Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #22, 12 pp.
Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Set Partitions, Fermions, and Skein Relations

Jesse Kim[1*] and Brendon Rhoades[1†]

[1] *Department of Mathematics, University of California, San Diego*

Abstract. The second author defined an action of the symmetric group $\mathfrak{S}_n$ on the vector space spanned by noncrossing partitions of $\{1, \ldots, n\}$ by introducing new skein relations which resolve local crossings in set partitions. On the other hand, the second author and Jongwon Kim defined and studied the *fermionic diagonal coinvariant ring* FDR_n which has a definition analogous to the Garsia-Haiman diagonal coinvariant ring DR_n, but with fermionic (anticommuting) variables. We prove that set partition skein relations arises naturally in the context of FDR_n. This clarifies and sharpens results on the skein action and gives an $\mathfrak{S}_n$-equivariant way to resolve an arbitrary set partition into a linear combination of noncrossing partitions.

Keywords: noncrossing set partition, exterior algebra, skein relation

1 Introduction

This extended abstract relates two representations of the symmetric group $\mathfrak{S}_n$ — one combinatorial and one algebraic. We describe the combinatorial module first, and then turn to the algebraic one.

A set partition π of $[n] := \{1, \ldots, n\}$ is *noncrossing* if for all $1 \leq a < b < c < d \leq n$ such that $a \sim c$ and $b \sim d$ in π, we have $a \sim b \sim c \sim d$. We let $\mathrm{NC}(n)$ denote the family of noncrossing set partitions of $[n]$ and $\mathrm{NC}(n,k) \subseteq \mathrm{NC}(n)$ denote the subfamily of noncrossing set partitions with k blocks. It is well-known that these families are counted by the Catalan and Narayana numbers

$$|\mathrm{NC}(n)| = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}, \qquad |\mathrm{NC}(n,k)| = \mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \tag{1.1}$$

Let $\Pi(n)$ be the family of **all** set partitions of $[n]$ and $\Pi(n,k)$ be the set partitions of $[n]$ with k blocks. The sets $\Pi(n)$ and $\Pi(n,k)$ carry a permutation action $\pi \mapsto w(\pi)$ of the symmetric group $\mathfrak{S}_n$. However, this action does not preserve the noncrossing property; the subsets $\mathrm{NC}(n) \subseteq \Pi(n)$ and $\mathrm{NC}(n,k) \subseteq \Pi(n,k)$ are not $\mathfrak{S}_n$-closed. Nevertheless, the second author introduced [12] new "set partition skein relations" to define an $\mathfrak{S}_n$-action on the linearized versions $\mathbb{C}[\mathrm{NC}(n)]$ and $\mathbb{C}[\mathrm{NC}(n,k)]$ of these sets.

*jvkim@math.ucsd.edu
†bprhoades@math.ucsd.edu. Partially supported by NSF grant DMS-1953781.

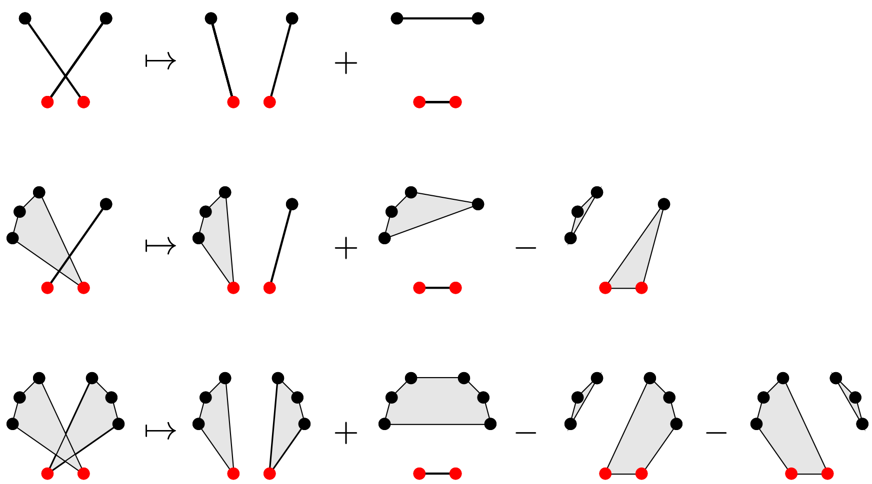


Figure 1: The three skein relations defining the action of $\mathfrak{S}_n$ on $\mathbb{C}[\mathrm{NC}(n)]$. The red vertices are adjacent and the shaded blocks have at least three elements. The 3-term relation obtained by reflecting the middle relation across the y-axis is not shown.

