A statistic for regions of braid deformations

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Abstract. An arrangement of hyperplanes in \mathbb{R}^n is a finite collection of hyperplanes. The regions are the connected components of the complement of the union of these hyperplanes. By a theorem of Zaslavsky, the number of regions of a hyperplane arrangement is the sum of the absolute values of the coefficients of its characteristic polynomial. Arrangements that contain hyperplanes parallel to subspaces whose defining equations are $x_i - x_j = 0$ form an important class called the deformations of the braid arrangement. In a recent work, Bernardi showed regions of certain deformations are in one-to-one correspondence with certain labeled trees. In this article, we define a statistic on these trees such that the distribution is given by the coefficients of the characteristic polynomial. In particular, our statistic applies to the well-studied families like extended Catalan, Shi, Linial and semiorder.

Keywords: Hyperplane arrangement, combinatorial statistic, braid arrangement, labeled trees

1 Introduction

A hyperplane arrangement \mathcal{A} is a finite collection of affine hyperplanes (*i.e.*, codimension 1 subspaces and their translates) in \mathbb{R}^n . A region of \mathcal{A} is a connected component of $\mathbb{R}^n \setminus \bigcup \mathcal{A}$. The number of regions of \mathcal{A} is denoted by $r(\mathcal{A})$. The poset of non-empty intersections of hyperplanes in an arrangement \mathcal{A} ordered by reverse inclusion is called its *intersection poset* denoted by $L(\mathcal{A})$. The ambient space of the arrangement (*i.e.*, \mathbb{R}^n) is an element of the intersection poset; considered as the intersection of none of the hyperplanes. The *characteristic polynomial* of \mathcal{A} is defined as

$$\chi_{\mathcal{A}}(t) := \sum_{x \in \mathcal{L}(\mathcal{A})} \mu(\widehat{0}, x) t^{\dim(x)},$$

where μ is the Möbius function of the intersection poset and $\hat{0}$ corresponds to \mathbb{R}^n . Using the fact that every interval of the intersection poset of an arrangement is a geometric lattice, we have

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i,$$
(1.1)

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where c_i is a non-negative integer for all $0 \le i \le n$ [10, Corollary 3.4]. The characteristic polynomial is a fundamental combinatorial and topological invariant of the arrangement and plays a significant role throughout the theory of hyperplane arrangements.

In this article our focus is on the enumerative aspects of (rational) arrangements in \mathbb{R}^n . In that direction we have the following seminal result by Zaslavsky.

Theorem 1.1 ([12]). Let \mathcal{A} be an arrangement in \mathbb{R}^n . The number of regions of \mathcal{A} is given by

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$$
$$= \sum_{i=0}^n c_i.$$

When the regions of an arrangement are in bijection with a certain combinatorially defined set, one could ask if there is a corresponding 'statistic' on the set whose distribution is given by the c_i 's. For example, the regions of the braid arrangement in \mathbb{R}^n (whose hyperplanes are given by the equations $x_i - x_j = 0$ for $1 \le i < j \le n$) correspond to the n! permutations of [n]. The characteristic polynomial of this arrangement is $t(t-1) \cdots (t-n+1)$ [10, Corollary 2.2]. Hence, c_i 's are the unsigned Stirling numbers of the first kind. Consequently, the distribution of the statistic 'number of cycles' on the set of permutations is given by the coefficients of the characteristic polynomial.

In this paper, we consider arrangements where each hyperplane is of the form $x_i - x_j = s$ for some $s \in \mathbb{Z}$. Such arrangements are called deformations of the braid arrangement. Recently, Bernardi [3] obtained a method to count the regions of any deformation of the braid arrangement using certain objects called *boxed trees*. For certain special deformations, which he calls *transitive*, he also obtained an explicit bijection between the regions of the arrangement and a certain set of trees. Our main aim is to obtain a statistic on such trees whose distribution is given by the coefficients of the characteristic polynomial of the corresponding arrangement.

