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Combinatorics of Newell–Littlewood Numbers

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Abstract. We give an exposition of recent developments in the study of Newell–Littlewood numbers. These are the tensor product multiplicities of Weyl modules in the stable range. They are also the structure coefficients of the Koike–Terada basis of the ring of symmetric functions. Two types of combinatorial results are exhibited, those obtained combinatorially starting from the definition of the numbers, and those that also employ geometric and/or representation theoretic methods.

Keywords: tensor product multiplicities, eigencones, Newell–Littlewood numbers

1 Introduction

The *Newell–Littlewood numbers* [23, 21] are defined by

$$N_{\mu,\nu,\lambda} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha,\beta} c^{\nu}_{\alpha,\gamma} c^{\lambda}_{\beta,\gamma}, \tag{1.1}$$

where the indices are partitions in

$$\mathsf{Par}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0} : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

Here, $c^{\mu}_{\alpha,\beta}$ is the *Littlewood–Richardson coefficient*; these are of interest in combinatorics, representation theory and algebraic geometry; see, *e.g.*, the books [6, 5, 24]. This extended abstract mostly summarizes [9, 8, 7] but we also mention related follow-up work.

For an n-dimensional vector space V over $\mathbb{C}$ and $\lambda \in \mathsf{Par}_n$, the *Weyl module* (or *Schur functor*) $\mathbb{S}_\lambda(V)$ is an irreducible $\mathsf{GL}(V)$-module (see, *e.g.*, [6, Lectures 6 and 15]). The Littlewood–Richardson coefficients are the tensor product multiplicities

$$\mathbb{S}_\mu(V) \otimes \mathbb{S}_\nu(V) \cong \bigoplus_{\lambda \in \mathsf{Par}_n} \mathbb{S}_\lambda(V)^{\oplus c^{\lambda}_{\mu,\nu}}.$$

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The Newell–Littlewood numbers arise similarly, where $\mathsf{GL}(V)$ is replaced by other classical Lie groups G. Suppose W is a complex vector space, with a fixed nondegenerate symplectic or orthogonal form ω. Let G be the subgroup of $\mathsf{SL}(W)$ preserving ω. Then $\mathsf{G} = \mathsf{SO}_{2n+1}$ if $\dim W = 2n+1$ and ω is orthogonal. It is $\mathsf{G} = \mathsf{Sp}_{2n}$ if $\dim W = 2n$ and ω is symplectic. Finally, $\mathsf{G} = \mathsf{SO}_{2n}$ if $\dim W = 2n$ and ω is orthogonal. These are groups in the B_n, C_n, D_n series of the Cartan-Killing classification, respectively.

If $\lambda \in \mathsf{Par}_n$, H. Weyl's construction [26] (see also [6, Lectures 17 and 19]) gives a G-module $\mathbb{S}_{[\lambda]}(W)$. These modules are irreducible, except in type D_n, where irreducibility holds if $\lambda_n = 0$. In the *stable range* $\ell(\mu) + \ell(\nu) \leq n$,

$$\mathbb{S}_{[\mu]}(W) \otimes \mathbb{S}_{[\nu]}(W) \cong \bigoplus_{\lambda \in \mathsf{Par}_n} \mathbb{S}_{[\lambda]}(W)^{\oplus N_{\mu,\nu,\lambda}}; \tag{1.2}$$

this is [19, Corollary 2.5.3]. In particular, $N_{\mu,\nu,\lambda}$ is independent of G [19, Theorem 2.3.4].

The Schur functions s_λ form a basis of the ring Λ of symmetric functions. It is the "universal character" of $\mathbb{S}_\lambda(V)$ for GL. In a similar fashion, [19, Section 2] establishes universal characters of $\mathbb{S}_{[\lambda]}(W)$ for Sp. This *Koike–Terada basis* $\{s_{[\lambda]}\}$ of Λ satisfies

$$s_{[\mu]} s_{[\nu]} = \sum_{\lambda} N_{\mu,\nu,\lambda} s_{[\lambda]}, \tag{1.3}$$

where μ, ν, λ are arbitrary partitions. (Now, [19] also defines a basis for SO with the same structure coefficients. Which one we use is simply a matter of choice.)

Section 2 outlines the combinatorially derived results found in [9]. Section 3 presents the results from [7] which resolve a conjecture from [8] and furthermore explains the connection to *eigencones*. A number of problems remain in this subject; some of these are stated in Section 4.

2 Results using purely combinatorial methods

In this section we summarize results that can be obtained directly from (1.1).