For $1 \leq i \leq n-1$, let $s_i = (i, i+1) \in \mathfrak{S}_n$ be the adjacent transposition. Given $\pi \in \mathrm{NC}(n)$, the *skein action* of s_i on π is given by[1]

$$s_i \cdot \pi := \begin{cases} -s_i(\pi) & \text{if } s_i(\pi) \text{ is noncrossing} \\ \sigma(s_i(\pi)) & \text{otherwise} \end{cases} \tag{1.2}$$

where $\sigma(s_i(\pi)) \in \mathbb{C}[\mathrm{NC}(n)]$ resolves the crossing at $i, i+1$ using the skein relations in Figure 1. More precisely, if B_i and B_{i+1} are the blocks of $s_i(\pi)$ containing i and $i+1$ we have

$$\sigma(s_i(\pi)) := \begin{cases} \pi_1 + \pi_2 & \text{if } |B_i| = |B_{i+1}| = 2, \\ \pi_1 + \pi_2 - \pi_3 & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ \pi_1 + \pi_2 - \pi_4 & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ \pi_1 + \pi_2 - \pi_3 - \pi_4 & \text{if } |B_i|, |B_{i+1}| > 2, \end{cases} \tag{1.3}$$

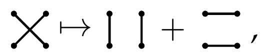
where the set partitions $\pi_1 = \pi$ and π_2, π_3, π_4 are obtained from $s_i(\pi)$ as follows:

- π_2 replaces B_i and B_{i+1} with $(B_i \cup B_{i+1}) - \{i, i+1\}$ and $\{i, i+1\}$,
- π_3 replaces B_i and B_{i+1} with $B_i - \{i\}$ and $B_{i+1} \cup \{i\}$, and
- π_4 replaces B_i and B_{i+1} with $B_i \cup \{i+1\}$ and $B_{i+1} - \{i+1\}$.

[1] This version of the skein action is slightly modified from that in [12]. The skein relations are the same, but the sign convention when $s_i(\pi)$ is noncrossing is different.

Since skein relations preserve the number of blocks $\mathbb{C}[\mathrm{NC}(n,k)] \subseteq \mathbb{C}[\mathrm{NC}(n)]$ is a submodule for this action and we have $\mathbb{C}[\mathrm{NC}(n)] = \bigoplus_{k=1}^{n} \mathbb{C}[\mathrm{NC}(n,k)]$.

The top skein relation in Figure 1 is the famous transformation



which appears in many contexts including Schubert calculus, centralizer algebras, invariant theory, cluster algebras, and knot theory. In contrast, the lower two skein relations do not seem to have been defined before the 2017 paper [12].

The skein action was defined to give representation-theoretic proofs of cyclic sieving results of Reiner–Stanton–White and Pechenik [10, 11]. While its basic properties were established in [12], its purely combinatorial definition made for very involved proofs – even verifying that the local action of s_i satisfies the Coxeter relations used a number of 'miraculous' 16-term identities. It was also unclear whether the skein relations of Figure 1 were tied to other areas of mathematics.

Our algebraic module is as follows. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ and $\Xi_n = (\xi_1, \ldots, \xi_n)$ be two lists of n variables and let $\wedge\{\Theta_n, \Xi_n\}$ be the rank $2n$ exterior algebra over these variables. This ring carries a diagonal action of $\mathfrak{S}_n$, viz.

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \xi_i := \xi_{w(i)} \qquad (w \in \mathfrak{S}_n,\ 1 \leq i \leq n)$$

Adopting the language of physics, we refer to the anticommuting variables θ_i and ξ_i as *fermionic* and general elements $f \in \wedge\{\Theta_n, \Xi_n\}$ as *fermions*.

The second author and Jongwon Kim [5] defined the *fermionic diagonal coinvariant ring* to be the quotient

$$FDR_n := \wedge\{\Theta_n, \Xi_n\}/I \tag{1.4}$$

where $I \subseteq \wedge\{\Theta_n, \Xi_n\}$ is the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. The ring FDR_n is a bigraded $\mathfrak{S}_n$-module, with one grading coming from the θ-variables and the other from the ξ-variables. The ring FDR_n is analogous to the Garsia-Haiman diagonal coinvariant ring DR_n (see [4]) but uses anticommuting rather than commuting variables. In recent years, various authors [1, 2, 3, 8, 15, 14, 17, 18, 20, 19] have considered versions of DR_n involving mixtures of commuting and anticommuting variables.

We recall some $\mathfrak{S}_n$-module terminology. For a partition $\lambda \vdash n$, let S^λ be the corresponding $\mathfrak{S}_n$-irreducible. The *Frobenius image* of an $\mathfrak{S}_n$-module $V = \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$ is the symmetric function $\mathrm{Frob}\, V = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where s_λ is the Schur function of shape λ.

The ring FDR_n was shown [5] to have total dimension $\binom{2n-1}{n}$, proving a conjecture of Zabrocki [19]. The bigraded piece $(FDR_n)_{i,j}$ was shown [5] to vanish unless $i + j < n$. When $i + j < n$ we have

$$\mathrm{Frob}\, (FDR_n)_{i,j} = s_{(i,1^{n-i})} * s_{(j,1^{n-j})} - s_{(i+1,1^{n-i-1})} * s_{(j+1,1^{n-j-1})} \tag{1.5}$$

where $*$ denotes Kronecker product of Schur functions.

The combinatorics of FDR_n went largely unexplored in [5]. A hint at its combinatorial interest is that in the 'extremal bidegrees' $i + j = n - 1$ it has dimension

$$\dim(FDR_n)_{n-k,k-1} = \mathrm{Nar}(n,k) \quad \text{so that} \quad \sum_{k=1}^{n} \dim(FDR_n)_{n-k,k-1} = \mathrm{Cat}(n) \tag{1.6}$$

which is a consequence of Equation (1.5). Our contributions are as follows.

