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An Involution on Derangements Preserving Excedances and Right-to-Left Minima

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Abstract. We give a bijective proof of a result by R. Mantaci and F. Rakotondrajao from 2003 regarding even and odd derangements with a fixed number of excedances. We refine their result by also considering the set of right-to-left minima.

Keywords: derangement, excedance, right-to-left minimum

1 Introduction and Notations

A permutation π is a bijection from the set $[n] = \{1, 2, \dots, n\}$ to itself and we will write it in standard representation as $\pi = \pi(1)\ \pi(2) \cdots \pi(n)$, or as the product of disjoint cycles. We let $\mathfrak{S}_n$ be the the symmetric group, the set of all permutations, acting on $[n]$. A *fixed point* of a permutation π is an integer $i \in [n]$ such that $\pi(i) = i$. Let $\mathfrak{D}_n \subseteq \mathfrak{S}_n$ denote the set of permutations with no fixed points, which are called *derangements*. An *inversion* of a permutation π is a pair (i, j) such that $\pi(i) > \pi(j)$, where $1 \leq i < j \leq n$. The parity of a permutation π is defined as the parity of the number of inversions of π, $\mathrm{inv}(\pi)$. That is, π is called an *even* if $\mathrm{inv}(\pi)$ is even, and an *odd* permutation otherwise. The set of even permutations in $\mathfrak{S}_n$ is denoted $\mathfrak{S}_n^e$, and the set of odd permutations is $\mathfrak{S}_n^o$. Similarly, $\mathfrak{D}_n^e$ and $\mathfrak{D}_n^o$ represent the sets of even and odd derangements, respectively, in $\mathfrak{D}_n$.

In order to state our results, we need to recall some standard terminology and notations. For any function $g\colon [n] \to [n]$, let the set of *excedances*, the set of *excedance values*, the set of *right-to-left minima indices*, the set of *right-to-left minima values*, and the fixed point set respectively, are defined as

$$\begin{aligned}
\mathrm{EXCi}(g) &:= \{j \in [n] : g(j) > j\},\\
\mathrm{EXCv}(g) &:= \{g(j) : j \in \mathrm{EXCi}(g)\},\\
\mathrm{RLMi}(g) &:= \{i \in [n] : g(i) < g(j) \text{ for all } j \in \{i+1, \dots, n\}\},\\
\mathrm{RLMv}(g) &:= \{g(i) : i \in \mathrm{RLMi}(g)\},\\
\mathrm{FIX}(g) &:= \{i \in [n] : g(i) = i\}.
\end{aligned}$$

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Moreover, we denote $\mathrm{exc}(g) := |\mathrm{EXCi}(g)|$ and $\mathrm{rlm}(g) := |\mathrm{RLMi}(g)| = |\mathrm{RLMv}(g)|$. Note that, $|\mathrm{EXCv}(\sigma)| = |\mathrm{EXCi}(\sigma)| = \mathrm{exc}(\sigma)$, for any $\sigma \in \mathfrak{S}_n$.

Example 1. Consider the following three permutations in $\mathfrak{S}_7$. The first is not a derangement since it has 3 and 6 as fixed-points, while the remaining two are derangements.

Permutation, π	$\mathrm{inv}(\pi)$	$\mathrm{EXCi}(\pi)$	$\mathrm{RLMi}(\pi)$	$\mathrm{RLMv}(\pi)$
2135764	5	{1,4,5}	{2,3,7}	{1,3,4}
2153746	5	{1,3,5}	{2,4,6,7}	{1,3,4,6}
6713245	11	{1,2}	{3,5,6,7}	{1,2,4,5}

Note that whenever $S = \{s_1, \dots, s_m\}$ is a finite set of positive integers, we shall let $\mathbf{x}_S$ denote the product $x_{s_1} x_{s_2} \cdots x_{s_m}$. By definition, $\mathbf{x}_\emptyset := 1$.