2.1 Basic observations

This lemma is stated as [9, Lemma 2.2] without claims of originality by the authors:

Lemma 2.1 (Facts about the Newell–Littlewood numbers)**.**

(I) $N_{\mu,\nu,\lambda}$ *is invariant under any of the* $3!$*-permutations of the indices* (μ, ν, λ).

(II) $N_{\mu,\nu,\lambda} = c^{\lambda}_{\mu,\nu}$ *if* $|\mu| + |\nu| = |\lambda|$.

(III) $N_{\mu,\nu,\lambda} = 0$ *unless* $|\mu|, |\nu|, |\lambda|$ *satisfy the triangle inequalities (possibly with equality),* i.e., $|\mu| + |\nu| \geq |\lambda|$, $|\mu| + |\lambda| \geq |\nu|$, *and* $|\lambda| + |\nu| \geq |\mu|$.

(IV) $N_{\mu,\nu,\lambda} = 0$ *if* $|\nu \wedge \lambda| + |\mu \wedge \nu| < |\nu|$, *where* $\nu \wedge \lambda$ *is the partition whose i-th part is* $\min(\nu_i, \lambda_i)$.

(V) $N_{\mu,\nu,\lambda} = 0$ *unless* $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$.

(VI) $N_{\mu,\nu,\lambda} = N_{\mu',\nu',\lambda'}$ *where* μ' *is the conjugate partition of* μ, etc.

This is an observation used in [9]:

Proposition 2.2 ([9, Proposition 2.3])**.** $N_{\mu,\nu,\lambda} = \sum_{\alpha \subseteq \mu \wedge \nu} \langle s_{\mu/\alpha} s_{\nu/\alpha}, s_\lambda \rangle$.

2.2 Shape of $s_{[\mu]} s_{[\nu]}$

Let $\mu \Delta \nu = (\mu \setminus \nu) \cup (\nu \setminus \mu)$ be the symmetric difference of the Young diagrams of λ and μ. Define Par to be the set of all integer partitions. This theorem is proved in [9] using Young tableau combinatorics based on a *demotion* procedure. In [7] it is further studied in connection to the Robinson–Schensted–Knuth correspondence to prove Theorem 3.2.

Theorem 2.3 ([9, Theorem 3.1])**.** *Fix* $\mu, \nu \in \mathsf{Par}$.

(I) Let $k \in \mathbb{Z}_{\geq 0}$. *There exists* $\lambda \in \mathsf{Par}$ *with* $|\lambda| = k$ *and* $N_{\mu,\nu,\lambda} > 0$ *if and only if*

$$k \equiv |\mu \Delta \nu| \pmod 2 \ \ \textit{and} \ \ |\mu \Delta \nu| \leq k \leq |\mu| + |\nu|.$$

(II) If $N_{\mu,\nu,\lambda} > 0$ *with* $|\lambda| > |\mu \Delta \nu|$, *there exists* $\lambda^{\downarrow\downarrow}$ *such that* $N_{\mu,\nu,\lambda^{\downarrow\downarrow}} > 0$, $\lambda^{\downarrow\downarrow} \subset \lambda$ *and* $|\lambda^{\downarrow\downarrow}| = |\lambda| - 2$.

(III) If $N_{\mu,\nu,\lambda} > 0$ *with* $|\lambda| < |\mu| + |\nu|$, *there exists* $\lambda^{\uparrow\uparrow}$ *such that* $N_{\mu,\nu,\lambda^{\uparrow\uparrow}} > 0$, $\lambda \subset \lambda^{\uparrow\uparrow}$ *and* $|\lambda^{\uparrow\uparrow}| = |\lambda| + 2$.

2.3 Newell–Littlewood polytopes

We now turn to "polytopal" aspects of the Newell–Littlewood numbers. Fix $\lambda, \mu, \nu \in \mathsf{Par}_n$. Let $\alpha_i^j, \beta_i^j, \gamma_i^j \in \mathbb{R}$ for $1 \leq i, j \leq n$ and consider the linear constraints:

1. *Non-negativity*: For all $1 \leq i, j \leq n$, $\alpha_i^j, \beta_i^j, \gamma_i^j \geq 0$.
2. *Shape constraints*: For all k,

   (a) $\sum_j \alpha_k^j + \sum_i \beta_i^k = \mu_k$,

(b) $\sum_j \gamma_k^j + \sum_i \alpha_i^k = \nu_k$,

(c) $\sum_j \beta_k^j + \sum_i \gamma_i^k = \lambda_k$.