- We enhance (1.6) by establishing $\mathfrak{S}_n$-module isomorphisms

$$(FDR_n)_{n-k,k-1} \cong \mathbb{C}[\mathrm{NC}(n,k)] \quad \text{so that} \quad \bigoplus_{k=1}^{n}(FDR_n)_{n-k,k-1} \cong \mathbb{C}[\mathrm{NC}(n)] \tag{1.7}$$

  between the extreme components of FDR_n and the skein modules. We prove these isomorphisms by attaching fermions $F_\pi, f_\pi \in \wedge\{\Theta_n, \Xi_n\}$ to any set partition $\pi \in \Pi(n)$ which satisfy the skein relations in Figure 1.

- The fermions F_π give a natural method of resolving crossings in set partitions which extends the usual crossing resolution in chord diagrams. We describe this resolution combinatorially.

The skein relations of Figure 1 have appeared in invariant theory. Given two integers $\ell \le n$ which are both ≥ 2, Patrias, Pechenik, and Striker [9] studied the projective variety X of two-step flags $V_\bullet = (0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$ with $\dim V_1 = 2$ and $\dim V_2 = \ell$. They found certain natural elements $g_\pi \in \mathbb{C}[X]$ of the homogeneous coordinate ring of X indexed by set partitions which satisfy the relations of Figure 1. More general flag varieties G/P give a natural setting for possible generalizations of the relations in Figure 1. We are hopeful that our skein relations will see further application to algebra, geometry, and topology.

2 Fermions for Set Partitions

We recall a notion of 'differentiation' in exterior algebras. Let $\Omega_m = (\omega_1, \ldots, \omega_m)$ be a list of fermionic variables and let $\wedge\{\Omega_m\}$ be the exterior algebra over these variables. For $1 \le i \le m$ we define

$$\omega_i \odot (\omega_{j_1} \cdots \omega_{j_r}) := \begin{cases} (-1)^{s-1}\omega_{j_1} \cdots \widehat{\omega_{j_s}} \cdots \omega_{j_r} & \text{if } i = j_s, \\ 0 & \text{if } i \notin \{j_1, \ldots, j_r\}, \end{cases} \tag{2.1}$$

whenever $1 \le j_1, \ldots, j_r \le n$ are distinct indices. Linear extension gives an action $\odot$ of $\wedge\{\Omega_m\}$ on itself called *contraction*. Using our alphabet $(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n)$ of variables, we have an action

$$\odot\colon\ \wedge\{\Theta_n, \Xi_n\} \otimes \wedge\{\Theta_n, \Xi_n\} \longrightarrow \wedge\{\Theta_n, \Xi_n\} \tag{2.2}$$

of $\wedge\{\Theta_n, \Xi_n\}$ on itself.

Certain derivations of $\wedge\{\Theta_n, \Xi_n\}$ will be key to our constructions. For any nonempty subset $B \subseteq [n]$, define the *block operator* $\rho_B \colon \wedge\{\Theta_n, \Xi_n\} \to \wedge\{\Theta_n, \Xi_n\}$ by

$$\rho_B(f) := \begin{cases} \xi_i \cdot (\theta_i \odot f) & B = \{i\} \text{ is a singleton,} \\ \sum_{\substack{i,j \in B \\ i \neq j}} \xi_j \cdot (\theta_i \odot f) & \text{otherwise,} \end{cases} \tag{2.3}$$

for $f \in \wedge\{\Theta_n, \Xi_n\}$. The operator ρ_B raises ξ-degree and lowers θ-degree by one. It bears formal similarity to the *polarization operators* $y_1 \partial^j_{x_1} + \cdots + y_n \partial^j_{x_n}$ (see, *e.g.*, [14]) in the theory of harmonic spaces.

Lemma 1. *For any nonempty subsets A and B of $[n]$, we have $\rho_A \circ \rho_B = \rho_B \circ \rho_A$ as operators on $\wedge\{\Theta_n, \Xi_n\}$.*

Our aim is to attach fermions to set partitions. The following construction is valid by Lemma 1.

Definition 1. Let $\pi = \{B_1 \mid \cdots \mid B_k\} \in \Pi(n)$ be a set partition. We define two fermions $F_\pi, f_\pi \in \wedge\{\Theta_n, \Xi_n\}$ by

$$F_\pi := (\rho_{B_1} \circ \cdots \circ \rho_{B_k})(\theta_1 \cdots \theta_n) \qquad \text{and} \qquad f_\pi := (\xi_1 + \cdots + \xi_n) \odot F_\pi \tag{2.4}$$

As an example of these objects, for $\pi = \{1, 3 \mid 2\}$ we have

$$\begin{aligned} F_{\{1,3 \mid 2\}} &= \rho_{\{1,3\}} \circ \rho_{\{2\}}(\theta_1\theta_2\theta_3) = \rho_{\{1,3\}}(-\xi_2 \cdot \theta_1\theta_3) = \xi_3\xi_2\theta_3 - \xi_1\xi_2\theta_1, \\ f_{\{1,3 \mid 2\}} &= (\xi_1 + \xi_2 + \xi_3) \odot (\xi_3\xi_2\theta_3 - \xi_1\xi_2\theta_1) = \xi_2\theta_3 - \xi_3\theta_3 - \xi_2\theta_1 + \xi_1\theta_1. \end{aligned}$$

If $\pi \in \Pi(n,k)$ has k blocks, the fermion F_π has bidegree $(n-k, k)$ and the fermion f_π has bidegree $(n-k, k-1)$. The F_π are cleaner to work with, but the f_π are useful for the study of FDR_n.

