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Weighted Ehrhart Series and a Type-B Analogue of a Formula of MacMahon

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Abstract. We present a formula for the joint distribution of major index and descent statistic on signed multiset permutations. It allows a description in terms of the h^*-polynomial of a certain polytope. We associate a family of polytopes to (generalised) permutations of types A and B. We use this connection to study properties of the (generalised) Eulerian numbers, such as palindromicity and unimodality, by identifying certain properties of the associated polytope.

Keywords: descents, signed multiset permutations, coloured multiset permutations, Ehrhart theory, h^*-polynomials, q-analogues

In this note we present some of the material of the preprint [17]. We develop a signed analogue of a formula of MacMahon to which we refer in the following as MacMahon's formula. Throughout we denote by $\eta = (\eta_1, \ldots, \eta_r)$ a composition of $n \in \mathbb{N}$. The formula reads as the following identity of rational functions [13]

$$\frac{\sum_{w \in S_\eta} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^{n}(1 - q^i t)} = \sum_{k \geq 0} \prod_{i=1}^{r} \binom{k + \eta_i}{\eta_i}_q t^k \in \mathbb{Q}(q, t), \tag{MM}$$

where S_η denotes the set of multiset permutations and $\binom{n}{k}_q$ the q-binomial coefficient. By work of Chapoton [6] and Stanley [14], the series on the right hand side of (MM) can be interpreted as a q-Ehrhart series of products of simplices, counting weighted lattice points in a polytope. In the special case $\eta = (1, \ldots, 1)$ and $q = 1$, we obtain the Eulerian polynomial on the left hand side and the Ehrhart series of products of one-dimensional simplices, *i.e.* of the cube, on the right hand side of (MM):

$$\frac{\sum_{w \in S_n} t^{\mathrm{des}(w)}}{(1 - t)^{n+1}} = \sum_{k \geq 0} (k + 1)^n t^k.$$

Since S_n is a Coxeter group of type A, we refer to (MM) as MacMahon's formula of type A. By passing from permutations to signed permutations we obtain the hyperoctahedral group B_n, a Coxeter group of type B, and its generalisation B_η, the set of signed

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multiset permutations. We give new definitions of major index and descent statistic on signed multiset permutations. This generalisation of MacMahon's formula proceeds on the side of the (q-) Ehrhart series: instead of counting (weighted) lattice points in products of simplices, we count those in products of cross polytopes, which can be seen as signed analogues of simplices. Our main theorem (Theorem 2.2), a type-B analogue of MacMahon's formula, establishes the following identity:

$$\frac{\sum_{w\in B_\eta} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^{n}(1-q^i t)} = \sum_{k\geq 0}\left(\prod_{i=1}^{r}\sum_{j=0}^{\eta_i}\left(q^{\frac{j(j-1)}{2}}\binom{\eta_i}{j}_q\binom{k-j+\eta_i}{\eta_i}_q\right)\right)t^k\mathbb{Q}(q,t). \tag{MB}$$

Furthermore it interprets the right hand side as a q-Ehrhart series of products of η_i-dimensional cross polytopes. We refer to Section 2 for further details.

We will explain what is needed to understand both sides of each (MM) and (MB) in Section 1. Our main result is stated in Section 2. We will use the connection between permutations statistics and Ehrhart theory to study properties like palindromicity and unimodality of the (generalised) Eulerian numbers of types A and B. Further possible generalisations of our result are discussed in Section 3.

1 Preliminaries

1.1 Permutations statistics

The left hand side of MacMahon's formula of types A and B is described in terms of statistics on (signed) multiset permutations.

1.1.1 Multiset permutations

A multiset permutation w is a rearrangement of the letters of the multiset

$$\{\{\underbrace{1,\dots,1}_{\eta_1},\underbrace{2,\dots,2}_{\eta_2},\dots,\underbrace{r,\dots,r}_{\eta_r}\}\}.$$

We write $w = w_1 \dots w_n$ (using the one-line notation) for such a permutation and denote by S_η the set of all permutations of the multiset given by η. The *descent set* is defined to be $\mathrm{Des}(w) = \{i \in \{1,\dots,n-1\} : w_i > w_{i+1}\}$. The *major index* and the *descent* statistic are

$$\mathrm{maj}(w) = \sum_{i\in\mathrm{Des}(w)} i \quad \text{and} \quad \mathrm{des}(w) = |\,\mathrm{Des}(w)|.$$

If, for example, $\eta = (2,3)$, then $w = 22121$ is a permutation of the corresponding multiset $\{\{1,1,2,2,2\}\}$. Here, $\mathrm{Des}(w) = \{2,4\}$ and therefore $\mathrm{maj}(w) = 6$ and $\mathrm{des}(w) = 2$.