R. Mantaci and F. Rakotondrajao [5] have proven[1] the identity

$$|\{\pi \in \mathfrak{D}_n^e : \mathrm{exc}(\pi) = k\}| - |\{\pi \in \mathfrak{D}_n^o : \mathrm{exc}(\pi) = k\}| = (-1)^{n-1}, \tag{1.1}$$

for every $n \geq 1$ and $1 \leq k \leq n-1$. This refines a result by Chapman, stating that $|\mathfrak{D}_n^e| - |\mathfrak{D}_n^o| = (-1)^{n-1}(n-1)$, see [2].

We provide a proof for a refinement of (1.1), namely

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} \mathbf{y}_{\mathrm{EXCv}(\pi)} = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 \cdots x_j y_{j+1} \cdots y_n,$$

in Section 2, by exhibiting a bijection and by using generating functions. The bijection $\widehat{\Psi} \colon \mathfrak{D}_n \to \mathfrak{D}_n$ with exactly $(n-1)$ fixed-elements, is a sign-reversing involution outside the set of fixed-elements. Moreover, it preserves the excedance value and right-to-left minima permutation statistics, which gives the desired result. We use the code obtained in [4], which defined as follows.

Definition 2. A *subexcedant function* f on $[n]$ is a map $f \colon [n] \to [n]$ such that

$$1 \leq f(i) \leq i \text{ for all } 1 \leq i \leq n.$$

We let $\mathcal{F}_n$ denote the set of all subexcedant functions on $[n]$. The *image* of $f \in \mathcal{F}_n$ is defined as $\mathrm{IM}(f) := \{f(i) : i \in [n]\}$.

We write subexcedant functions as words, $f(1)f(2)\dots f(n)$. For example, the subexcedant function $f = 112352$ has $\mathrm{IM}(f) = \{1,2,3,5\}$.

From each subexcedant function $f \in \mathcal{F}_{n-1}$, one can obtain n distinct subexcedant functions in $\mathcal{F}_n$ by appending any integer $i \in [n]$ at the end of the word representing f.

[1] Their proof uses a recursion rather than an explicit involution.

Hence, the cardinality of $\mathcal{F}_n$ is $n!$. The bijection $\texttt{sefToPerm}\colon \mathcal{F}_n \to \mathfrak{S}_n$, described in [4], is defined by the product:

$$\texttt{sefToPerm}(f) := (n\ f(n)) \cdots (2\ f(2))(1\ f(1)).$$

For $\sigma \in \mathfrak{S}_n$ and $j \in [n]$, the j^{th} entry of $\texttt{sefToPerm}^{-1}(\sigma)$ is express in the recursive formula:

$$\texttt{sefToPerm}^{-1}(\sigma)_j := \begin{cases} \sigma(n) & \text{if } j = n, \\ \texttt{sefToPerm}^{-1}\big((n\ \sigma(n)) \circ \sigma\big)_j & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that $\sigma' := (n\ \sigma(n)) \circ \sigma$ is the result after interchanging n and the image of n in σ. Therefore, $\sigma'(n) = n$ and, by a slight abuse of notation, σ' can be considered as a permutation in $\mathfrak{S}_{n-1}$. For simplicity, we use the shorthand $f_\sigma := \texttt{sefToPerm}^{-1}(\sigma)$.

Example 3. The corresponding subexcedant function of the permutation $\sigma = 6\,1\,2\,9\,3\,5\,4\,8\,7$ is $f_\sigma = 112435487 \in \mathcal{F}_9$.

Since subexcedant functions are maps on $[n]$, we have the notion of excedance, right-to-left minima, fixed points, etc., as defined above.

Proposition 4 (See [4, Proposition 3.5])**.** *For $f_\sigma \in \mathcal{F}_n$ we have that $[n] \setminus \mathrm{IM}(f_\sigma) = \mathrm{EXCv}(\sigma)$. In particular,* $\mathrm{exc}(\sigma) = n - |\,\mathrm{IM}(f_\sigma)|$.

We say that a subexcedant function f has a *strict anti-excedance* at i if $f(i) < i$.

Proposition 5 (See [4, Proposition 4.1])**.** *The permutation σ is even (odd) if and only if the number of strict anti-excedances in f_σ even (odd).*

A fixed point of $f \in \mathcal{F}_n$ is an integer $i \in [n]$ such that $f(i) = i$. Moreover, i is a *multiple fixed point* of f if $f(i) = i$ and there is some $j > i$ such that $f(j) = i$.