How do the F_π and f_π interact with the combinatorics of set partitions? Recall the natural permutation action $w(\pi)$ of $\mathfrak{S}_n$ on $\Pi(n)$. Denote by $\star$ the sign twist of this action on $\mathbb{C}[\Pi(n)]$. Explicitly, we have

$$w \star \pi := \mathrm{sign}(w)\, w(\pi) \qquad (w \in \mathfrak{S}_n,\ \pi \in \Pi(n)). \tag{2.5}$$

Proposition 1. *The assignments $\pi \mapsto F_\pi$ and $\pi \mapsto f_\pi$ both induce $\mathfrak{S}_n$-module homomorphisms*

$$\mathbb{C}[\Pi(n)] \longrightarrow \wedge\{\Theta_n, \Xi_n\},$$

where $\mathbb{C}[\Pi(n)]$ is endowed with the $\star$-action.

The map in Proposition 1 is neither surjective (for degree reasons) or injective (by our next result). Let $\pi \in \mathrm{NC}(n)$ be a noncrossing partition and let $1 \leq i \leq n$ be such that $s_i(\pi)$ is not noncrossing. Let B_i and B_{i+1} be the blocks of $s_i(\pi)$ containing i and $i+1$. For brevity, define a fermion $F_{\sigma(s_i(\pi))} \in \wedge\{\Theta_n, \Xi_n\}$ by

$$F_{\sigma(s_i(\pi))} := \begin{cases} F_{\pi_1} + F_{\pi_2} & \text{if } |B_i| = |B_{i+1}| = 2, \\ F_{\pi_1} + F_{\pi_2} - F_{\pi_3} & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ F_{\pi_1} + F_{\pi_2} \qquad\quad - F_{\pi_4} & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ F_{\pi_1} + F_{\pi_2} - F_{\pi_3} - F_{\pi_4} & \text{if } |B_i|, |B_{i+1}| > 2, \end{cases} \tag{2.6}$$

where $\pi_1, \ldots, \pi_4 \in \mathrm{NC}(n)$ are as in (1.3). We also define $f_{\sigma(s_i(\pi))} \in \wedge\{\Theta_n, \Xi_n\}$ by

$$f_{\sigma(s_i(\pi))} := (\xi_1 + \cdots + \xi_n) \odot F_{\sigma(s_i(\pi))}. \tag{2.7}$$

The following result states that the fermions F_π and f_π satisfy the skein relations. For complete versions of the proofs in this abstract, see [6].

Theorem 1. *Let $\pi \in \mathrm{NC}(n)$ be a noncrossing partition and $1 \leq i \leq n-1$. Let B_i and B_{i+1} be the blocks of $s_i(\pi)$ containing i and $i+1$, respectively. Then*

$$F_{s_i(\pi)} + F_{\sigma(s_i(\pi))} = 0 \quad \textit{and} \quad f_{s_i(\pi)} + f_{\sigma(s_i(\pi))} = 0. \tag{2.8}$$

in $\wedge\{\Theta_n, \Xi_n\}$ where we interpret $\sigma(s_i(\pi)) = \pi$ if $s_i(\pi)$ is noncrossing.

Proof sketch. Assume $s_i(\pi)$ is not noncrossing. Introduce a variant ψ_B of the block operator ρ_B on $\wedge\{\Theta_n, \Xi_n\}$ by

$$\psi_B(f) := \sum_{\substack{i,j \in B \\ i \neq j}} \xi_j \cdot (\theta_j \odot f) \tag{2.9}$$

for $f \in \wedge\{\Theta_n, \Xi_n\}$. In particular, we have $\psi_B = 0$ when B is a singleton. Like the ρ-operators, the ψ-operators commute. The ψ-operators remove the branching in the definition of $F_{\sigma(s_i(\pi))}$. Writing $A := B_i - \{i\}$ and $B := B_{i+1} - \{i+1\}$, one checks the operator identity

$$\psi_{A \sqcup \{i+1\}} \circ \psi_{B \sqcup \{i\}} + \psi_{A \sqcup \{i\}} \circ \psi_{B \sqcup \{i+1\}} + \psi_{A \sqcup B} \circ \psi_{\{i,i+1\}} - \psi_A \circ \psi_{B \sqcup \{i,i+1\}} - \psi_{A \sqcup \{i,i+1\}} \circ \psi_B = 0. \tag{2.10}$$

Applying both sides of Equation (2.10) to $\theta_1 \cdots \theta_n$ (together with the block operators ρ_C for blocks $C \in s_i(\pi)$ other than B_i, B_{i+1}) yields $F_{s_i(\pi)} + F_{\sigma(s_i(\pi))} = 0$. The further application of $(\xi_1 + \cdots + \xi_n) \odot (-)$ gives $f_{s_i(\pi)} + f_{\sigma(s_i(\pi))} = 0$. $\square$

Singleton blocks play a special role in the theory of skein actions. To this end, we define subsets $\Pi(n,k,m) \subseteq \Pi(n,k)$ and $\mathrm{NC}(n,k,m) \subseteq \mathrm{NC}(n,k)$ by

$$\Pi(n,k,m) := \{\pi \in \Pi(n,k) \,:\, \pi \text{ has } m \text{ singletons}\},$$
$$\mathrm{NC}(n,k,m) := \{\pi \in \mathrm{NC}(n,k) \,:\, \pi \text{ has } m \text{ singletons}\},$$