3. *Tableau/semistandardness constraints*: For all k, l:

   (a) $\sum_j \alpha_{k+1}^j + \sum_{i \le l} \beta_i^{k+1} \le \sum_j \alpha_k^j + \sum_{i<l} \beta_i^k$,

   (b) $\sum_j \gamma_{k+1}^j + \sum_{i \le l} \alpha_i^{k+1} \le \sum_j \gamma_k^j + \sum_{i<l} \alpha_i^k$,

   (c) $\sum_j \beta_{k+1}^j + \sum_{i \le l} \gamma_i^{k+1} \le \sum_j \beta_k^j + \sum_{i<l} \gamma_i^k$.

4. *Ballot constraints*: For all k, l:

   (a) $\sum_{i<k} \alpha_l^i \ge \sum_{i \le k} \alpha_{l+1}^i$,

   (b) $\sum_{i<k} \beta_l^i \ge \sum_{i \le k} \beta_{l+1}^i$,

   (c) $\sum_{i<k} \gamma_l^i \ge \sum_{i \le k} \gamma_{l+1}^i$.

Definition 2.4 ([9, Section 5])**.** The *Newell–Littlewood polytope* is

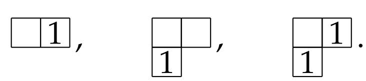
$$\mathcal{P}_{\mu,\nu,\lambda} = \{(\alpha_i^j, \beta_i^j, \gamma_i^j) \in \mathbb{R}^{3n^2} : (1)\text{-}(4) \text{ hold}\} \subset \mathbb{R}^{3n^2}.$$

Theorem 2.5 ([9, Theorem 5.1])**.** *$N_{\mu,\nu,\lambda} = \#(\mathcal{P}_{\mu,\nu,\lambda} \cap \mathbb{Z}^{3n^2})$.*

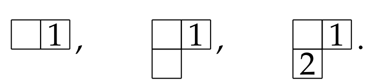
Example 2.6. We illustrate the correspondence asserted by Theorem 2.5. Let $\mu = (2)$, $\nu = (2,1)$, and $\lambda = (2,1)$. Write α_i^j, β_i^j and γ_i^j in terms of matrices $[\alpha], [\beta]$ and $[\gamma]$ so that $[\alpha]_{i,j} = \alpha_i^j$, $[\beta]_{i,j} = \beta_i^j$ and $[\gamma]_{i,j} = \gamma_i^j$. Then $\mathcal{P}_{\mu,\nu,\lambda} \cap \mathbb{Z}^{12}$ would be the two triples

$$([\alpha],[\beta],[\gamma]) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) \text{ or } \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

implying $N_{\mu,\nu,\lambda} = 2$. The first triple corresponds to the triple of LR tableaux contributing, respectively, to $c_{\alpha,\beta}^{\mu}, c_{\gamma,\alpha}^{\nu}$ and $c_{\beta,\gamma}^{\lambda}$ where $\alpha = (1), \beta = (1), \gamma = (2)$:



Similarly, the second triple corresponds to these LR tableaux



which contribute, respectively, to $c_{\alpha,\beta}^{\mu}, c_{\gamma,\alpha}^{\nu}$ and $c_{\beta,\gamma}^{\lambda}$ with $\alpha = (1), \beta = (1), \gamma = (1,1)$. □

That $N_{\lambda,\mu,\nu}$ counts lattice points in a polytope can be proved using A. Berenstein–A. Zelevinsky [2, Section 2.2] on more general tensor product multiplicities, together with [19, Corollary 2.5.3]. The proof of Theorem 2.5 in [9] uses a self-contained approach, similar to one in the preprint version of [22] for the Littlewood–Richardson coefficients.

2.4 Streched Newell–Littlewood numbers

A conjecture of W. Fulton (proved in [18]) states that

$$c_{\mu,\nu}^{\lambda} = 1 \implies c_{k\mu,k\nu}^{k\lambda} = 1, \text{ for all } k \geq 1.$$

Example 2.7 ([9, Example 5.24]). One checks that

$$N_{(1,1),(1,1),(1,1)} = (c_{(1),(1)}^{(1)})^3 = 1 \text{ but } N_{(2,2),(2,2),(2,2)} = (c_{(1,1),(1,1)}^{(1,1)})^3 + (c_{(2),(2)}^{(2)})^3 = 2.$$

Therefore, the analogue of Fulton's conjecture for $N_{\nu,\mu,\lambda}$ does not hold. □

Define a function

$$\mathfrak{c}_{\mu,\nu}^{\lambda} \colon \mathbb{Z}_{\geq 1} \to \mathbb{N} \text{ by } k \mapsto c_{k\mu,k\nu}^{k\lambda}.$$

R. C. King–C. Tollu–F. Toumazet [12] conjecture that this function is interpolated by a polynomial with nonnegative rational coefficients. The polynomiality property was proved by H. Derksen–J. Weyman [3]. Consequently, the polynomial $\mathfrak{c}_{\mu,\nu}^{\lambda}$ is called the *Littlewood–Richardson polynomial*. The positivity conjecture is still open.

