 5.3].

Remark 9. Theorem 6 for a procedure (D) does not hold if β is a simple root; see [6, Remark 5.4].

4 Schubert calculus of semi-infinite flag manifolds

In this section, we apply generalized quantum Yang–Baxter moves to the study of the equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$. For this purpose, we need additional notation. Following [9, Sections 4.1, 4.3], for $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}$, we define $\overline{\mathrm{Par}}(\lambda)$ by

$$\overline{\mathrm{Par}}(\lambda) := \left\{ \chi = (\chi^{(i)})_{i \in I} \;\middle|\; \begin{array}{l} \chi^{(i)} \text{ is a partition of length less than or} \\ \text{equal to } \max\{m_i, 0\} \end{array} \right\};$$

here I is the index set for the simple roots of $\mathfrak{g}$. Also, by writing the partition $\chi^{(i)}$ for $i \in I$ as $(\chi_1^{(i)} \geq \cdots \geq \chi_{l_i}^{(i)})$, we set

$$|\chi| := \sum_{i \in I} \sum_{k=1}^{l_i} \chi_k^{(i)}, \quad \iota(\chi) := \sum_{i \in I} \chi_1^{(i)} \alpha_i^{\vee};$$

if $\chi^{(i)} = \emptyset$, then we understand that $l_i = 0$ and $\chi_1^{(i)} = 0$.

By making use of our generalized quantum Yang–Baxter moves, we can give a combinatorial proof of the Chevalley formula in the equivariant K-group of semi-infinite flag manifolds, first proved by Lenart–Naito–Sagaki based on the Yang–Baxter equation for quantum Bruhat operators.

Theorem 10 ([9, Theorem 33])**.** *Let $\lambda \in P$ and $x \in W_{\mathrm{af}}^{\geq 0}$. Write $x = wt_{\xi}$ for $w \in W$ and $\xi \in Q^{\vee,+}$. Let Γ be a reduced λ-chain. Then, in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$, the following identity holds:*

$$\begin{aligned}&[\mathcal{O}(-w_\circ\lambda)]\cdot[\mathcal{O}_x]\\ &= \sum_{A\in\mathcal{A}(w,\Gamma)}\sum_{\chi\in\overline{\mathrm{Par}}(\lambda)}(-1)^{n(A)}q^{-\mathrm{height}(A)-\langle\lambda,\xi\rangle-|\chi|}e^{\mathrm{wt}(A)}[\mathcal{O}_{\mathrm{end}(A)t_{\xi+\mathrm{down}(A)+\iota(\chi)}}],\end{aligned}$$

where $w_\circ \in W$ denotes the longest element of W.

To prove this identity, we employ certain generating functions.

Definition 11 ([6, Definition 5.1])**.** Let $\lambda \in P$, and take a λ-chain Γ. For $x = wt_{\xi} \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^{\vee}$, we define the *generating function* $\mathbf{G}_{\Gamma}(x) \in (\mathbb{Z}[q,q^{-1}])[P][W_{\mathrm{af}}]$ by

$$\mathbf{G}_{\Gamma}(x) := \sum_{A\in\mathcal{A}(w,\Gamma)}(-1)^{n(A)}q^{-\mathrm{height}(A)-\langle\lambda,\xi\rangle}e^{\mathrm{wt}(A)}\,\mathrm{end}(A)t_{\xi+\mathrm{down}(A)}.$$

The generalized quantum Yang–Baxter moves in Theorem 6 imply the following preservation of generating functions (Theorem 12).

Theorem 12 ([6, Propositions 5.2, 5.3])**.** *Let $\lambda \in P$ and $x \in W_{\mathrm{af}}$.*

(1) *Take λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by applying (YB). Then we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

(2) *Take λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by applying (D) in which the segment $(\beta,-\beta)$, with β not a simple root, is deleted. Then we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

Remark 13. In the same way as Theorem 6, Theorem 12 (2) does not hold if β is a simple root; see [6, Remark 5.4].

Proof of Theorem 10. Theorem 12 plays a crucial role in our proof of Theorem 10. First, we write λ as the sum $\lambda^- + \lambda^+$, with $\lambda^- \in -P^+$ and $\lambda^+ \in P^+$. Then, we take a "lex" $\lambda^{\pm}$-chain $\Gamma^{\pm}$ (see [11, Proposition 4.2] and [9, Section 4.2]). Note that we already know from [9, Theorems 29, 32] (see also [4, Theorem 5.13] and [12, Corollaries C.3, 9.1]) the Chevalley formulas for the dominant weight λ^+ and the anti-dominant weight λ^-. Let Γ_0 denote the concatenation of Γ^- and Γ^+ in this order. Then we deduce that $\mathbf{G}_{\Gamma^-} \circ \mathbf{G}_{\Gamma^+}(x) = \mathbf{G}_{\Gamma_0}(x)$ (see [6, Theorem 5.10]). Here, by [9, Claim 46.1] and [4, Proposition 5.8], we obtain the given reduced λ-chain Γ from the Γ_0 by repeated application of (YB), and by that of (D) in which the segment $(\beta,-\beta)$, with β not a simple root, is deleted. Therefore, Theorem 12 yields the equality $\mathbf{G}_{\Gamma_0}(x) = \mathbf{G}_{\Gamma}(x)$. This equality immediately implies Theorem 10. For the details of this proof, see [6, Section 5]. □

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