Our families of set partitions give rise to six subspaces of $\wedge\{\Theta_n, \Xi_n\}$ as follows:

$$\begin{aligned} V(n) &:= \mathrm{span}\{F_\pi \,:\, \pi \in \mathrm{NC}(n)\}, & W(n) &:= \mathrm{span}\{f_\pi \,:\, \pi \in \mathrm{NC}(n)\}, \\ V(n,k) &:= \mathrm{span}\{F_\pi \,:\, \pi \in \mathrm{NC}(n,k)\}, & W(n,k) &:= \mathrm{span}\{f_\pi \,:\, \pi \in \mathrm{NC}(n,k)\}, \\ V(n,k,m) &:= \mathrm{span}\{F_\pi \,:\, \pi \in \mathrm{NC}(n,k,m)\}, & W(n,k,m) &:= \mathrm{span}\{f_\pi \,:\, \pi \in \mathrm{NC}(n,k,m)\}, \end{aligned}$$

Theorem 1 guarantees that these subspaces are closed under the action of $\mathfrak{S}_n$. The next result states that they are isomorphic to the skein modules and implies that their defining spanning sets are in fact bases.

Theorem 2. *Let $m \leq k \leq n$. The action of s_i on $\mathbb{C}[\mathrm{NC}(n)]$ defined in Equation (1.2) satisfies the Coxeter relations, and so extends to an action of $\mathfrak{S}_n$ on $\mathbb{C}[\mathrm{NC}(n)]$ for which $\mathbb{C}[\mathrm{NC}(n,k)]$ and $\mathbb{C}[\mathrm{NC}(n,k,m)]$ are submodules.*

Furthermore, the assignments $F_\pi \leftrightarrow f_\pi \leftrightarrow \pi$ induce $\mathfrak{S}_n$-module isomorphisms

$$V(n) \cong W(n) \cong \mathbb{C}[\mathrm{NC}(n)], \qquad V(n,k) \cong W(n,k) \cong \mathbb{C}[\mathrm{NC}(n,k)],$$
$$V(n,k,m) \cong W(n,k,m) \cong \mathbb{C}[\mathrm{NC}(n,k,m)].$$

for any $m \leq k \leq n$. The common Frobenius images of these modules are

$$\mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n)] = \sum_{k=1}^{n} \mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n,k)], \quad \mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n,k)] = \sum_{m=1}^{k} \mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n,k,m)],$$
$$\mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n,k,m)] = s_{(k-m,k-m,1^{n-2k+m})} \cdot s_{(1^m)}.$$

Proof sketch. By Theorem 1, the map $\pi \mapsto F_\pi$ gives $\mathfrak{S}_n$-epimorphisms from the skein modules to the V-modules. One then proves, using parabolic symmetrizers and antisymmetrizers in $\mathbb{C}[\mathfrak{S}_n]$, that $V(n,k,0) \cong S^{(k,k,1^{n-2k})}$ is the $\mathfrak{S}_n$-irreducible of *flag shape* $(k,k,1^{n-2k}) \vdash n$. Next, one checks directly that

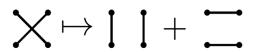
$$V(n,k,m) \cong \mathrm{Ind}_{\mathfrak{S}_{n-m} \times \mathfrak{S}_m}^{\mathfrak{S}_n} V(n-m,k-m,0) \otimes \mathrm{sign}_{\mathfrak{S}_m} \tag{2.11}$$

as $\mathfrak{S}_n$-modules. This proves $\dim V(n,k,m) = |\mathrm{NC}(n,k,m)|$ so that the epimorphism $V(n,k,m) \twoheadrightarrow \mathbb{C}[\mathrm{NC}(n,k,m)]$ is an isomorphism. The other isomorphisms are proven in a similar way. □

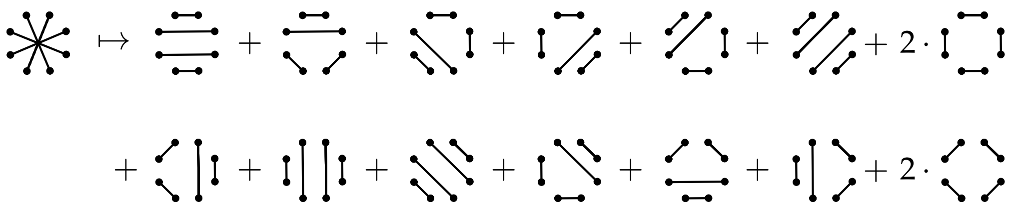
As mentioned earlier, the proof in [12] that the skein action of s_i on $\mathbb{C}[\mathrm{NC}(n)]$ satisfied the Coxeter relations was lengthy and calculation intensive. By contrast, fermions made the proof of Theorem 2 much cleaner. The simplicity of the block operators ρ_B give conceptual reason for 'why' the skein action should exist.