The coefficients of the descent polynomial given by $\sum_{w \in S_\eta} t^{\mathrm{des}(w)}$ are called generalised Eulerian numbers (of type A). Note that for $\eta = (1,\ldots,1)$ we have $S_\eta = S_n$ and the coefficients of $\sum_{w \in S_n} t^{\mathrm{des}(w)}$ are the Eulerian numbers.

1.1.2 Signed multiset permutations

In the following we introduce signed multiset permutations and give definitions of the major index and descent statistic generalising those discussed in Section 1.1.1.
Recall that *signed permutations* are obtained from permutations $w = w_1 \cdots w_n \in S_n$ (in one-line notation), where each letter w_i is independently equipped with a sign ± 1. We denote by B_n the set of signed permutations on the letters $1,\ldots,n$.

Similarly, we obtain the set of *signed multiset permutations* B_η from the set of multiset permutations S_η by 'adding signs': more precisely, the elements of B_η are given by a multiset permutation $w \in S_\eta$ and $\epsilon : \{1,\ldots,n\} \to \{\pm 1\}$, a sign vector which attaches every i (or w_i) with a positive or negative sign. It is sometimes useful to write an element of B_η as a pair $w^\epsilon := (w,\epsilon)$, where $w \in S_\eta$ and $\epsilon\colon \{1,\ldots,n\} \to \{\pm 1\}$ encodes the signs appearing in w^ϵ.

In one-line notation, we write $\bar{i}$ instead of $-i$. For example, for $\eta = (2)$ the set of signed multiset permutations is

$$B_\eta = \{(11,(1,1)),(11,(1,-1)),(11,(-1,1)),(11,(-1,-1))\}$$

which we abbreviate by $\{11,\bar{1}1,1\bar{1},\bar{1}\bar{1}\}$.

For the definition of a descent set of an element $w \in B_\eta$ we need a notion of standardisation. We use the map $\mathrm{std}\colon S_\eta \to S_n$, which is known for multiset permutations, defined as follows: for an element $w \in S_\eta$ we obtain $\mathrm{std}(w) \in S_n$ by substituting the η_1 1s from left to right with $1,\ldots,\eta_1$, the η_2 2s from left to right with $\eta_1+1,\ldots,\eta_1+\eta_2$ and so on. In [17] we extend this *standardisation* to signed multiset permutations

$$\begin{aligned} B_\eta &\to B_n, \\ (w,\epsilon) &\mapsto (\mathrm{std}(w),\epsilon). \end{aligned}$$

We denote both the standardisation on S_η and the one on B_η by std. For instance, $\mathrm{std}(\bar{2}\bar{2}12\bar{1}) = \bar{3}\bar{4}15\bar{2}$.

In [17] we define the *descent set* of a signed multiset permutation $w^\epsilon \in B_\eta$ to be

$$\mathrm{Des}(w^\epsilon) := \{i \in \{0,\ldots,n-1\} : \mathrm{std}(w^\epsilon)_i > \mathrm{std}(w^\epsilon)_{i+1}\},$$

where $\operatorname{std}(w^\epsilon)_0 := 0$. In other words, for $w^\epsilon = (w, \epsilon) \in B_\eta$

$$\begin{aligned}\operatorname{Des}(w^\epsilon) = \{i \in \{0, \dots, n-1\} :\ & \epsilon(i) = \epsilon(i+1) = 1 \text{ and } w_i > w_{i+1},\\ & \text{or } \epsilon(i) = \epsilon(i+1) = -1 \text{ and } w_i \leq w_{i+1}, \text{ or } \epsilon(i) = 1 \text{ and } \epsilon(i+1) = -1\},\end{aligned}$$

where $w_0 := 0$ and $\epsilon(0) := 1$. In particular, $0 \in \operatorname{Des}(w^\epsilon)$ if and only if $\epsilon(1) = -1$. Note that on elements in B_n our definition of the descent set coincides with the Coxeter-theoretic one; see [2, Proposition 8.1.2].