For any finite set of integers *S*, we associate a deformation of the braid arrangement $\mathcal{A}_S(n)$ in \mathbb{R}^n with hyperplanes

$${x_i - x_j = k \mid k \in S, \ 1 \le i < j \le n}.$$

Important examples of such arrangements are the Catalan, Shi, Linial and semiorder arrangements. These correspond to $S = \{-1, 0, 1\}, \{0, 1\}, \{1\}, \text{ and } \{-1, 1\}$ respectively. For any $m \ge 1$, the extended Catalan arrangement, or *m*-Catalan arrangement, in \mathbb{R}^n is $\mathcal{A}_S(n)$ where $S = \{-m, \ldots, m\}$. Similarly, the extended Shi, Linial, and semiorder arrangements correspond to $S = \{-m + 1, \ldots, m\}, \{-m + 1, \ldots, m\} \setminus \{0\},$ and $\{-m, \ldots, m\} \setminus \{0\}$ respectively.

If the set *S* satisfies certain conditions (see Definition 2.5), then the arrangements $A_S(n)$ are called transitive. The extended Catalan, Shi, Linial, and semiorder arrangements are all transitive. We note here that Bernardi [3] considers a larger class of arrangements to be transitive, but we only focus on such arrangements.

From [3, Theorem 3.8], we know that if *S* is transitive, then the regions of $A_S(n)$ are in bijection with a certain set of trees $\mathcal{T}_S(n)$ (see Definition 2.4). For example, when $S = \{0, 1\}$ which corresponds to the Shi arrangement, $\mathcal{T}_{\{0,1\}}(n)$ is the set of labeled binary trees with *n* nodes where any right node has a label smaller than its parent. *Example* 1.2. A tree in $\mathcal{T}_{\{0,1\}}(4)$ is shown in Figure 1.



Figure 1: A tree in $T_{\{0,1\}}(4)$

For such a tree, we define the *trunk* to be the path from the root to the leftmost leaf. Using the nodes on the trunk, we obtain a sequence of numbers. A node in this sequence that is greater than all the nodes after it is called a *branch node*. For the tree in Figure 1, the sequence on the trunk is 4, 2, 3 and the branch nodes are 4 and 3. These definitions can be generalized to trees in $T_S(n)$ for other sets *S*.

The main theorem of this article is:

Theorem 1.3. For a transitive set *S*, the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_S(n)}(t)$ is the number of trees in $\mathcal{T}_S(n)$ with *j* branch nodes.

The article begins with a short account of Bernardi's work [3] in Section 2. In Section 3 the branch statistic is introduced and the main theorem is proved. In Section 4 we discuss the class of extended Catalan arrangements and derive some properties of the coefficients of the characteristic polynomial.

2 Preliminaries

A *tree* is a graph with no cycles. A *rooted tree* is a tree with a distinguished vertex called the root. We will draw rooted trees with their root at the bottom. Children of a vertex v in a rooted tree are those vertices w that are adjacent to v and such that the unique path from the root to w passes through v. Similarly, we can define the parent of a vertex v to be the vertex w for which v is the child of w. Any non-root vertex has a unique parent. All the vertices that have at least one child are called *nodes* and those that do not are called *leaves*.

A *rooted plane tree* is a rooted tree with a specified ordering for the children of each node. When drawing a rooted plane tree, the children of any node will be ordered from left to right. The *left siblings* of a vertex v are the vertices that are also children of the parent of v but are to the left of v. We denote the number of left siblings of v as lsib(v).

Definition 2.1. An (m + 1)-*ary tree* is a rooted plane tree where each node has exactly (m + 1) children. We will denote by $\mathcal{T}^{(m)}(n)$ the set of all (m + 1)-ary trees with n nodes labeled with distinct elements from [n].

For trees in $\mathcal{T}^{(m)}(n)$, we will denote the node having label $i \in [n]$ by just *i*.