Definition 2.8 ([9, Section 5.4]). The *Newell–Littlewood function* is $\mathfrak{N}_{\mu,\nu,\lambda} \colon \mathbb{Z}_{\geq 1} \to \mathbb{N}$ by $k \mapsto N_{k\mu,k\nu,k\lambda}$.

$\mathfrak{N}_{\mu,\nu,\lambda}(k)$ cannot always be interpolated by a single polynomial:

Theorem 2.9 ([9, Theorem 5.25]). *There exist λ, μ, ν such that $\mathfrak{N}_{\mu,\nu,\lambda}(k) \notin \mathbb{R}[k]$.*

Proof. One argues that $\mathfrak{N}_{(1,1),(1,1),(1,1)}(k) = \left\lceil \frac{k+1}{2} \right\rceil$, which is clearly non-polynomial. □

Recent work of R. C. King [11] extensively studies $\mathfrak{N}_{\mu,\nu,\lambda}(k)$. One of his results is

Theorem 2.10 ([11, Corollary 2.2]). *For any $\lambda, \mu, \nu \in \mathsf{Par}_n$, there exist $P_e(k), P_o(k) \in \mathbb{Q}[k]$ such that*

$$\mathfrak{N}_{\mu,\nu,\lambda}(k) = \begin{cases} P_e(k) & \text{for } k \text{ even}, \\ P_o(k) & \text{for } k \text{ odd}. \end{cases}$$

2.5 Multiplicity-freeness

Definition 2.11 ([9, Section 6]). A pair $(\mu, \nu) \in \mathsf{Par}^2$ is *NL-multiplicity-free* if (1.3) contains no multiplicity, *i.e.*, each $N_{\mu,\nu,\lambda} \in \{0, 1\}$ for all $\lambda \in \mathsf{Par}$.

Theorem 2.12 ([9, Theorem 6.1]). *A pair $(\mu, \nu) \in \mathsf{Par}^2$ is NL-multiplicity-free if and only if*

(I) μ or ν is either a single box or $\emptyset$;

(II) μ is a single row and ν is a rectangle (or vice versa); or

(III) μ is a single column and ν is a rectangle (or vice versa).

Theorem 2.12 is an analogue of a theorem of [25] for Schur functions.

2.6 Version of T. Lam–A. Postnikov–P. Pylyavskyy's theorems

If $\alpha, \beta \in \mathsf{Par}$ then $\alpha \vee \beta \in \mathsf{Par}$ has parts $\max(\alpha_i, \beta_i)$ (where one postpends 0's to α or β as necessary). Given two skew shapes ν/α and μ/β, let

$$(\nu/\alpha) \wedge (\mu/\beta) := (\nu \wedge \mu)/(\alpha \wedge \beta) \text{ and } (\nu/\alpha) \vee (\mu/\beta) := (\nu \vee \mu)/(\alpha \vee \beta).$$

Set $\mathrm{sort}_1(\nu, \mu) := (\rho_1, \rho_3, \rho_5, \ldots)$ and $\mathrm{sort}_2(\nu, \mu) := (\rho_2, \rho_4, \rho_6, \ldots)$, where $(\rho_1, \rho_2, \ldots) := \nu \cup \mu$. In what follows, $\frac{\nu+\mu}{2}$ means coordinate-wise addition and division. Moreover, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are taken coordinate-wise.

Define $f \in \Lambda$ to be *Schur nonnegative* if $f = \sum_\lambda a_\lambda s_\lambda$ with $a_\lambda \geq 0$ for all $\lambda \in \mathsf{Par}$.