Further, the *major index* and *descent* statistics are

$$\operatorname{maj}(w^\epsilon) := \sum_{i \in \operatorname{Des}(w^\epsilon)} i \quad \text{and} \quad \operatorname{des}(w^\epsilon) := |\operatorname{Des}(w^\epsilon)|.$$

For instance, for $\overline{2}\overline{2}12\overline{1} \in B_{(2,3)}$ we have $\operatorname{Des}(\overline{2}\overline{2}12\overline{1}) = \operatorname{Des}(\overline{3}\overline{4}152) = \{0, 1, 4\}$, hence $\operatorname{maj}(\overline{2}\overline{2}12\overline{1}) = 5$ and $\operatorname{des}(\overline{2}\overline{2}12\overline{1}) = 3$. Different definitions of major index and descent for signed multiset permutations appear in [9] and [12].

We call the coefficients of the descent polynomial $\sum_{w \in B_\eta} t^{\operatorname{des}(w)}$ *generalised Eulerian numbers of type* B. Note that for $\eta = (1, \dots, 1)$, we have $B_\eta = B_n$ and the coefficients of $\sum_{w \in B_n} t^{\operatorname{des}(w)}$ are the Eulerian numbers of type B.
Our goal is to construct for each η and $X \in \{S_\eta, B_\eta\}$ a polytope such that the generating function

$$\frac{\sum_{w \in X} q^{\operatorname{maj}(w)} t^{\operatorname{des}(w)}}{\prod_{i=0}^{n} (1 - q^i t)}$$

is a weighted Ehrhart series. For $X = S_\eta$ this is MacMahon's formula (of type A).

1.2 Ehrhart theory

As we shall now explain, both the rational functions in (MM) and in (MB) may be interpreted as weighted Ehrhart series of certain polytopes. We start with the special case where $q = 1$, viz. classical Ehrhart theory.

1.2.1 Classical Ehrhart theory

Throughout, let $\mathcal{P} = \mathcal{P}_n$ be an n-dimensional lattice polytope in $\mathbb{R}^n$. The *lattice point enumerator* of $\mathcal{P}$ is the function $\mathrm{L}_\mathcal{P} \colon \mathbb{N} := \mathbb{N} \cup \{0\} \to \mathbb{N}_0$ given by $\mathrm{L}_\mathcal{P}(k) := |k\mathcal{P} \cap \mathbb{Z}^n|$. For details on polytopes and Ehrhart theory see [1] and [18].
A fundamental result in this theory is Ehrhart's Theorem [8], which states that the function $\mathrm{L}_\mathcal{P}(k)$ is a polynomial in k, the so-called *Ehrhart polynomial*. Equivalently, its generating function, the *Ehrhart series* of $\mathcal{P}$, is of the form

$$\operatorname{Ehr}_\mathcal{P}(t) := \sum_{k \geq 0} \mathrm{L}_\mathcal{P}(k) t^k = \frac{h^*(t)}{(1-t)^{n+1}} \in \mathbb{Q}(t),$$

where the numerator, the so-called h^*-*polynomial* of $\mathcal{P}$, has degree at most n.

Ehrhart series of products of polytopes can be described in terms of Hadamard products. For series $A(t) = \sum_{k\geq 0} a_k t^k, B(t) = \sum_{k\geq 0} b_k t^k \in \mathbb{Q}(t)$ we denote their Hadamard product (with respect to t) by $(A * B)(t) := \sum_{k\geq 0} a_k b_k t^k$.

Remark 1.1. For $\eta = (\eta_1, \ldots, \eta_r)$ a composition of n, let $\mathcal{P}_{\eta_i}$ be an η_i-dimensional polytope for $i \in \{1, \ldots, r\}$. Let further $\mathrm{L}_{\mathcal{P}_{\eta_i}}(k)$ be the Ehrhart polynomial and $\mathrm{Ehr}_{\mathcal{P}_{\eta_i}}(t)$ the Ehrhart series of $\mathcal{P}_{\eta_i}$. The product $\mathcal{P}_\eta := \mathcal{P}_{\eta_1} \times \cdots \times \mathcal{P}_{\eta_r}$ is a $\sum_{i=1}^r \eta_i$-dimensional polytope with Ehrhart polynomial $\prod_{i=1}^r \mathrm{L}_{\mathcal{P}_{\eta_i}}(k)$. Therefore its Ehrhart series is given by

$$\mathrm{Ehr}_{\mathcal{P}_\eta}(t) = \sum_{k\geq 0} \prod_{i=1}^r \mathrm{L}_{\mathcal{P}_{\eta_i}}(k) t^k = \mathop{\ast}_{i=1}^{r} \mathrm{Ehr}_{\mathcal{P}_{\eta_i}}(t).$$

The polytopes which are relevant for us are products of simplices or cross polytopes.