Definition 2.2. If a node *i* in a tree $T \in \mathcal{T}^{(m)}(n)$ has at least one child that is a node, the *cadet* of *i* is the rightmost such child, which we denote by cadet(i).

Example 2.3. Figure 1 shows an element of $\mathcal{T}^{(1)}(4)$ where

- 4 is the root,
- lsib(2) = 0, lsib(3) = 0, lsib(1) = 1,
- cadet(4) = 2, and cadet(2) = 1.

Definition 2.4. For any finite set of integers *S* with $m = \max\{|s| | s \in S\}$, define $\mathcal{T}_S(n)$ to be the set of trees in $\mathcal{T}^{(m)}(n)$, such that if cadet(i) = j:

- $lsib(j) \notin S \cup \{0\}$ implies i < j and
- $-\operatorname{lsib}(j) \notin S$ implies i > j.

Definition 2.5. A finite set of integers *S* is said to be *transitive* if for any $s, t \notin S$,

- st > 0 implies $s + t \notin S$ and
- s > 0 and $t \le 0$ implies $s t \notin S$ and $t s \notin S$.

Example 2.6. As mentioned in Section 1, for any $m \ge 1$, the sets $\{-m, \ldots, m\}$, $\{-m + 1, \ldots, m\}$, $\{-m, \ldots, m\} \setminus \{0\}$, and $\{-m + 1, \ldots, m\} \setminus \{0\}$ are all transitive.

We can now state the result for arrangements $A_S(n)$ where *S* is transitive. Though Bernardi [3] derived results for more general deformations, we will only be focused on these.

Theorem 2.7 ([3, Theorem 3.8]). For any transitive set of integers *S*, the regions of the arrangement $A_S(n)$ are in bijection with the trees in $T_S(n)$.

Before looking at the characteristic polynomials of such arrangements, we recall a few results from [9]. Suppose that $c: \mathbb{N} \to \mathbb{N}$ is a function and for each $n, j \in \mathbb{N}$, we define

$$c_j(n) = \sum_{\{B_1,\dots,B_j\}\in\Pi_n} c(|B_1|)\cdots c(|B_j|),$$

where Π_n is the set of partitions of [n]. Define for each $n \in \mathbb{N}$,

$$h(n) = \sum_{j=0}^{n} c_j(n).$$

From [9, Example 5.2.2], we know that in such a situation,

$$\sum_{n,j\geq 0} c_j(n) t^j \frac{x^n}{n!} = \left(\sum_{n\geq 0} h(n) \frac{x^n}{n!}\right)^t.$$

Informally, we consider h(n) to be the number of "structures" that can be placed on an *n*-set where each structure can be uniquely broken up into a disjoint union of "connected sub-structures". Here c(n) denotes the number of connected structures on an *n*-set and $c_j(n)$ denotes the number of structures on an *n*-set with exactly *j* connected sub-structures.

We now consider the characteristic polynomials of arrangements of the form $A_S(n)$. For a fixed set *S*, the sequence of arrangements ($A_S(1), A_S(2), ...$) forms what is called an *exponential sequence of arrangements* (ESA).

Definition 2.8 ([10, Definition 5.14]). A sequence of arrangements $(A_1, A_2, ...)$ is called an ESA if

- A_n is a braid deformation in \mathbb{R}^n and
- for any *k*-subset *I* of [*n*], the arrangement

$$\mathcal{A}_n^I = \{ H \in \mathcal{A}_n \mid H \text{ is of the form } x_i - x_j = s \text{ for some } i, j \in I \}$$

satisfies $L(\mathcal{A}_n^I) \cong L(\mathcal{A}_k)$ (isomorphic as posets).

The result on ESAs that we will need is the following.

Theorem 2.9 ([10, Theorem 5.17]). If $(A_1, A_2, ...)$ is an ESA, then

$$\sum_{n\geq 0}\chi_{\mathcal{A}_n}(t)\frac{x^n}{n!}=\left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-t}.$$

Remark 2.10. We note that this is also a special case of [3, Theorem 5.2].