Proposition 6 (See [4, Proposition 3.8])**.** *We have that $\sigma \in \mathfrak{D}_n$ if and only if all fixed points of f_σ are multiple.*

Proposition 7. *Let $\pi \in \mathfrak{S}_n$ and f_π be the corresponding subexcedant function. Then*

(a) $i \in \mathrm{RLMi}(\pi)$ *implies* $\pi(i) = f_\pi(i)$*,*

(b) $\mathrm{RLMv}(\pi) = \mathrm{RLMv}(f_\pi)$*,*

(c) $\mathrm{RLMi}(\pi) = \mathrm{RLMi}(f_\pi)$*.*

2 An involution and its consequences

A subexcedant function f is *matchless* if it is of the form

$$f := 1\,1\,2\,3\,4\ldots k{-}1\;k\;k\ldots k \quad \text{for } 1 \leq k \leq n{-}1.$$

There are $n-1$ matchless subexcedant functions of length n. For example, for $n = 10$, the following subexcedant functions are matchless:

$$\begin{array}{lll} 1111111111, & 1122222222, & 1123333333, \\ 1123444444, & 1123455555, & 1123456666, \\ 1123456777, & 1123456788, & 1123456789. \end{array}$$

Let $\mathcal{DF}_n$ be the set of subexcedant functions corresponding to derangements of $[n]$. Note that every $f \in \mathcal{DF}_n$ must have at least two 1's in its row representation.

For any matchless $f_\sigma \in \mathcal{DF}_n$

$$\sigma = \texttt{sefToPerm}(f_\sigma) = (1\; k{+}1\; k{+}2\; \ldots\; n\; k\; k{-}1\; \ldots\; 2).$$

Since σ has only one cycle, its sign is $(-1)^{n-1}$. Looking directly at the definition of f_σ, we have that

$$\mathrm{IM}(f_\sigma) = [k] \;\text{ implies }\; \mathrm{EXCv}(\sigma) = [n] \setminus [k],$$

by Proposition 4. Similarly, from Proposition 7 we have $\mathrm{RLMv}(\sigma) = [k]$.

Definition 8. Define a mapping $\Psi\colon \mathcal{DF}_n \to \mathcal{DF}_n$ below, where f_τ is short for $\Psi(f_\sigma)$. First, if f_σ is matchless, we set $f_\tau := f_\sigma$. Now we assume that f_σ is non-matchless and let

$$\mathrm{IM}(f_\sigma) = \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \ldots, \mathbf{m}_\ell\}.$$

Note that $\mathbf{m}_1 = 1$ and since f_σ is non-matchless, we know that $\ell \geq 2$ in $\mathrm{IM}(f_\sigma)$. With these preparations, we define two auxiliary maps, $\texttt{fix}_i$, $\texttt{unfix}_i$ on subexcedant functions. For $i \in \{2, \ldots, \ell\}$,

$$\texttt{fix}_i(f_\sigma)(\mathbf{m}_i) := \mathbf{m}_i, \qquad \texttt{unfix}_i(f_\sigma)(\mathbf{m}_i) := \mathbf{m}_{i-1}$$

while the remaining entries of f_σ are untouched. For $i \in \{2, \ldots, \ell\}$, we say that f_σ satisfies $\circledast_i$ if the three conditions

$$f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell, \quad f_\sigma^{-1}(1) = \{1, 2\}, \text{ and } \{\mathbf{m}_i + 1\} \subsetneq f_\sigma^{-1}(\mathbf{m}_i), \tag{$\circledast_i$}$$

hold. Note that

$$\{\mathbf{m}_i + 1\} \subsetneq f_\sigma^{-1}(\mathbf{m}_i) \text{ if and only if } f_\sigma(\mathbf{m}_i + 1) = \mathbf{m}_i \text{ and } |f_\sigma^{-1}(\mathbf{m}_i)| \geq 2.$$

Now let $i \in \{2, \ldots, \ell\}$ be the *smallest* element satisfying one of the cases below, and let f_τ be given as described in each case.