Example 1.2. The h^*-polynomials of products of simplices and cross polytopes can be described through permutations statistics:

(a) The n-dimensional standard simplex is the convex hull of zero and the unit vectors denoted by $\Delta_n := \mathrm{conv}\{0, e_1, \ldots, e_n\}$. Its Ehrhart series is given by

$$\mathrm{Ehr}_{\Delta_n}(t) = \sum_{k\geq 0} \binom{n+k}{n} t^k = \frac{1}{(1-t)^{n+1}} = \frac{\sum_{w\in S_{(n)}} t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

For the n-dimensional unit cube $\square_n := [0,1]^n$, which is the product of n one-dimensional simplices, we obtain

$$\mathrm{Ehr}_{\square_n}(t) = \sum_{k\geq 0} (k+1)^n t^k = \frac{\sum_{w\in S_n} t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

(b) For the n-dimensional cross polytope $\Diamond_n := \mathrm{conv}\{0, e_1, -e_1, \ldots, e_n, -e_n\}$, the Ehrhart series is given by

$$\mathrm{Ehr}_{\Diamond_n}(t) = \sum_{k\geq 0} \sum_{j=0}^{n} \binom{n}{j} \binom{k+n-j}{n} t^k = \frac{(1+t)^n}{(1-t)^{n+1}} = \frac{\sum_{w\in B_{(n)}} t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

For the n-dimensional cube (centered at the origin) $\boxdot_n := [-1,1]^n$, which is the product of n one-dimensional cross polytopes, we obtain

$$\mathrm{Ehr}_{\boxdot_n}(t) = \sum_{k\geq 0} (2k+1)^n t^k = \frac{\sum_{w\in B_n} t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

The first three Ehrhart series can be found in [1, Section 2], the last one follows from (2.1), a special case of Theorem 2.2, which was already shown in [3, Theorem 3.4].

1.2.2 Weighted Ehrhart theory

We obtain q-analogues of Ehrhart series of simplices and cross polytopes by refining the lattice point enumeration. Inspired by [6] and [14] we define for each $k \in \mathbb{N}$ a *weight function* $\mu_{k,n} \colon k\Delta_n \to \mathbb{N}_0$ on the kth dilate of Δ_n such that

$$\mathrm{Ehr}_{\Delta_n,\boldsymbol{\mu}_n}(q,t) := \sum_{k\geq 0} \sum_{x \in k\Delta_n \cap \mathbb{Z}^n} q^{\mu_{k,n}(x)} t^k = \frac{1}{\prod_{i=0}^n (1-q^i t)} \left(= \frac{\sum_{w \in S_{(n)}} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^n (1-q^i t)} \right). \tag{1.1}$$

We denote the family of weight functions $(\mu_{k,n})$ by $\boldsymbol{\mu}_n$. We call the series in (1.1) the *q-Ehrhart series* (or *weighted Ehrhart series*) of Δ_n. Note that for $q = 1$ we obtain the classical Ehrhart series.

We omit the precise definition of the $\boldsymbol{\mu}_n$, which utilises a bijection between $k\Delta_n$ and the kth dilate of the so-called order polytope of an n-chain and some linear form on $\mathbb{Z}^n$, see [17] for details. The weight functions we define are no longer linear forms, so this gives rise to a different approach defining weighted Ehrhart series than Chapoton uses. It turns out that the weight function behaves well under taking products, *i.e.*

$$\mathrm{Ehr}_{\Delta_\eta,\boldsymbol{\mu}_n}(q,t) = \mathop{\ast}_{i=1}^{r} \mathrm{Ehr}_{\Delta_{\eta_i},\boldsymbol{\mu}_{\eta_i}}(q,t). \tag{1.2}$$

This leads to a q-analogue of Example 1.2 (a):

Example 1.3. For the n-dimensional standard simplex this procedure yields

$$\mathrm{Ehr}_{\Delta_n,\boldsymbol{\mu}_n}(q,t) = \sum_{k\geq 0} \binom{n+k}{n}_q t^k = \frac{1}{\prod_{i=0}^n (1-q^i t)}.$$

The q-Ehrhart series of the n-dimensional unit cube is given by

$$\mathrm{Ehr}_{\square_n,\boldsymbol{\mu}_n}(q,t) = \sum_{k\geq 0} \binom{1+k}{1}_q^n t^k = \frac{\sum_{w \in S_n} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^n (1-q^i t)}.$$

Generalising Example 1.3 to products of η_i-dimensional simplices and to permutation statistics on S_η, we obtain MacMahon's formula of type A (Theorem 2.1).