Using this result, the form of a characteristic polynomial given in (1.1), and the above discussion on connected structures, we note that interpreting the coefficients of the polynomial $\chi_{\mathcal{A}_S(n)}(t)$ is equivalent to defining a notion of "connected structures" for trees in $\mathcal{T}_S(n)$. We do this in the next section.

3 A branch statistic

A *label set* is a finite set of positive integers. For any label set *V*, we define $\mathcal{T}^{(m)}(V)$ to be the set of (m + 1)-ary trees with |V| nodes labeled distinctly using *V*. Note that $\mathcal{T}^{(m)}([n]) = \mathcal{T}^{(m)}(n)$.

We now describe the method we use to break up a tree in $\mathcal{T}^{(m)}(V)$ into "connected sub-structures", which we call *branches*.

Definition 3.1. The *trunk* of a tree in $\mathcal{T}^{(m)}(V)$ is the path from the root to the leftmost leaf. The nodes on the trunk of the tree break up the tree into sub-trees, which we call *twigs* (see Figure 2).

Let the nodes on the trunk of a tree be $v_1, v_2, ..., v_k$, where v_1 is the root and v_{i+1} is the leftmost child of v_i for any $i \in [k-1]$. If $v_i = \max\{v_1, ..., v_k\}$, then the first branch of the tree consists of the twigs corresponding to the nodes $v_1, ..., v_i$. If $v_j = \max\{v_{i+1}, ..., v_k\}$, then the second branch of the tree consists of the twigs corresponding to the nodes $v_1, ..., v_i$. If $v_j = \max\{v_{i+1}, ..., v_k\}$, then the second branch of the tree consists of the twigs corresponding to the nodes $v_{i+1}, ..., v_i$. Continuing this way, we break up the tree into branches.

Note that the number of branches of the tree is just the number of right-to-left maxima of the sequence $v_1, v_2, ..., v_k$ of nodes on the trunk, *i.e.*, the number of v_i such that $v_i > v_j$ for all j > i. We will call such v_i the *branch nodes* of the trunk.

Example 3.2. The tree in Figure 2 has 3 twigs and 2 branches. The first branch consists of just the first twig since 6 is the largest node in the trunk. The second branch consists of the second and third twigs since 5 is larger than 4. Here 6 and 5 are the branch nodes.



Figure 2: A labeled 3-ary tree with twigs and branches specified.

We use the notation $\mathcal{T}_{j}^{(m)}(V)$ to denote the trees in $\mathcal{T}^{(m)}(V)$ having *j* branches. To prove that this is indeed a break-up of trees into connected sub-structures, we have to

prove that

$$|\mathcal{T}_{j}^{(m)}(V)| = \sum_{\{B_{1},\dots,B_{j}\}\in\Pi_{V}} |\mathcal{T}_{1}^{(m)}(B_{1})|\cdots|\mathcal{T}_{1}^{(m)}(B_{j})|$$

Hence, "connected" trees are those with exactly one branch, *i.e.*, trees where the last node of the trunk is the one with the largest label.

The connected components associated to a given tree are the branches of the tree.

Example 3.3. The connected components associated to the tree in Figure 2 are given in Figure 3.



Figure 3: Connected components of the tree in Figure 2.

Comparing the labels on the trunks, it can be checked that a collection of connected trees (with disjoint label sets) can be put together in exactly one way to form a tree for which they are the branches.

Example 3.4. The labels on the trunks of the connected trees in Figure 3 are 4,5 and 6, of which 6 is the largest. This means that the tree on the right, call it T_1 , must form the first branch. Hence, the only tree whose branches are these trees is the one formed by gluing the tree on the left, call it T_2 , to T_1 by replacing the leftmost leaf of T_1 with the root of T_2 . This gives back the tree in Figure 2.

We define the set $\mathcal{T}_S(V)$ analogously to Definition 2.4. We set $\mathcal{T}_S := \bigcup_V \mathcal{T}_S(V)$ where the union is over all label sets *V*. We now show that

1. the connected components of any tree in T_S are also in T_S and

2. trees that are built using connected trees in T_S are also in T_S .