Case $\heartsuit_i$: If $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$, then $f_\tau := \texttt{unfix}_i(f_\sigma)$.

Case $\spadesuit_i$: If $f_\sigma(\mathbf{m}_i) < \mathbf{m}_i$ and $|f_\sigma^{-1}(1)| \geq 3$, then $f_\tau := \texttt{fix}_i(f_\sigma)$.

Case $\diamondsuit_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$, then $f_\tau := \texttt{unfix}_{i+1}(f_\sigma)$.

Case $\clubsuit_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) < \mathbf{m}_{i+1}$, then $f_\tau := \texttt{fix}_{i+1}(f_\sigma)$.

Note that for the same i, the four cases are mutually exclusive. We emphasize that by saying that a case with subscript i holds, this particular $i \geq 2$ is the smallest i for which the conditions one of the four cases hold.

Remark 9. Suppose $\spadesuit_i$ applies for f_σ. Then, for sure $f_\sigma(\mathbf{m}_2) < \mathbf{m}_2$, since otherwise, we would be in the case $\heartsuit_2$. Hence, $\spadesuit_i$ may only apply when $i = 2$.

Theorem 10. *The map $\Psi\colon \mathcal{DF}_n \to \mathcal{DF}_n$ is an involution with the following properties.*

(i) The image is preserved, $\mathrm{IM}(f_\sigma) = \mathrm{IM}(\Psi(f_\sigma))$.

(ii) If $f_\tau = \Psi(f_\sigma)$, then $\mathrm{EXCv}(\sigma) = \mathrm{EXCv}(\tau)$.

(iii) The set of right-to-left minima is preserved, $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(\Psi(f_\sigma))$.

(iv) Ψ changes the parity of a non-matchless subexcedant function.

The complete proof of this theorem can be found in [1].

Example 11. Consider the following four subexcedant functions in $\mathcal{DF}_7$.

1. Let $f_\sigma = 1133535$. Then $\mathrm{IM}(f_\sigma) = \{1, 3, 5\}$ and 2 is the smallest index greater than 1 with $f_\sigma(\mathbf{m}_2) = f_\sigma(3) = 3$. Hence, f_σ is in case $\heartsuit_2$ and $f_\tau = \texttt{unfix}_2(f_\sigma) = 1113535$.

2. Now let $f_\sigma = 1121355$. Then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 5\}$. Since $f_\sigma(2) < 2$ and $|f_\sigma^{-1}(1)| = 3$, then f_σ is in case $\spadesuit_2$. Thus, $f_\tau = \texttt{fix}_2(f_\sigma) = 1221355$.

3. Suppose that $f_\sigma = 1123535$, then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 5\}$. The index 2 does not satisfy any of the four cases. So, we consider the next integer $i = 3$. We note that $\circledast_3$ holds and in addition, $f_\sigma(\mathbf{m}_4) = f_\sigma(5) = 5$. Hence, f_σ fulfills $\diamondsuit_3$ and $f_\tau = \texttt{unfix}_{i+1}(f_\sigma) = \texttt{unfix}_4(f_\sigma) = 1123335$.

4. Now take $f_\sigma = 1123445$. Then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 4, 5\}$. None of the four cases for f_σ are fulfilled with $i \in \{2, 3\}$. However, f_σ satisfies $\circledast_4$ and $f_\sigma(\mathbf{m}_5) = f_\sigma(5) = 4 < \mathbf{m}_5$. Thus, we are in $\clubsuit_4$ and $f_\tau = \texttt{fix}_5(f_\sigma) = 1123545$.

We now have an involution on derangements $\widehat{\Psi} : \mathfrak{D}_n \to \mathfrak{D}_n$ by setting

$$\widehat{\Psi}(\sigma) := (\texttt{sefToPerm} \circ \Psi \circ \texttt{sefToPerm}^{-1})(\sigma), \text{ for } \sigma \in \mathfrak{D}_n.$$

Corollary 12. *The involution $\widehat{\Psi}$ satisfies the properties below:*

(i) The excedance value set is preserved: $\mathrm{EXCv}(\widehat{\Psi}(\sigma)) = \mathrm{EXCv}(\sigma)$.