Next we extend the weight functions above to cross polytopes. Lattice points of the cross polytope and its dilates can be described as lattice points in disjoint unions of (shifted) simplices. Figure 1 illustrates how we obtain the new weight function $\overline{\boldsymbol{\mu}}_n$ from $\boldsymbol{\mu}_n$ for the third dilate of the two-dimensional cross polytope: we map a lattice point, *e.g.*, a point in the red simplex in the left cross polytope in Figure 1, to one in the standard simplex by taking absolute values of the entries. This point then lies in the red simplex

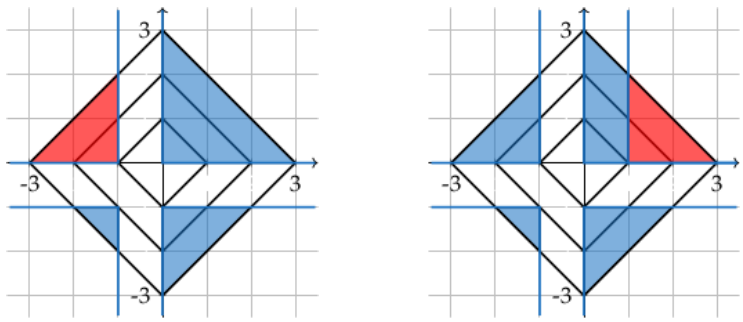

Figure 1: A subdivision of $3\Diamond_2 \cap \mathbb{Z}^2$ into (shifted) standard simplices. The red simplex in the cross polytope on the left is identified with the red one on the right.

in the cross polytope on the right of Figure 1. Since the points in red simplex on the right are contained in $3\Delta_2$, we associate a weight to this point by applying the weight function $\mu_{3,2}$ on the $3\Delta_2$ we determined above.

As sketched above we define weight functions $\overline{\mu}_{k,n} \colon k\Diamond_n \to \mathbb{N}_0$ which fulfill

$$\mathrm{Ehr}_{\Diamond_n,\overline{\boldsymbol{\mu}}_n}(q,t) := \sum_{k\geq 0}\sum_{x\in k\Diamond_n\cap\mathbb{Z}^n} q^{\overline{\mu}_{k,n}(x)}t^k = \frac{\prod_{i=0}^{n-1}(1+q^i t)}{\prod_{i=0}^{n}(1-q^i t)} \left(= \frac{\sum_{w\in B_{(n)}} q^{\mathrm{maj}(w)}t^{\mathrm{des}(w)}}{\prod_{i=1}^{r}(1-q^i t)} \right). \tag{1.3}$$

Analogously to (1.2) we obtain

$$\mathrm{Ehr}_{\Diamond_\eta,\overline{\boldsymbol{\mu}}_n}(q,t) = \mathop{\ast}_{i=1}^{r} \mathrm{Ehr}_{\Diamond_{\eta_i},\overline{\boldsymbol{\mu}}_{\eta_i}}(q,t). \tag{1.4}$$

The essence of our type-B analogue of MacMahon's formula (see Theorem 2.2) is an explicit description of the numerator of (1.4).

2 MacMahon's formula of type B

We obtain an interpretation of MacMahon's formula and its generalisation of type B in terms of weighted Ehrhart series. We use this connection to study properties like palindromicity of the sequences of (generalised) Eulerian numbers of types A and B.

Theorem 2.1 (MacMahon's formula of type A, [6, 13, 14])**.** *The joint distribution of major index and descent on the set of multiset permutations is a q-analogue of the h^*-polynomial of products of standard simplices, i.e.*

$$\frac{\sum_{w\in S_\eta} q^{\mathrm{maj}(w)}t^{\mathrm{des}(w)}}{\prod_{i=0}^{n}(1-q^i t)} = \sum_{k\geq 0}\prod_{i=1}^{r}\binom{k+\eta_i}{\eta_i}_q t^k = \mathrm{Ehr}_{\Delta_\eta,\boldsymbol{\mu}_n}(q,t).$$