3 Resolving Set Partition Crossings

A *matching* of size n is a set partition of $[n]$ in which every block has size 1 or 2. Any matching may be transformed into a linear combination of noncrossing matchings by repeated use of the local transformation



For example, resolving crossings in the 'asterisk of order 4' yields



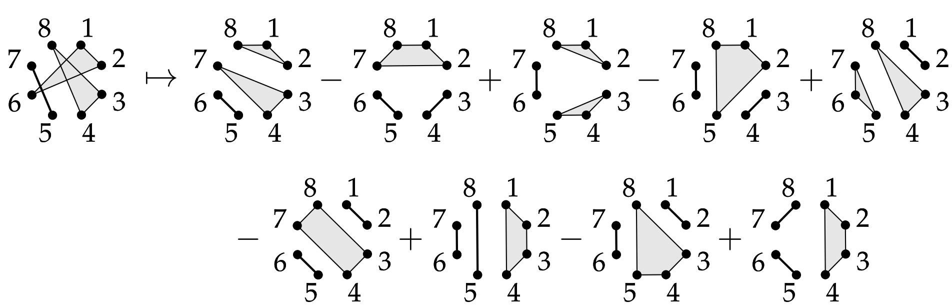
Among other things, this resolution combinatorializes the representation theory of the Temperley-Lieb algebra TL_n, the Kazhdan-Lusztig cellular basis of the $\mathfrak{S}_n$-irreducible of 2-row rectangular shape, $\mathfrak{sl}_2$-web bases, and the coordinate ring of the Grassmannian of 2-planes in n-space [7, 13, 16]. Thanks to skein actions, we can extend this resolution from matchings to arbitrary set partitions.

Definition 2. Let $\pi \in \Pi(n)$ be a set partition. We define $p(\pi) \in \mathbb{C}[\mathrm{NC}(n)]$ by

$$p(\pi) = \sum_{\tau \in \mathrm{NC}(n)} c_{\pi,\tau} \cdot \tau,$$

where the $c_{\pi,\tau}$ are the unique coefficients so that $F_\pi = \sum_{\tau \in \mathrm{NC}(n)} c_{\pi,\tau} \cdot F_\tau$.

When π is a matching, $p(\pi)$ agrees with the resolution described above up to a global sign. For a non-matching example, if $\pi = \{1, 2, 6 \mid 3, 4, 8 \mid 5, 7\} \in \Pi(8)$, applying p yields



The map p fixes noncrossing set partitions and is equivariant with respect to $\mathfrak{S}_n$-actions.

Theorem 3. *The linear map $p\colon \mathbb{C}[\Pi(n)] \twoheadrightarrow \mathbb{C}[\mathrm{NC}(n)]$ is an $\mathfrak{S}_n$-equivariant projection where $\mathbb{C}[\Pi(n)]$ carries the $\star$-action and $\mathbb{C}[\mathrm{NC}(n)]$ carries the skein action.*

Proof. This follows from Theorem 1 and Theorem 2. □

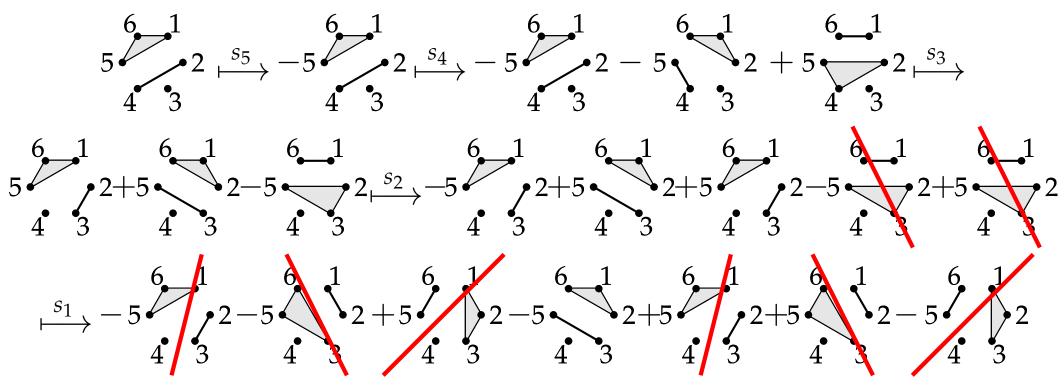
Corollary 1. *Let $w \in \mathfrak{S}_n$ and $\pi \in \mathrm{NC}(n)$ be such that $w(\pi)$ is noncrossing. In the skein action on $\mathbb{C}[\mathrm{NC}(n)]$ we have $w \cdot \pi = \mathrm{sign}(w) w(\pi)$.*

Proof. By Theorem 3 we calculate

$$\mathrm{sign}(w)w(\pi) = \mathrm{sign}(w)p(w(\pi)) = p(w \star \pi) = w \cdot p(\pi) = w \cdot \pi, \tag{3.1}$$

where we used that $w(\pi)$ and π are noncrossing, hence fixed by p. □

To see Corollary 1 in action, consider applying the long cycle $c = (1,2,3,4,5,6) \in \mathfrak{S}_6$ to $\pi = \{1, 5, 6 \mid 2\,4 \mid 3\} \in \mathrm{NC}(6)$. Using the skein action, we calculate the action of $c = s_1 s_2 s_3 s_4 s_5$, cancelling terms along the way:



As Corollary 1 says, the end result is $\mathrm{sign}(c) \cdot c(\pi) = -c(\pi) = -\{1, 2, 6 \mid 3, 5 \mid 4\}$. Corollary 1 implies a cyclic sieving result of Reiner, Stanton, and White [11]. The *q-Narayana number* is

$$\mathrm{Nar}_q(n,k) := \frac{q^{(n-k)(n-k+1)}}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{3.2}$$

where we use the standard q-analog notation

$$[n] := 1 + q + \cdots + q^{n-1}, \quad [n]!_q := [n]_q [n-1]_q \cdots [1]_q, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}.$$

Corollary 2. *The triple $(\mathrm{NC}(n,k), \mathbb{Z}_n, \mathrm{Nar}_q(n,k))$ exhibits the cyclic sieving phenomenon where $\mathbb{Z}_n$ acts on $\mathrm{NC}(n,k)$ by rotation.*

Proof sketch. Corollary 1 implies that the skein action of $c = (1, \ldots, n)$ on $\mathbb{C}[\mathrm{NC}(n,k)]$ is the scalar $\mathrm{sign}(c) = (-1)^{n-1}$ times rotation. Theorem 2 gives the $\mathfrak{S}_n$-isomorphism type of $\mathbb{C}[\mathrm{NC}(n,k)]$. Now apply Springer's Theorem on regular elements. □

Given $\pi \in \Pi(n)$, calculating $p(\pi) \in \mathbb{C}[\mathrm{NC}(n)]$ using Definition 2 involves expanding F_π as a linear combination of $\{F_\tau \,:\, \tau \in \mathrm{NC}(n)\}$. There is a purely combinatorial way to compute $p(\pi)$ as follows.

If $\pi \in \Pi(n)$ is not noncrossing, let $A, B \in \pi$ be two blocks whose convex hulls cross on the circle labeled clockwise with $1, 2, \ldots, n$. The union $A \cup B$ may be expressed as a cyclic sequence $(C_1, C_2, \ldots, C_{2m})$ of nonempty sets where each C_{2i} is a cyclically contiguous subset of A and each C_{2i+1} is a cyclically contiguous subset of B. For example, if $\pi = \{A \mid B\} \in \Pi(16)$ is the two-block set partition with

$$A = \{1, 2, 4, 8, 9, 10, 12, 13, 14, 15, 16\} \text{ and }]B = \{3, 5, 6, 7, 11\}$$

then $m = 3$, and we may take

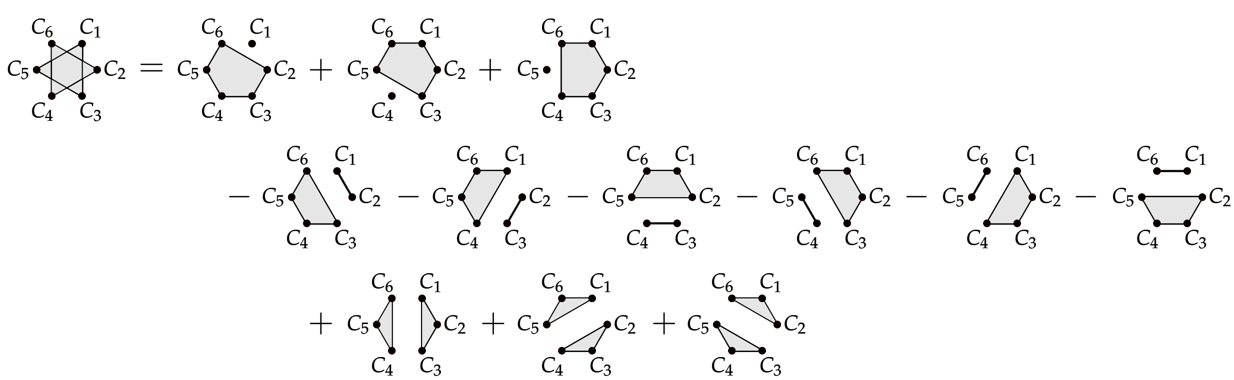
$$(C_1, \ldots, C_6) = (\{1, 2, 12, 13, 14, 15, 16\}, \{3\}, \{4\}, \{5, 6, 7\}, \{8, 9, 10\}, \{11\}).$$

Theorem 4. *With π, A, B, and $(C_1, C_2, \ldots, C_{2m})$ as above we have*

$$F_\pi = \sum_{1 \le i \le j \le 2m} \epsilon(i,j) \cdot F_{\pi(i,j)}$$

where $\pi(i,j)$ is obtained from π by replacing A and B with $C := C_i \cup C_{i+1} \cup \cdots \cup C_j$ and $D := (A \cup B) - C$ and the coefficient $\epsilon(i,j)$ is 0 if either of C or D are singletons and $(-1)^{j-i}$ otherwise.

Theorem 4 resolves the local crossing between the blocks $A, B \in \pi$. In the example above, this resolution looks like



where we have compressed the sets C_i to vertices and terms with an isolated C_2, C_3, and C_6 do not appear since these sets are singletons. For arbitrary $\pi \in \Pi(n)$, repeated applications of this rule yield the resolution $p(\pi)$ of the crossings in π.

4 Fermionic Diagonal Coinvariants and the Skein Action

Thus far, we have considered the fermions F_π and f_π as members of the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. The next theorem establishes that $\{f_\pi : \pi \in \mathrm{NC}(n)\}$ descends to a basis of the 'extreme bidegree part' of its quotient FDR_n.