We first note that (1) follows since the condition for a tree to be in T_S is a local condition. To prove (2), we only have to check that conditions in Definition 2.4, which we call Condition *S*, are satisfied for the branch nodes of a tree built using connected trees in T_S . If a branch node does not have a cadet, Condition *S* is trivially satisfied. If a branch node *u* has a cadet *v*, we consider two cases:

- If the cadet is not the first child, then Condition *S* is satisfied since it is satisfied by the connected components of the tree.
- If the cadet is the first child, then we must have *u* > *v* since *u* is a branch node. This makes sure that Condition *S* is satisfied since we have lsib(*v*) = 0 and hence lsib(*v*) ∈ *S* ∪ {0}.

Hence, from the discussion in Section 2, we get the following result.

Theorem 3.5. For a transitive set of integers S, the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_S(n)}(t)$ is the number of trees in $\mathcal{T}_S(n)$ with j branches.

Example 3.6. When $S = \{0\}$, we obtain the braid arrangement. Here, $\mathcal{T}_{\{0\}}(n)$ corresponds to permutations of [n] and Theorem 3.5 states that the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_{\{0\}}(n)}(t)$ is the number of permutations of [n] with j right-to-left maxima. By [8, Corollary 1.3.11], this agrees with the observation in Section 1 that the coefficients are the Stirling numbers of the first kind.

Example 3.7. The Linial arrangement \mathcal{L}_n in \mathbb{R}^n is the deformation $\mathcal{A}_{\{1\}}(n)$. The trees in $\mathcal{T}_{\{1\}}(n)$, called Linial trees, are those labeled binary trees where

- any right node has a label less than that of its parent and
- any left node whose sibling is a leaf has smaller label than that of its parent.

The Linial trees for n = 3 are given in Figure 4. Counting the branches in these trees, we get $\chi_{\mathcal{L}_3}(t) = t^3 + 3t^2 + 3t$, which agrees with the known formula for the characteristic polynomial (for example, see [2, Theorem 4.2]).

4 Extended Catalan arrangement

We now focus on the case when $S = \{-m, -m + 1, ..., m - 1, m\}$ for some $m \ge 1$. The corresponding arrangement $\mathcal{A}_S(n)$ is called the *m*-Catalan arrangement in \mathbb{R}^n . Here $\mathcal{T}_S(n) = \mathcal{T}^{(m)}(n)$ and from Theorem 3.5 we get that the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_S(n)}(t)$ is

$$C(m,n,j) := \sum_{k=j}^{n} T_m(n,k) \binom{n}{k} c(k,j)(n-k)!$$

where

• *c*(*k*, *j*) is the number of permutations of [*k*] with *j* right-to-left maxima (unsigned Stirling number of first kind), and



Figure 4: Linial trees for n = 3.

• $T_m(n,k)$ is the number of unlabeled (m + 1)-ary trees with n nodes, k of which are on its trunk, given by

$$\frac{mk}{(m+1)n-k}\binom{(m+1)n-k}{n-k}.$$

This follows from the bijection between trees and Dyck paths (for example, see [3, Section 8]) and the discussion about returns in Dyck paths in [4, Page 4].

Noting that for any $m, n, k \ge 1$,

$$B_m(n,k) := \frac{(n-1)!}{(k-1)!} \binom{(m+1)n}{n-k}$$

is the number of ways to partition [n] into k blocks and associate to each block B a tree in $\mathcal{T}^{(m)}(B)$ (for example, using [9, Theorem 5.3.10]), we get another expression for C(m, n, j).

Proposition 4.1. *For any* $m, n, j \ge 1$ *,*

$$C(m, n, j) = \sum_{k=j}^{n} (-1)^{k-j} B_m(n, k) c(k, j).$$

Remark 4.2. The triangle of numbers C(1, n, j) is listed in the OEIS [6] as A038455. For $m \ge 2$, the triangle C(m, n, j) does not seem to be listed.