The first identity was proven by MacMahon [13, §462, Vol. 2, Ch. IV, Sect. IX], the second is due to Chapoton [6, Section 4] and Stanley [14, Section 8]. Our main result is a 'signed analogue' of Theorem 2.1:

Theorem 2.2 (MacMahon's formula of type B, [17]). *The joint distributions of major index and descent on signed multiset permutations is a q-analogue of the h^*-polynomial of products of cross polytopes, i.e.*

$$\frac{\sum_{w\in B_\eta} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^{n}(1-q^i t)} = \sum_{k\geq 0}\left(\prod_{i=1}^{r}\sum_{j=0}^{\eta_i}\left(q^{\frac{j(j-1)}{2}}\binom{\eta_i}{j}_q\binom{k-j+\eta_i}{\eta_i}_q\right)\right)t^k = \mathrm{Ehr}_{\diamond_\eta,\overline{\mu}_n}(q,t).$$

We obtain the following corollaries as special cases of Theorem 2.2, which are (the missing) q-analogues of Example 1.2 (b).

For $r = 1$ we recover (1.3) as the q-Ehrhart series of a n-dimensional cross polytope on the right hand side of Theorem 2.2:

$$\frac{\sum_{w\in B_{(n)}} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^{n}(1-q^i t)} = \frac{\prod_{i=0}^{n-1}(1+q^i t)}{\prod_{i=0}^{n}(1-q^i t)} = \mathrm{Ehr}_{\diamond_n,\overline{\mu}_n}(q,t).$$

Let $\eta_i = 1$ for all i, so $B_\eta = B_n$. Then

$$\frac{\sum_{w\in B_n} t^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}}{\prod_{i=0}^{n}(1-q^i t)} = \sum_{k\geq 0}\left(\binom{k+1}{1}_q+\binom{k}{1}_q\right)^n t^k = \mathrm{Ehr}_{\boxdot_n,\overline{\mu}_n}(q,t). \tag{2.1}$$

The first identity is also known by [7, Equation 26].

We use the special case of Theorem 2.1 and Theorem 2.2 where $q = 1$ to re-prove palindromicity of the generalised Eulerian numbers of type A and real-rootedness of the Eulerian numbers of types A and B. These properties are known by [5, Proposition 2.12], [10], and [3]. Moreover, we obtain new results for the generalised Eulerian numbers of type B, which turn out to be palindromic and unimodal.

The h^*-polynomial of a polytope $\mathcal{P}$ is palindromic, *i.e.* $h^*_k = h^*_{n-k}$ for all $0 \leq k \leq \frac{n}{2}$, if and only is $\mathcal{P}$ is Gorenstein; see [1, Section 4]. Analysing products of simplices and cross products yields:

Proposition 2.3.

(a) The generalised Eulerian numbers of type A *are palindromic if and only if η is a rectangle (i.e. all parts are equal).*

(b) The generalised Eulerian numbers of type B *are palindromic for every η.*

Corollary 2.4.

(a) The generalised Eulerian polynomial (of type A*) satisfies a functional equation*

$$\sum_{w \in S_\eta} (t^{-1})^{\mathrm{des}(w)} = t^{-(r-1)m} \sum_{w \in S_\eta} t^{\mathrm{des}(w)}$$

for some $m \in \mathbb{N}_0$ *if and only if* η *is the rectangle* $\eta = (m, \dots, m)$.

(b) The generalised Eulerian polynomial of type B *satisfies a functional equation for all* η

$$\sum_{w \in B_\eta} (t^{-1})^{\mathrm{des}(w)} = t^{-n} \sum_{w \in B_\eta} t^{\mathrm{des}(w)}.$$

In [17] we show that products of cross polytopes are *reflexive* and *anti-blocking* (see also [11] for the relevant definitions). The h^*-polynomial of such polytopes are known to be unimodal due to [11, Theorem 3.4].

Proposition 2.5. *The generalised Eulerian numbers of type* B *are unimodal.*

Further it is known that the Eulerian numbers (of type A) can be interpreted as the h-vector of the barycentric subdivision of the boundary of the simplex; see [4, Theorem 2.2]. Interpreting the simplex as a type-A polytope and the cross polytope a type-B analogue is supported by the following proposition. Supplementing the Coxeter-theoretic proof, we show the result in an elementary and self-contained way also in [17].

Proposition 2.6 ([3, Theorem 2.3])**.** *The h-vector of the barycentric subdivision of the boundary of the cross polytope is given by the Eulerian numbers of type* B.