(ii) The set of right-to-left minima is preserved: $\mathrm{RLMv}(\widehat{\Psi}(\sigma)) = \mathrm{RLMv}(\sigma)$.

(iii) Whenever σ is a non-matchless derangement (the corresponding f_σ is non-matchless), $\widehat{\Psi}$ changes the parity of σ.

Theorem 13. *We have that*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} \mathbf{y}_{\mathrm{EXCv}(\pi)} = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 \cdots x_j \cdot y_{j+1} \cdots y_n. \tag{2.1}$$

Moreover,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMi}(\pi)} \mathbf{y}_{\mathrm{EXCi}(\pi)} = (-1)^{n-1} \sum_{j=1}^{n-1} y_1 \cdots y_j \cdot x_{j+1} \cdots x_n. \tag{2.2}$$

Proof. By applying the involution $\widehat{\Psi}$ and using all the properties listed in Corollary 12, all terms in the left-hand side of (2.1) that are non-matchless derangements cancel. Thus, the left-hand side of (2.1) is equal to

$$\sum_{k=1}^{n-1} (-1)^{n-1} \mathbf{x}_{[k]} \mathbf{y}_{[n]\setminus[k]},$$

using properties of matchless derangements, which is the right-hand side of (2.1).

Equation (2.2) follows by applying the change of variables $i \mapsto n+1-i$ on both sides of (2.1) and then use the bijection $\zeta : \mathfrak{D}_n \to \mathfrak{D}_n$, where

$$\zeta(\sigma)(k) := n+1-\sigma^{-1}(n+1-k), \quad \text{for } \sigma \in \mathfrak{D}_n \text{ and } k \in [n],$$

on the left-hand side. □

Corollary 14. *By letting $x_j \to 1$ and $y_j \to t$, we have that*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{exc}(\pi)} = (-1)^{n-1} (t + t^2 + \cdots + t^{n-1}).$$

By comparing coefficients of t^k, we get (1.1). In a similar manner,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{rlm}(\pi)} = (-1)^{n-1} (t + t^2 + \cdots + t^{n-1}).$$

3 A proof using generating functions

Mantaci, in [3], proved Proposition 15 (albeit stated in a slightly different manner) by introducing a bijection on $\mathfrak{S}_n$ that preserves the set of excedances and changes the sign of non-fixed elements of the bijection. There is a unique fixed element for each excedance set and its parity is the same as the parity of the cardinality of its excedance set.

Proposition 15. *Let $n \geq 1$, then*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \prod_{j \in [n-1]} (1 - x_j) = \sum_{E \subseteq [n-1]} (-1)^{|E|} \mathbf{x}_E. \tag{3.1}$$

In particular, by setting all x_i equal to t, we have

$$\sum_{\pi \in \mathfrak{S}_n^e} t^{\mathrm{exc}(\pi)} - \sum_{\pi \in \mathfrak{S}_n^o} t^{\mathrm{exc}(\pi)} = (1-t)^{n-1}.$$

Proposition 16. *Let $n \geq 1$ and let $T \subseteq [n]$. Let $m \leq n$ be the largest integer not in T and set $E = \{1, 2, \dots, m-1\} \setminus T$. Then*

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ T \subseteq \mathrm{FIX}(\pi)}} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \prod_{j \in E} (1 - x_j), \tag{3.2}$$

where the empty product has value 1.

Setting all x_i to be t, we have

$$\sum_{\substack{\pi \in \mathfrak{S}_n^e \\ T \subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} - \sum_{\substack{\pi \in \mathfrak{S}_n^o \\ T \subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} = \begin{cases} 1 & \text{if } |T| = n, \\ (1-t)^{n-1-|T|} & \text{otherwise.} \end{cases}$$

Proof. if $T = [n]$, then $E = \emptyset$ and (3.2) follows. Now assume $|T| < n$. From formation of E, we can easily see that $|E| = n - 1 - |T|$. Now suppose $\pi \in \mathfrak{S}_n$ is a permutation such that $T \subseteq \mathrm{FIX}(\pi)$. We then construct $\pi' \in \mathfrak{S}_{n-|T|}$, by only considering the positions not in T, and the relative ordering of the entries at these positions. For example, for $\pi = 127436589$ we have $T = \{2, 4, 6, 8, 9\}$, $[n] \setminus T = \{1, 3, 5, 7\}$ and $\pi' = 1423$.