Using the known formula for $|\mathcal{T}^{(m)}(n)|$ (see [9, Section 5.3]), we have

$$C^{(m)}(n) := \sum_{j=1}^{n} C(m, n, j) = \frac{n!}{mn+1} \binom{(m+1)n}{n}.$$

The following properties of C(m, n, j) are easy consequences of Theorem 3.5.

Proposition 4.3. *For any* $m, n, j \ge 1$ *, we have*

- 1. $C(m, n, j) \leq C(m + 1, n, j)$,
- 2. $C^{(m)}(n) \leq C(m+1, n, 1),$
- 3. $C(m, n, j) \leq C(m, n + 1, j)$,
- 4. $C^{(m)}(n) \leq C(m, n+1, 1).$

Using a slightly different break-up of trees into "connected components", we obtain the following results. The details and a generalization are given in [5].

Proposition 4.4. *For any* $m, n \ge 1$ *, we have*

$$C(m,n,1) \geq \sum_{j=2}^{n} C(m,n,j).$$

Proposition 4.5. *For any* $m, n, j \ge 1$ *, we have*

$$C(m, n, j) \ge C(m, n, j+1).$$

There are several combinatorial objects that correspond to the regions of the extended Catalan arrangement (especially in the case m = 1, see [11]). One such is the generalized Dyck paths. We now describe a corresponding statistic for these Dyck paths.

A labeled *m*-Dyck path on [n] is a sequence of (m + 1)n terms where

- *n* terms are '+m',
- mn terms are '-1',
- the sum of any prefix of the sequence is non-negative, and
- each +m term is given a distinct label from [n].

A labeled *m*-Dyck path on [n] can be drawn in \mathbb{R}^2 in the natural way. Start the path at (0,0), read the labeled *m*-Dyck path and for each term move by (1,m) if it is +m and by (1,-1) if it is -1. Also, label each +m step with its corresponding label in [n].

A Dyck path breaks up into *primitive* parts based on when it touches the *x*-axis. If a labeled Dyck path has *k* primitive parts, then we break the path into *compartments* as follows. If the number *n* is in the i_1^{th} primitive part, then the primitive parts up to the i_1^{th} form the first compartment. Let *j* be the largest number in $[n] \setminus A$ where *A* is the set of numbers in first compartment. If *j* is in the i_2^{th} primitive part then the primitive parts after the i_1^{th} up to the i_2^{th} form the second compartment. Continuing this way, we break up a labeled Dyck path into compartments.

Example 4.6. The labeled 1-Dyck path on [7] given in Figure 5 has 3 primitive parts and 2 compartments.



Figure 5: A labeled 1-Dyck path with compartments specified.

It can be checked that this is a valid break-up of Dyck paths into connected structures. Thus have the following.

Result 4.7. The number of labeled m-Dyck paths on [n] with j compartments is C(m, n, j).

We say that a labeled Dyck path has *j* right-to-left maxima if the string of labels before its first down step has *j* right-to-left maxima. For example, the string of labels before the first down step in the Dyck path in Figure 5 is 4,7,2. Hence, it has 2 right-to-left maxima.

Using the bijection between labeled trees and labeled Dyck paths given in [3] and the result in Section 3, we get the another statistic on labeled Dyck paths with the same distribution.

Result 4.8. The number of labeled *m*-Dyck paths on [n] with *j* right-to-left maxima is C(m, n, j).

5 Concluding remarks

We note that a combinatorial interpretation for the coefficients of the characteristic polynomial of the Linial arrangement is already given in [7, Corollary 4.2]. This is in terms of alternating trees.

For various deformations of the braid arrangement, expressions for the characteristic polynomials are known (for example, see [1, 2]). Hence, for transitive sets *S*, these can be used to extract coefficients and hence give formulas for the number of trees in T_S according to number nodes and branches.

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