Using [4, Theorem 3.1] this leads to the following.

Corollary 2.7. *The Eulerian polynomials of types* A *and* B *only have real roots. In particular, the sequences of their coefficients are log-concave and unimodal.*

3 Further generalisations: coloured multiset permutations

It may seem natural to seek a type D analogue of MacMahon's formula. A first step towards this would be to find an n-dimensional polytope $\mathcal{P}_n$ such that

$$\frac{\sum_{w \in D_n} t^{\mathrm{des}(w)}}{(1-t)^{n+1}} = \mathrm{Ehr}_{\mathcal{P}_n}(t).$$

However, at present, we do not know how to generalise elements in D_n to even signed multiset permutations without losing the product structure of the corresponding Ehrhart

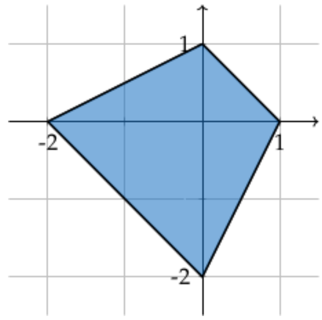

Figure 2: The distorted cross polytope $\mathcal{C}_2^3$.

series.

A natural way for a generalisation comes from considering B_n as the wreath product of the cyclic group of order two by the symmetric group. This leads to the study of coloured permutations $S_n^c := \mathbb{Z}/c\mathbb{Z} \wr S_n$.

Generalising this further in [17] we define coloured multiset permutations. We denote a *coloured multiset permutation* $w^\gamma := (w, \gamma)$ by the indexed permutation $w^\gamma = w_1^{\gamma_1} \cdots w_n^{\gamma_n}$, where $w \in S_\eta$ and $\gamma\colon \{1, \ldots, n\} \to \{0, \ldots, c-1\}$. Denote by S_η^c the set of all coloured multiset permutations. Fixing an ordering, *e.g.*, $1^{c-1} < \cdots < r^{c-1} < \cdots < 1^0 < \cdots < r^0$, we define a *descent* statistic as

$$\begin{aligned} \operatorname{des}(w^\gamma) = |\{i \in \{0, \ldots, n-1\} : {} & \gamma_i = \gamma_{i+1} = 0 \text{ and } w_i > w_{i+1}, \\ & \text{or } \gamma_i = \gamma_{i+1} > 0 \text{ and } w_i \geq w_{i+1}, \\ & \text{or } \gamma_i < \gamma_{i+1}\}|, \end{aligned} \tag{3.1}$$

where $w_0^{\gamma_0} := 0^0$. For instance, for $\eta = (2, 3)$ and $c = 3$ we obtain $\operatorname{des}(2^1 2^1 1^2 2^2 1^0) = 3$. In the special case of $S_\eta^c = S_n^c$ our definition of descents coincides with the one in [16] and for $c = 2$ we obtain the descent statistic from Section 1.1.2.

We are able to show that in a special case the descent polynomial of S_η^c is an h^*-polynomial of a polytope. Intuitively, compared to B_η we increase the number of negatives by adding colours. This leads to a product of distorted cross polytopes

$$\mathcal{C}_n^c := \operatorname{conv}\{e_1, \ldots, e_n, -(c-1)e_1, \ldots, -(c-1)e_n\}.$$

As an example, $\mathcal{C}_2^3$ is illustrated in Figure 2.

Question 3.1. *Does there exist a description of a function* stat *as a permutation statistic on* S_η^c *such that*

$$\frac{\sum_{w \in S_\eta^c} t^{\operatorname{stat}(w)}}{(1-t)^{n+1}} = \mathop{\ast}_{i=1}^{r} \operatorname{Ehr}_{\mathcal{C}_{\eta_i}^c}(t)? \tag{3.2}$$

In the special case that $\eta = (\eta_1, \ldots, \eta_r)$ with $\eta_i \leq 2$ for every i, we answer Question 3.1 affirmatively.

Proposition 3.2 ([17]). *For $\eta = (\eta_1, \ldots, \eta_r)$ with $\eta_i \leq 2$ for every i, we have*

$$\frac{\sum_{w \in S_\eta^c} t^{\mathrm{des}(w)}}{(1-t)^{n+1}} = \mathop{\ast}_{i=1}^{r} \mathrm{Ehr}_{\mathcal{C}_{\eta_i}^c}(t). \tag{3.3}$$

In the special case where $S_\eta^c = S_n^c$ the corresponding polytope is the cth dilate of the n-dimensional unit cube.