Theorem 2.13 ([20])**.** *Let ν/α and μ/β be skew shapes. These are Schur nonnegative:*

1. $s_{(\nu/\alpha)\wedge(\mu/\beta)} s_{(\nu/\alpha)\vee(\mu/\beta)} - s_{\nu/\alpha} s_{\mu/\beta}$
2. $s_{\lfloor \frac{\nu+\mu}{2} \rfloor / \lfloor \frac{\alpha+\beta}{2} \rfloor} s_{\lceil \frac{\nu+\mu}{2} \rceil / \lceil \frac{\alpha+\beta}{2} \rceil} - s_{\nu/\alpha} s_{\mu/\beta}$
3. $s_{\mathrm{sort}_1(\nu,\mu)/\mathrm{sort}_1(\alpha,\beta)} s_{\mathrm{sort}_2(\nu,\mu)/\mathrm{sort}_2(\alpha,\beta)} - s_{\nu/\alpha} s_{\mu/\beta}$

We refer the reader to [19, Definition 2.1.1] for a definition of $s_{[\lambda]} \in \Lambda$ as a determinant in terms of complete homogeneous symmetric functions.

Definition 2.14 ([9, Section 7.3])**.** $f \in \Lambda$ is *Koike–Terada nonnegative* if $f = \sum_\lambda b_\lambda s_{[\lambda]}$ has $b_\lambda \geq 0$ for every $\lambda \in \mathsf{Par}$.

Combining Theorem 2.13 with Proposition 2.2 implies:

Theorem 2.15 ([9, Theorem 7.4])**.** *The following are Koike–Terada nonnegative:*

1. $s_{[\nu\wedge\mu]} s_{[\nu\vee\mu]} - s_{[\nu]} s_{[\mu]}$
2. $s_{[\lfloor \frac{\nu+\mu}{2} \rfloor]} s_{[\lceil \frac{\nu+\mu}{2} \rceil]} - s_{[\nu]} s_{[\mu]}$
3. $s_{[\mathrm{sort}_1(\nu,\mu)]} s_{[\mathrm{sort}_2(\nu,\mu)]} - s_{[\nu]} s_{[\mu]}$

3 Nonvanishing results using geometric methods; connection to eigencones

We now turn to the results of [7], whose proofs rely on a mix of geometry and combinatorics. Fix $n \in \mathbb{N}$. Let $\text{NL-semigroup}(n) = \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3 : N_{\lambda,\mu,\nu} > 0\}$. Indeed, NL-semigroup is a finitely generated semigroup [9, Section 5.2]. An approximation of it is the saturated semigroup:

$$\text{NL-sat}(n) = \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3 : \text{there exists } t > 0 \quad N_{t\lambda,t\mu,t\nu} \neq 0\},$$

where $\mathrm{Par}_n^{\mathbb{Q}} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n : \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}$. By Lemma 2.1(II), a subproblem asks to study

$$\text{LR-sat}(n) = \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3 : \text{there exists } t > 0 \quad c_{t\lambda, t\mu}^{t\nu} > 0\}.$$

In fact, A. Klyachko [16] characterized LR-sat(n). For $I = \{i_1 < \cdots < i_d\} \subseteq \mathbb{Z}_{>0}$, set

$$\tau(I) := (i_d - d \geq \cdots \geq i_2 - 2 \geq i_1 - 1) \in \mathsf{Par}_d.$$

Theorem 3.1 ([16]). *$(\lambda, \mu, \nu) \in$ LR-sat(n) if and only if $|\lambda| = |\mu| + |\nu|$, and for every $d < n$, and every triple of subsets $I, J, K \subseteq [n]$ of cardinality d such that $c_{\tau(I),\tau(J)}^{\tau(K)} > 0$,*

$$\sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j.$$

For our next result, we need a definition. Let $\lambda^{(1)}, \ldots, \lambda^{(s)} \in \mathrm{Par}_n$ for $s \geq 3$. Think of the indices $1, \ldots, s$ as elements of $\mathbb{Z}/s\mathbb{Z}$. The *multiple Newell–Littlewood number* [7] is

$$N_{\lambda_1, \ldots, \lambda_s} = \sum_{(\alpha_1, \ldots, \alpha_s) \in (\mathrm{Par}_n)^s} \prod_{i \in \mathbb{Z}/s\mathbb{Z}} c_{\alpha_i\, \alpha_{i+1}}^{\lambda^{(i)}}.$$

When $s = 3$, we recover (1.1). We have a representation-theoretic interpretation of these numbers. The second author is pursuing a study of these numbers defined for any graph $G = (V, E)$ (the multiple Newell–Littlewood numbers being those for a cycle).

In [8, Conjecture 1.4], three of the authors conjectured a description of the semigroup NL-semigroup(n). That assertion subsumes Conjecture 4.1 and a description of NL-sat using *extended Horn inequalities* [8, Definition 1.2]. In [7] one finds a resolution of the latter part of the conjecture, giving a second description of NL-sat(n); this is Theorem 3.2.