Theorem 5. *Let $k \leq n$. The set $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ descends to a basis of $(FDR_n)_{n-k,k-1}$ and the composite map*

$$W(n,k) \hookrightarrow \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1} \twoheadrightarrow (FDR_n)_{n-k,k-1}$$

is an isomorphism of $\mathfrak{S}_n$-modules.

Proof sketch. We need only prove that that $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ descends to a basis. Since $\dim (FDR_n)_{n-k,k-1} = \mathrm{Nar}(n,k)$, it suffices to show linear independence. In [5, 8] it is proven that the defining ideal I of FDR_n is generated by three elements:

$$\theta_1 + \cdots + \theta_n, \quad \xi_1 + \cdots + \xi_n, \quad \text{and} \quad \theta_1\xi_1 + \cdots + \theta_n\xi_n.$$

One deduces that $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ is orthogonal to $I \cap \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1}$ under the inner product $\langle -, - \rangle$ on $\wedge\{\Theta_n, \Xi_n\}_{n-k,k-1}$ which declares the 'fermionic monomial basis' $\{\theta_{i_1} \cdots \theta_{i_{n-k}} \xi_{j_1} \cdots \xi_{j_{k-1}} : 1 \leq i_1 < \cdots < i_{n-k} \leq n, 1 \leq j_1 < \cdots < j_{k-1} \leq n\}$ to be orthonormal. The linear independence of $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ in $(FDR_n)_{n-k,k-1}$ follows from Theorem 2. □

Theorem 5 gives a combinatorial basis for the portion of $FDR_n = \bigoplus_{i+j<n} (FDR_n)_{i,j}$ in extreme bidegrees $i + j = n - 1$. In a forthcoming paper, the first author will extend our results to all bidegrees $i + j < n$.

Acknowledgements

The authors are grateful to Alessandro Iraci, Jongwon Kim, Rebecca Patrias, Oliver Pechenik, and Jessica Striker for helpful conversations.

References

[1] F. Bergeron. "The bosonic-fermionic diagonal coinvariant modules conjecture". 2020. arXiv: 2005.00924.

[2] S. Billey, B. Rhoades, and V. Tewari. "Boolean product polynomials, Schur positivity, and Chern plethysm". *Int. Math. Res. Not. IMRN* (2019). rnz261.

[3] M. D'Adderio, A. Iraci, and A. Wyngaerd. "Theta operators, refined Delta conjectures, and coinvariants". *Adv. Math.* **376** (2021). 107477.

[4] M. Haiman. “Conjectures on the quotient ring by diagonal invariants”. *J. Algebraic Combin.* **3** (1994), pp. 17–76.

[5] J. Kim and B. Rhoades. “Lefschetz theory for exterior algebras and fermionic diagonal coinvariants”. *Int. Math. Res. Not. IMRN* (2020). rnaa203, pp. 334–338.

[6] J. Kim and B. Rhoades. “Set partitions, fermions, and skein relations”. 2021. arXiv: 2109.06373.

[7] J. P. S. Kung and G.-C. Rota. “The invariant theory of binary forms”. *Bull. Amer. Math. Soc.* **19**.1 (1984), pp. 27–85.

[8] R. Orellana and M. Zabrocki. “A combinatorial model for the decomposition of multivariate polynomial rings as an S_n-module”. *Electron. J. Combin.* **27**.3 (2020). P3.24.

[9] R. Patrias, O. Pechenik, and J. Striker. A web basis of invariant polynomials from noncrossing partitions. In preparation, 2021.

[10] O. Pechenik. “Cyclic sieving of increasing tableaux and small Schröder paths”. *J. Combin. Theory Ser. A* **125** (2014), pp. 357–378.

[11] V. Reiner, D. Stanton, and D. White. “The cyclic sieving phenomenon”. *J. Combin. Theory Ser. A* **108** (2004), pp. 17–50.

[12] B. Rhoades. “A skein action of the symmetric group on noncrossing partitions”. *J. Algebraic Combin.* **45**.1 (2017), pp. 81–127.

[13] B. Rhoades. “The polytabloid basis expands positively into the web bases”. *Forum Math. Sigma* **7** (2019). e26.

[14] B. Rhoades and A. T. Wilson. “Vandermondes in superspace”. *Trans. Amer. Math. Soc.* **373**.6 (2020), pp. 4483–4516.

[15] B. Rhoades and A. T. Wilson. “Set superpartitions and superspace duality modules”. 2021. arXiv:2104.05630.

[16] H. Russell and J. Tymoczko. “The transition matrix between the Specht and web bases is unipotent with additional vanishing entries”. *Int. Math. Res. Not. IMRN* **2019**.5 (), pp. 1479–1502–4516.

[17] J. Swanson. “Tanisaki witness relations for harmonic differential forms”. 2021. arXiv: 2109.05080.

[18] J. Swanson and N. Wallach. “Harmonic differential forms for pseudo-reflection groups I. Semi-invariants”. *J. Combin. Theory Ser. A* (2021). To appear. arXiv:2001.06076.

[19] M. Zabrocki. “Coinvariants and harmonics”. Blog for Open Problems in Algebraic Combinatorics, 2020.

[20] M. Zabrocki. “A module for the Delta conjecture”. 2019. arXiv:1902.08966.