Corollary 3.3 ([16, Theorem 32]).

$$\frac{\sum_{w \in S_n^c} t^{\mathrm{des}(w)}}{(1-t)^{n+1}} = \mathop{\ast}_{i=1}^{n} \mathrm{Ehr}_{\mathcal{C}_1^c}(t) = \mathrm{Ehr}_{\prod_{i=1}^n \mathcal{C}_1^c}(t) = \mathrm{Ehr}_{c\square_n}(t).$$

The statement of (3.3) of Proposition 3.2 fails for larger η, even for $\eta = (3)$. Computations with SageMath [15] show that in general the descent polynomial defined by (3.1) is not an h^*-polynomial of a polytope.
At least a necessary condition for a positive answer to Question 3.1 is satisfied. Indeed, a lemma in Ehrhart theory ([1, Corollary 3.21]) states that the coefficients of the h^*-polynomial sum up to the normalised volume of the n-dimensional polytope, *i.e.* $h_n^* + \cdots + h_0^* = n! \ \mathrm{vol}(\mathcal{P})$.

Proposition 3.4 ([17]). *For $t = 1$, the numerator of both sides in (3.2) coincide, that is*

$$|S_\eta^c| = n! \,\mathrm{vol}\left(\prod_{i=1}^{r} \mathcal{C}_{\eta_i}^c\right).$$

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References

[1] M. Beck and S. Robins. *Computing the Continuous Discretely*. Undergraduate Texts in Mathematics. Springer, New York, 2015, pp. xx+285.

[2] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[3] F. Brenti. "q-Eulerian polynomials arising from Coxeter groups". *European J. Combin.* **15**.5 (1994), pp. 417–441.

[4] F. Brenti and V. Welker. "f-vectors of barycentric subdivisions". *Math. Z.* **259**.4 (2008), pp. 849–865.

[5] A. Carnevale and C. Voll. "Orbit Dirichlet series and multiset permutations". *Monatsh. Math.* **186**.2 (2018), pp. 215–233.

[6] F. Chapoton. "q-analogues of Ehrhart polynomials". *Proc. Edinb. Math. Soc. (2)* **59**.2 (2016), pp. 339–358.

[7] C.-O. Chow and I. M. Gessel. "On the descent numbers and major indices for the hyperoctahedral group". *Adv. in Appl. Math.* **38**.3 (2007), pp. 275–301.

[8] E. Ehrhart. "Sur les polyèdres rationnels homothétiques à n dimensions". *C. R. Acad. Sci. Paris* **254** (1962), pp. 616–618.

[9] D. Foata and G.-N. Han. "Signed words and permutations. III. The MacMahon Verfahren". *Sém. Lothar. Combin.* **54** (2005/07), Art. B54a, 20.

[10] G. Frobenius. "Über die Bernoullischen Zahlen und die Eulerschen Polynom". *Sitzungsber. Preuss. Akad. Wiss.* (1910), pp. 809–847.

[11] F. Kohl, M. Olsen, and R. Sanyal. "Unconditional reflexive polytopes". *Discrete Comput. Geom.* **64**.2 (2020), pp. 427–452.

[12] Z. Lin. "On the descent polynomial of signed multipermutations". *Proc. Amer. Math. Soc.* **143**.9 (2015), pp. 3671–3685.

[13] P. A. MacMahon. *Combinatory Analysis*. Vol. I, II (bound in one volume). Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY, 2004, pp. ii+761. Reprint of *An Introduction to Combinatory Analysis* (1920) and *Combinatory Analysis. Vol. I, II* (1915, 1916).

[14] R. P. Stanley. *Ordered Structures and Partitions*. Memoirs of the American Mathematical Society, No. 119. American Mathematical Society, Providence, R.I., 1972, pp. iii+104.

[15] W. Stein et al. *Sage Mathematics Software (Version 9.3)*. The Sage Development Team. 2021. Link.

[16] E. Steingrímsson. "Permutation statistics of indexed permutations". *European J. Combin.* **15**.2 (1994), pp. 187–205.

[17] E. Tielker. "Weighted Ehrhart series and a type-B analogue of a formula of MacMahon". 2022. arXiv:2203.15774.

[18] G. M. Ziegler. *Lectures on Polytopes*. Vol. 152. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. x+370.