For $A = \{i_1 < \ldots < i_r\} \subseteq [n]$, let λ_A be the partition using the only parts indexed by A; namely, $\lambda_A = (\lambda_{i_1}, \ldots, \lambda_{i_r})$. Let $|\lambda_A| = \sum_{i \in A} \lambda_i$.

Theorem 3.2 ([7, Theorem 1.5]). *$(\lambda, \mu, \nu) \in$ NL-sat(n) if and only if*

$$0 \leq |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}|$$

for any subsets $A, A', B, B', C, C' \subset [n]$ such that

1. *$A \cap A' = B \cap B' = C \cap C' = \emptyset$;*
2. *$|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'|$;*
3. *$N_{\tau(A'),\tau(B),\tau(C'),\tau(A),\tau(B'),\tau(C)} \neq 0$.*

Example 3.3. For $n = 2$, the table below gives the Horn inequalities (together with $|\nu| = |\lambda| + |\mu|$) and the Extended Horn inequalities:

Horn inequalities	Extended Horn/Klyachko inequalities
$\nu_1 \leq \lambda_1 + \mu_1$	$\nu_1 \leq \lambda_1 + \mu_1$, $\lambda_1 \leq \mu_1 + \nu_1$, $\mu_1 \leq \nu_1 + \lambda_1$
$\nu_2 \leq \lambda_1 + \mu_2$, $\nu_2 \leq \lambda_2 + \mu_1$	$\nu_2 \leq \lambda_1 + \mu_2$, $\lambda_2 \leq \mu_1 + \nu_2$, $\mu_2 \leq \nu_1 + \lambda_2$, $\nu_2 \leq \lambda_2 + \mu_1$, $\lambda_2 \leq \mu_2 + \nu_1$, $\mu_2 \leq \nu_2 + \lambda_1$
$\|\nu\| = \|\lambda\| + \|\mu\|$,	$\|\nu\| \leq \|\lambda\| + \|\mu\|$, $\|\lambda\| \leq \|\mu\| + \|\nu\|$, $\|\mu\| \leq \|\nu\| + \|\lambda\|$
	$\lambda_1 + \mu_2 \leq \lambda_2 + \mu_1 + \|\nu\|$, $\mu_1 + \nu_2 \leq \mu_2 + \nu_1 + \|\lambda\|$ $\nu_1 + \lambda_2 \leq \nu_2 + \lambda_1 + \|\mu\|$, $\lambda_1 + \nu_2 \leq \lambda_2 + \nu_1 + \|\mu\|$ $\mu_1 + \lambda_2 \leq \mu_2 + \lambda_1 + \|\nu\|$, $\nu_1 + \mu_2 \leq \nu_2 + \mu_1 + \|\lambda\|$

In this case, both lists are minimal, but this is not true for larger n.

We now derive minimal inequalities.

Definition 3.4. For $A, A' \subset [n]$, write $A = \{\alpha_1 < \cdots < \alpha_a\}$ and $A' = \{\alpha'_1 < \cdots < \alpha'_{a'}\}$. Define $\tau^0(A, A'), \tau^2(A, A') \in \mathrm{Par}_{a+a'}$ as follows:

$$\begin{array}{ll}
\tau^2(A,A')_k = a + |A' \cap [\alpha_k, n]| & \text{for all } k = 1, \ldots, a; \\
\tau^2(A,A')_{l+a} = |A \cap [\alpha'_{a'+1-l}]| & \text{for all } l = 1, \ldots, a'; \\
\tau^0(A,A')_k = n - a - a' + |[\alpha_k, n] - (A \cup A')| & \text{for all } k = 1, \ldots, a; \\
\tau^0(A,A')_{l+a} = |[\alpha'_{a'+1-l}] - (A \cup A')| & \text{for all } l = 1, \ldots, a'.
\end{array}$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (a^b)$, *i.e.*, the rectangle with a columns and b rows. Define $\lambda^\vee$ *with respect to* (a^b) to be the partition $(a - \lambda_b, a - \lambda_{b-1}, \ldots, a - \lambda_1)$ where we set $\lambda_i = 0$ for $i > k$. We will denote this by $\lambda^{\vee[a^b]}$. This is an NL-generalization of Theorem 3.1.