Observe that $\mathrm{exc}(\pi) = \mathrm{exc}(\pi')$ and $(-1)^{\mathrm{inv}(\pi)} = (-1)^{\mathrm{inv}(\pi')}$. Hence, the sum in the left-hand side of (3.2), can be taken as a sum over permutations $\pi' \in \mathfrak{S}_{n-|T|}$, but with a reindexing of the variables using values in $[n] \setminus T$. Now, this sum can be computed using Proposition 15 which finally gives (3.2). □

Using inclusion-Exclusion and Proposition 16, the following theorem is obtained.

Theorem 17. *Let $n \geq 1$. Then*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 x_2 \cdots x_j. \tag{3.3}$$

The following follows directly by comparing coefficients of degree k in (3.3).

Corollary 18. *For $n, k \geq 1$, we have that*

$$|\{\pi \in \mathfrak{D}_n^e : \mathrm{exc}(\pi) = k\}| - |\{\pi \in \mathfrak{D}_n^o : \mathrm{exc}(\pi) = k\}| = (-1)^{n-1}.$$

3.1 A right-to-left minima analog

Definition 19. Let $\kappa : \mathfrak{S}_n \to \mathfrak{S}_n$ be defined as follows. Given $\pi \in \mathfrak{S}_n$, let $i \in [n]$ be the smallest *odd* integer such that $\pi(i\ i+1)$ and π have the same sets of right-to-left minima, if such an i exists. That is, we swap the entries at positions i and $i+1$ in π. We then set $\kappa(\pi) := \pi(i\ i+1)$, and $\kappa(\pi) := \pi$ otherwise. We say that π is *decisive*[2] if it is a fixed-element of κ.

Example 20. In $\mathfrak{S}_7$, there are 8 decisive permutations:

$$1234567,\ 1234657,\ 1243567,\ 1243657,\ 2134567,\ 2134657,\ 2143567,\ 2143657.$$

Note that $\{1, 3, 5, 7\}$ are always right-to-left minima (but there might be more).

Lemma 21. *The map $\kappa : \mathfrak{S}_n \to \mathfrak{S}_n$ has the following properties:*

(i) κ is an involution.

(ii) κ preserves the number of right-to-left minima.

(iii) κ changes sign of non-fixed elements.

(iv) For each subset $T \in [n] \cap \{2, 4, 6, \dots\}$, there is a unique decisive permutation with $\{1, 3, 5, \dots\} \cup T$ as right-to-left minima set.

(v) There are $\binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}$ decisive permutations with exactly k right-to-left minima, and they all have sign $(-1)^{n-k}$.

The following is a right-to-left minima analog of Proposition 15.

Corollary 22. *We have that for any $n \geq 1$*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} = \Big(\prod_{\substack{i \in [n] \\ i \text{ odd}}} x_i \Big) \Big(\prod_{\substack{j \in [n] \\ j \text{ even}}} (x_j - 1) \Big). \tag{3.4}$$

In particular, for any $k = 1, \dots, n$ we have that

$$|\{\pi \in \mathfrak{S}_n^e : \mathrm{rlm}(\pi) = k\}| - |\{\pi \in \mathfrak{S}_n^o : \mathrm{rlm}(\pi) = k\}| = (-1)^{n-k} \binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}.$$

[2] As a nod to the word *critical*.

We conclude with the following problem.

Problem 23. *Is it possible to state an analog of Proposition 16? In particular, for $T \subseteq [n]$, is there a nice expression for the sum*

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ T \subseteq \mathrm{FIX}(\pi)}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{rlm}(\pi)}?$$

Computer experiments suggest that this sum is either 0 *or of the form* $\pm t^a (t+1)^b (t-1)^c$*, where* a*,* b*, and* c *depend on* T *in some manner.*

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