Theorem 3.5 ([7, Theorem 1.2])**.** *$(\lambda, \mu, \nu) \in \text{NL-sat}(n)$ if and only if*

$$0 \leq |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}|$$

for any subsets $A, A', B, B', C, C' \subset [n]$ such that

1. *$A \cap A' = B \cap B' = C \cap C' = \emptyset$;*
2. *$|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| =: r$;*
3. *$c^{\tau^0(C,C')}_{\tau^0(A,A')^{\vee[(2n-2r)^r]}\ \tau^0(B,B')^{\vee[(2n-2r)^r]}} = c^{\tau^2(C,C')}_{\tau^2(A,A')^{\vee[r^r]}\ \tau^2(B,B')^{\vee[r^r]}} = 1.$*

Moreover, this list of inequalities is irredundant.

We now state a result that factors the NL-coefficients on the boundary of NL-sat(n). It is an analogous to [4, Theorem 7.4] and [13, Theorem 1.4] for $c^{\nu}_{\lambda,\mu}$. Let $\lambda \in \mathrm{Par}_n$ and $A, A' \subset [n]$. Write $A' = \{i'_1 < \cdots < i'_s\}$ and $A = \{i_1 < \cdots < i_t\}$ and set

$$\lambda_{A,A'} = (\lambda_{i'_1}, \ldots, \lambda_{i'_s}, -\lambda_{i_t}, \ldots, -\lambda_{i_1}) \text{ and } \lambda^{A,A'} = \lambda_{[n]-(A \cup A')}, \textit{etc.}$$

Theorem 3.6 ([7, Theorem 1.3])**.** *Let $A, A', B, B', C, C' \subset [n]$ satisfy conditions 1, 2, and 3 from Theorem 3.5. For $(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3$ such that $0 = |\lambda_A| - |\lambda_{A'}| + |\mu_B| - |\mu_{B'}| + |\nu_C| - |\nu_{C'}|$,*

$$N_{\lambda,\mu,\nu} = c^{\nu^*_{C,C'}}_{\lambda_{A,A'},\mu_{B,B'}} N_{\lambda^{A,A'},\mu^{B,B'},\nu^{C,C'}}.$$

Famously, in [16] one finds a relation between LR-sat(n) and the *Hermitian eigencone*. Let $\mathcal{H}(n, \mathbb{C})$ be the set of $n \times n$ complex Hermitian matrices. For $M \in \mathcal{H}(n, \mathbb{C})$, let $\lambda(M) \in \{(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n) : \lambda_i \in \mathbb{R}\}$ be its eigenvalues in weakly decreasing order.

Theorem 3.7 ([16])**.** *Let $(\lambda, \mu, \nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3$. Then $(\lambda, \mu, \nu) \in$ LR-sat(n) if and only if there exists $M_1, M_2, M_3 \in \mathcal{H}(n, \mathbb{C})$ such that $(\lambda, \mu, \nu) = (\lambda(M_1), \lambda(M_2), \lambda(M_3))$ and $M_1 + M_2 = M_3$.*

For $\lambda \in \mathsf{Par}_n$, let $\widehat{\lambda} := (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$. This is an analogue of Theorem 3.7:

Theorem 3.8 ([7, Proposition 3.1])**.** *$(\lambda, \mu, \nu) \in$ NL-sat(n) if and only if there exists*

$$M_1, M_2, M_3 \in \left\{ \begin{pmatrix} A & B \\ \overline{B}^T & -A^T \end{pmatrix} : \overline{A}^T = A \text{ and } B^T = B \right\} \tag{3.1}$$

such that $M_1 + M_2 = M_3$ and $(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) = (\lambda(M_1), \lambda(M_2), \lambda(M_3))$.

The above results of [7] are proved using [7, Theorem 1.1] which shows NL-sat(n) is the truncation of the *saturated tensor cone* for Sp_{2m} whenever $m \geq n \geq 1$. The latter object was studied in [1] and minimal inequalities as well as an eigencone description were given. The set of matrices in (3.1) is $\mathfrak{sp}(2n, \mathbb{C}) \cap \mathcal{H}(2n, \mathbb{C})$ as used in [1]. The result [7, Theorem 1.1] is trivial if $m \geq 2n$ by (1.2); the content is the case $m < 2n$. The proof uses the third authors' work on "GIT-semigroups" and a dose of Schubert calculus.

Finally, the second author has proved a characterization of $N_{\lambda,\mu,\nu} > 0$ in terms of "short exact cycles" of abelian p-groups. This is analogous to the short exact sequences characterization for $c^{\nu}_{\lambda,\mu} > 0$ due to T. Klein [15]. Additionally, the second author has proved a characterization of non-zeroness of multiple Newell–Littlewood numbers in terms of "long exact cycles" of abelian p-groups. Details will appear elsewhere.

4 Some open problems

4.1 Saturation

Knutson–Tao's *Saturation Theorem* [17] states $c^{\nu}_{\lambda,\mu} > 0$ if and only if $c^{t\nu}_{t\lambda,t\mu} > 0$ for some $t \in \mathbb{Z}_{>0}$.

Conjecture 4.1 (NL-Saturation [9, Conjecture 5.5])**.** *Let $(\lambda, \mu, \nu) \in (\mathrm{Par}_n)^3$. Then $N_{\lambda,\mu,\nu} \neq 0$ if and only if $|\lambda| + |\mu| + |\nu|$ is even and there exists $t > 0$ such that $N_{t\lambda,t\mu,t\nu} \neq 0$.*

In [7], Theorem 3.5 is used to prove Conjecture 4.1 for $n \leq 5$, by computer-aided calculation of Hilbert bases. In earlier work, [9, Corollary 5.16] proved the $n = 2$ case, by combinatorial reasoning. In addition, we have:

Theorem 4.2 ([9, Theorem 5.7]). *Conjecture 4.1 holds if λ, μ, or ν is a row or a column.*

Conjecture 4.1 generalizes [9, Corollary 4.5] which is the case $\lambda = \mu = \nu$. That result addresses a matter in H. Hahn's notion of *detection* which is motivated by R. Langlands' *beyond endoscopy proposal* towards his functoriality conjecture; see [10].

4.2 Analogue of M. Kleber's conjecture

Fix a rectangle $R = a \times b$ and consider all products $s_\lambda s_{\lambda^{\vee[R]}}$. M. Kleber [14, Section 3] conjectured that these products, ranging over unordered pairs $(\lambda, \lambda^{\vee[R]})$ are linearly independent in Λ.

Problem 4.3 ([9, Problem 7.2]). *Are the products $s_{[\lambda]}s_{[\lambda^{\vee[R]}]}$, indexed over unordered pairs of partitions $(\lambda, \lambda^{\vee[R]})$ contained in R, linearly independent in Λ?*

By Lemma 2.1(II), M. Kleber's conjecture implies a "yes" answer to Problem 4.3.

4.3 A unimodality conjecture

There seems to be another "structural" aspect of (1.3). Define

$$h_t^{\mu,\nu} = \sum_{\lambda : |\lambda| = |\mu \Delta \nu| + 2t} N_{\mu,\nu,\lambda}.$$

A sequence $(a_k)_{k=0}^N$ is *unimodal* if there exists $0 \leq m \leq N$ such that

$$0 \leq a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots a_{N-1} \geq a_N.$$

Conjecture 4.4 ([9, Conjecture 3.7]). *The sequence $\{h_t^{\mu,\nu}\}_{t=0}^{|\mu \wedge \nu|}$ is unimodal.*

Conjecture 4.4 is true for all $s_{[\mu]}s_{[\nu]}$ where $0 \leq |\mu|, |\nu| \leq 7$, and many larger cases [9]. Theorem 2.3 (II) and (III) suggest a proof approach for Conjecture 4.4: construct chains in Young's poset, each element λ appearing $N_{\mu,\nu,\lambda}$-many times, "centered" at m.

4.4 The associativity relation

Since $N_{\mu,\nu,\lambda}$ are structure constants for the Koike–Terada basis, the associativity relation $(s_{[\mu]}s_{[\nu]})s_{[\lambda]} = s_{[\mu]}(s_{[\nu]}s_{[\lambda]})$, implies for any $\mu, \nu, \lambda, \tau \in \mathsf{Par}$ that:

$$\sum_\theta N_{\mu,\nu,\theta} N_{\theta,\lambda,\tau} = \sum_\theta N_{\nu,\lambda,\theta} N_{\mu,\theta,\tau}. \tag{4.1}$$

Problem 4.5 ([9, Problem 7.1])**.** *Give a bijective proof of (4.1) using the definition (1.1).*

Let $\bar{c}_{\lambda,\mu,\nu}$ be the structure constants for a ring R with basis $\{[\lambda] : \lambda \in \mathrm{Par}_n\}$. Define $\overline{N}_{\mu,\nu,\lambda}$ with these coefficients, as in (1.1). The $\overline{N}_{\mu,\nu,\lambda}$ are structure constants for an associative, commutative ring if $\bar{c}^{\nu}_{\lambda,\mu} = \alpha c^{\nu}_{\lambda,\mu}$ for a scalar α. *What are other examples?*

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